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SPACINGS BETWEEN SELECTED PRIME DIVISORS OF AN INTEGER

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Abstract

Given a large integer n, determining the relative size of each of its prime divisors as well as the spacings between these prime divisors has been the focus of several studies. Here, we examine the spacings between particular types of prime divisors of n, such as certain congruence classes of primes and various other subsets of the set of prime numbers.

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1. Introduction

Given a large integer n, determining the size of its distinct prime factors as well as their relative sizes has been the focus of several studies for the past 75 years. For instance, if we let $\omega(n)$ stand for the number of distinct prime divisors of an integer $n \geq 2$ and let

$$p_1(n) < p_2(n) < \cdots < p_{\omega(n)}(n) \quad \text{ or for short } \quad p_1 < p_2 < \cdots < p_{\omega(n)} \tag{1.1}$$

be these prime divisors, Paul Erdős [6] proved in 1946 that, letting $\xi(n) \to \infty$ as $n \to \infty$ and given any small number $\varepsilon > 0$, then

$$e^{e^{k(1-\varepsilon)}} < p_k(n) < e^{e^{k(1+\varepsilon)}} \quad (\xi(n) \leq k \leq \omega(n)) \quad \text{for almost all } n.$$

Thirty years later, Galambos [7] provided important information on the relative size of the consecutive prime factors of an integer by showing that, given any small $\varepsilon > 0$ and a function $j = j(x)$ which tends to infinity with x in such a manner that $j(x) \leq (1-\varepsilon)\log\log x$, then, for any fixed real number $z > 1$,

$$\lim_{x\to\infty} \frac{1}{x}\#\left\{n \leq x : \frac{\log p_{j+1}(n)}{\log p_j(n)} < z\right\} = 1 - \frac{1}{z}.$$

Further refinements of these results were provided by many others, such as De Koninck and Galambos [1] in 1987, and Granville [8], [9] in 2007.

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More recently, in [3] and [4], we further explored this topic by examining the spacings between those consecutive prime factors of an integer which are "close" to one another, in the following sense. Given a real number $\lambda \in (0,1]$, we introduced the arithmetic function

$$U_\lambda(n) := \#\{i \in [1, \omega(n)-1] : p_i < p_{i+1}^\lambda\}$$

and proved (see Theorem 1 in [3]) that

$$\sum_{n \le x} U_\lambda(n) = (1+o(1))\lambda\, x \log\log x \qquad (x \to \infty) \tag{1.2}$$

and that, for any given $\varepsilon > 0$,

$$\lim_{x\to\infty} \frac{1}{x} \#\left\{ n \le x : \left| \frac{U_\lambda(n)}{\log\log n} - \lambda \right| \ge \varepsilon \right\} = 0. \tag{1.3}$$

Here, we examine the spacings between particular types of prime divisors of n, such as certain congruence classes of primes and various other subsets of the set of prime numbers.

2. Setting the table

Let $\wp$ be the set of all primes. From here on, the letters p, q and π will stand for primes, whereas $\pi(x)$ will stand for the number of primes not exceeding x. Also, we will be using the logarithmic integral $\mathrm{li}(x) := \int_2^x \frac{dt}{\log t}$. Finally, given a real number $x \ge e^{e^e}$, we set

$$Y_1 := Y_1(x) = \exp\left\{(\log x)^{\varepsilon(x)}\right\} \quad \text{and} \quad Y_2 := Y_2(x) = \exp\left\{(\log x)^{1-\varepsilon(x)}\right\}, \tag{2.1}$$

where

$$\varepsilon(x) := \frac{1}{2} \frac{\log\log\log x}{\log\log x}.$$

Let $\mathcal{B}$ be a subset of $\wp$ whose corresponding counting function $B(x) := \#\{p \le x : p \in \mathcal{B}\}$ is such that, for some real number $\beta > 0$, we have

$$B(x) = \beta\, \mathrm{li}(x) + O\left(\frac{x}{\log^2 x}\right), \tag{2.2}$$

so that in particular, we have

$$\sum_{\substack{p \le x \\ p \in \mathcal{B}}} \frac{1}{p} = \beta \log\log x + \beta_2 + O\left(\frac{x}{\log x}\right) \quad \text{for some constant } \beta_2, \tag{2.3}$$

$$\sum_{\substack{u<p<v \\ p \in \mathcal{B}}} \frac{\log p}{p} = \beta \log \frac{v}{u} + O(1), \tag{2.4}$$

$$\prod_{\substack{u<p<v \\ p \in \mathcal{B}}} \left(1 - \frac{1}{p}\right) = \left(\frac{\log u}{\log v}\right)^\beta \left(1 + O\left(\frac{1}{\log u}\right)\right). \tag{2.5}$$

Any such set $\mathcal{B}$ satisfying (2.2) will be called a *B-set*.

Observe that given a B-set and considering its associated function $\omega_{\mathcal{B}}(n) := \sum_{\substack{p|n \\ p\in\mathcal{B}}} 1$, it is immediate that

$$\sum_{n\le x} \omega_{\mathcal{B}}(n) = \beta x \log\log x + O(x)$$

and one can show that it follows from the Turán-Kubilius inequality that

$$\sum_{n\le x} (\omega_{\mathcal{B}}(n) - \beta \log\log n)^2 \ll x \log\log x.$$

Moreover, one can establish that, for some positive constant β_3,

$$\lim_{x\to\infty} \frac{1}{x}\#\left\{n \le x : \frac{\omega_{\mathcal{B}}(n) - \beta\log\log n}{\beta_3\sqrt{\log\log n}} < y\right\} = \Phi(y),$$

where $\Phi(y) := \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{y} e^{-t^2/2}\,dt$.

Now, given positive real numbers $u < v$, let us set

$$Q(u,v) := \prod_{\substack{u<p<v \\ p\in\wp}} p.$$

Also, given an integer $n > 1$ and a prime divisor p of n which is smaller than $P(n)$, the largest prime divisor of n, we set

$$v_p = v_p(n) := \min\{q \mid n : q > p\}, \tag{2.6}$$

so that

$$\left(\frac{n}{p\, v_p}, Q(p, v_p)\right) = 1.$$

Now, consider the arithmetic function

$$U_{\lambda,\mathcal{B}}(n) := \sum_{\substack{p|n \\ p\in\mathcal{B} \\ \frac{\log p}{\log v_p(n)}<\lambda}} 1.$$

The technique we used in [3] to prove (1.2) can also be used to determine the mean value of the function $U_{\lambda,\mathcal{B}}(n)$ and in fact a more accurate result, namely the following.

Theorem 2.1. *Given $\lambda \in (0,1]$ and a set of primes $\mathcal{B}$ satisfying estimate (2.2), we have*

$$\sum_{n\le x} U_{\lambda,\mathcal{B}}(n) = (1+o(1))\lambda\beta x \log\log x \qquad (x\to\infty) \tag{2.7}$$

and, for an arbitrarily small $\varepsilon > 0$,

$$\lim_{x\to\infty} \frac{1}{x}\#\left\{n \le x : \left|\frac{U_{\lambda,\mathcal{B}}(n)}{\omega(n)} - \lambda\beta\right| \ge \varepsilon\right\} = 0. \tag{2.8}$$

Similarly, we have the following analog result for shifted primes.

THEOREM 2.2. *Given $\lambda \in (0,1]$, a set of primes $\mathcal{B}$ satisfying estimate (2.2) and a fixed integer $a \neq 0$, we have*

$$\frac{1}{\pi(x)} \sum_{p \leq x} U_{\lambda,\mathcal{B}}(p+a) = (1+o(1))\lambda\beta \log\log x \qquad (x \to \infty) \tag{2.9}$$

and moreover, for any arbitrarily small $\varepsilon > 0$,

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \# \left\{ p \leq x : \left| \frac{U_{\lambda,\mathcal{B}}(p+a)}{\omega(p+a)} - \lambda\beta \right| \geq \varepsilon \right\} = 0. \tag{2.10}$$

3. An extension of Theorem 2.1

Given $\lambda \in (0,1]$ and a set of primes $\mathcal{B}$ satisfying estimate (2.2), we introduce the arithmetic function $\widetilde{U}_{\lambda,\mathcal{B}}(n)$ which counts the number of prime divisors p of n which belong to $\mathcal{B}$ and which are such that the next prime divisor q of n which belongs to $\mathcal{B}$ is such that $\log p / \log q < \lambda$. In short, setting $Q_{\mathcal{B}}(u,v) := \prod_{\substack{u<p<v \\ p \in \mathcal{B}}} p$, we can write that

$$\widetilde{U}_{\lambda,\mathcal{B}}(n) := \sum_{\substack{p \mid n \\ \log p/\log q < \lambda \\ p,q \in \mathcal{B} \\ \left(\frac{n}{pq}, Q_{\mathcal{B}}(p,q)\right)=1}} 1 \tag{3.1}$$

and prove the following.

THEOREM 3.1. *Let $\lambda \in (0,1]$ and $\mathcal{B}$ a set of primes satisfying estimate (2.2). Then,*

$$\sum_{n \leq x} \widetilde{U}_{\lambda,\mathcal{B}}(n) = \beta\lambda^{\beta} x \log\log x + O(x \log\log\log x). \tag{3.2}$$

Moreover, for any arbitrarily small $\varepsilon > 0$,

$$\lim_{x\to\infty} \frac{1}{x} \# \left\{ n \leq x : \left| \frac{\widetilde{U}_{\lambda,\mathcal{B}}(n)}{\omega(n)} - \beta\lambda^{\beta} \right| \geq \varepsilon \right\} = 0. \tag{3.3}$$

Before we begin the proof of Theorem 3.1, let us recall Lemma A of our recent paper [4]:

Lemma A. *Let $x \geq e^{e^{100}}$ and let $Y_1 = Y_1(x)$ and $Y_2 = Y_2(x)$ be the functions defined in (2.1). Let $\pi_1 < \pi_2 < \cdots < \pi_s$ be s primes located in the interval (Y_1, Y_2). Write their product as $R = \pi_1\pi_2\cdots\pi_s$. Further set*

$$\eta := \sum_{i=1}^{s} \frac{1}{\pi_i}$$

and

$$S_R(x) := \sum_{\substack{n \leq x \\ (n,R)=1}} 1.$$

Assume that $\eta \le K$, where K is an arbitrary number, and let h be a positive integer satisfying $h \ge 3e^2K$. Then, letting ϕ stand for the Euler totient function, we have that

$$\left|S_R(x) - \frac{\phi(R)}{R}x\right| \le x\,(3e)^{-h} + 2Y_2^h,$$

so that in particular, choosing $h = \lfloor \log\log\log x \rfloor$, there exists a positive constant c such that

$$\left|S_R(x) - \frac{\phi(R)}{R}x\right| \le \frac{c\,x}{(\log\log x)^2}.$$

We are now ready for the proof of Theorem 3.1.

Proof of Theorem 3.1. Let $(p, q) \in \mathcal{B} \times \mathcal{B}$, with $p < q$, be a pair of prime divisors of an integer n which satisfy

$$\left(\frac{n}{pq}, Q_{\mathcal{B}}(p, q)\right) = 1.$$

Let us first assume that $Y_1 < p < q < Y_2$, where $Y_1 = Y_1(x)$ and $Y_2 = Y_2(x)$ are the functions defined in (2.1). Using Lemma A and estimate (2.5), one obtains that

$$\begin{aligned}
\sum_{n\le x} \widetilde{U}_{\lambda,\mathcal{B}}(n) &= \sum_{\substack{Y_1<p<q<Y_2\\ p,q\in\mathcal{B}\\ \frac{\log p}{\log q}<\lambda}} \frac{x}{pq} \prod_{\substack{p<\pi<q\\ \pi\in\mathcal{B}}} \left(1 - \frac{1}{\pi}\right)\left(1 + O\left(\frac{1}{\log p}\right)\right)\\
&= \sum_{\substack{Y_1<p<q<Y_2\\ p,q\in\mathcal{B}\\ \frac{\log p}{\log q}<\lambda}} \frac{x}{pq} \left(\frac{\log p}{\log q}\right)^{\beta} \left(1 + O\left(\frac{1}{\log p}\right)\right)\\
&= x \sum_{\substack{Y_1<p<q<Y_2\\ p,q\in\mathcal{B}\\ \frac{\log p}{\log q}<\lambda}} \frac{(\log p)^{\beta}}{p} \cdot \frac{1}{q(\log q)^{\beta}} \left(1 + O\left(\frac{1}{\log p}\right)\right). \qquad (3.4)
\end{aligned}$$

Fixing q and summing over p, we have

$$\begin{aligned}
\sum_{\substack{Y_1<p<q^{\lambda}\\ p\in\mathcal{B}}} \frac{(\log p)^{\beta}}{p} &= \int_{Y_1}^{q^{\lambda}} \frac{(\log u)^{\beta}}{u}\, d\,B(u)\\
&= \beta \int_{Y_1}^{q^{\lambda}} \frac{(\log u)^{\beta}}{u\log u}\, du + \int_{Y_1}^{q^{\lambda}} \frac{(\log u)^{\beta}}{u}\, d(B(u) - \beta \mathrm{li}(u))\\
&= I_1(x) + I_2(x), \qquad (3.5)
\end{aligned}$$

say.

On the one hand,

$$I_1(x) = \beta \int_{\log Y_1}^{\lambda \log q} v^{\beta-1}\, dv = \lambda^{\beta}(\log q)^{\beta} + O\left((\log Y_1)^{\beta}\right). \qquad (3.6)$$

On the other hand,

$$\begin{aligned} I_2(x) &\ll (B(u)-\beta \mathrm{li}(u))\,\frac{(\log u)^\beta}{u}\Big|_{Y_1}^{q^\lambda} - \int_{Y_1}^{q^\lambda} \frac{d}{du}\left(\frac{(\log u)^\beta}{u}\right) du \\ &\ll (\log q)^{\beta-2} + \int_{Y_1}^{q^\lambda} \frac{u}{\log^2 u}\cdot\frac{\beta(\log u)^{\beta-1}-(\log u)^\beta}{u^2}\,du \\ &\ll (\log q)^{\beta-2} + \int_{\log Y_1}^{\lambda\log q} v^{\beta-2}\,dv \\ &\ll (\log q)^{\beta-2} + (\log q)^{\beta-1} \ll (\log q)^{\beta-1}. \end{aligned} \tag{3.7}$$

Gathering estimates (3.6) and (3.7) in (3.5), we obtain that

$$\sum_{\substack{Y_1<p<q^\lambda \\ p\in\mathcal{B}}} \frac{(\log p)^\beta}{p} = \lambda^\beta(\log q)^\beta + O\left((\log q)^{\beta-1}\right) + O\left((\log Y_1)^\beta\right). \tag{3.8}$$

In light of estimate (3.8), the sum on the right-hand side of (3.4) can therefore be replaced by

$$\lambda^\beta \sum_{\substack{Y_1<q<Y_2 \\ q\in\mathcal{B}}} \frac{(\log q)^\beta}{q(\log q)^\beta} = \beta\,\lambda^\beta\, \log\log x + O(\log\log\log x), \tag{3.9}$$

where we could ignore the contribution of the last error term on the right-hand side of (3.8) because

$$\sum_{Y_1<q<Y_2} \frac{1}{q(\log q)^\beta} \ll \int_{Y_1}^{\infty} \frac{1}{t(\log t)^{\beta+1}}\,dt \ll \frac{1}{(\log Y_1)^\beta}.$$

Then, in light of (3.9) and with the help of (2.3), it is clear that estimate (3.4) can be replaced by

$$\sum_{n\le x} \widetilde{U}_{\lambda,\mathcal{B}}(n) = \beta\,\lambda^\beta\, x\,\log\log x + O(x\,\log\log\log x),$$

thus completing the proof of (3.2).

The proof of (3.3) rests on the evaluation of $\widetilde{U}_{\lambda,\mathcal{B}}(n)^2$, which can be handled using an approach similar to the one used to obtain (3.2). We will therefore omit this proof. □

4. Two B-sets involving congruence classes

Given an integer $k\ge 3$, let $\ell_1,\dots,\ell_r$ be the reduced residue system modulo k, with $r=\phi(k)$. Then, it is clear that each of the residue classes

$$\mathcal{B}_{\ell_j} := \{p\in\wp : p\equiv \ell_j \pmod{k}\} \qquad (j=1,\dots,r)$$

satisfies condition (2.2) and is therefore a B-set.

A second interesting B-set involves the sum-of-digits function $s_q(n)$ which stands for the sum of the base q digits of an integer n (here, $q \geq 2$). First note that it is known (see Mauduit and Rivat [11] as well as Drmota, Mauduit and Rivat [5]) that if $(k, q-1) = 1$, then there exists a real number $\sigma = \sigma_{q,k} > 0$ such that

$$\frac{1}{\pi(x)} \#\{p \leq x : s_q(p) \equiv \ell \pmod{k}\} = \frac{1}{k} + O_{q,k}\left(x^{-\sigma} \log x\right). \tag{4.1}$$

Clearly, estimate (4.1) guarantees that if for a given integer $k \geq 3$, we set

$$\mathcal{B}_\ell := \{p \in \wp : s_q(p) \equiv \ell \pmod{k}\} \qquad (\ell = 0, 1, \ldots, k-1),$$

then $\mathcal{B}_\ell$ is indeed a B-set for each $\ell = 0, 1, \ldots, k-1$.

5. Two B-sets involving fractional parts

We first identify a B-set involving fractional parts of prime powers. Let $\alpha, \sigma \in (0,1)$. In what follows, $\{y\}$ stands for the fractional part of y. Using the method of exponential sums initiated in 1940 by I.M. Vinogradov [13], one can prove that

$$\sum_{\substack{p \leq x \\ \{p^\alpha\} < \sigma}} 1 = \sigma\, \pi(x) + R(x) \tag{5.1}$$

(see the recent paper of Shubin [12] for a thorough discussion), where $R(x) \ll x^{\theta(\alpha)+\varepsilon}$ with

$$\theta(\alpha) = \begin{cases} 1 - \frac{2}{15}\alpha & \text{if } 0 < \alpha \leq 3/5, \\ \frac{4+\alpha}{5} & \text{if } 3/5 < \alpha < 1. \end{cases}$$

In light of (5.1), it is clear that indeed, given $\alpha, \sigma \in (0,1)$, the set

$$\mathcal{B} = \{p \in \wp : \{p^\alpha\} < \sigma\}$$

is a B-set.

Another B-set involving fractional parts is the following. Let α be an irrational number. As was shown in 1940 by I.M. Vinogradov [13],

$$\lim_{x \to \infty} \frac{1}{\pi(x)} \sum_{p \leq x} e^{2\pi i \alpha p} = 0.$$

Consequently the sequence $(\alpha p)_{p \in \wp}$ is uniformly distributed modulo 1. Thus,

$$E_\alpha(x) := \sup_{0 \leq u < v < 1} \left| \frac{1}{\pi(x)} \sum_{\substack{p \leq x \\ \{\alpha p\} \in [u,v)}} 1 - (v-u) \right| \to 0 \quad \text{as } x \to \infty.$$

Our next theorem will depend on the following result of Glyn Harman (Theorem 2.2 in his 2007 book [10]):

Theorem A (Harman). *Let $\alpha \in \mathbb{R}$ and suppose that for integers a, q with $(a, q) = 1$, inequality*

$$|q\alpha - a| < \frac{1}{q}$$

holds. Let $\delta \in (0, 1]$ be given. Write

$$\chi(\theta) = \begin{cases} 1 & \text{if } \|\theta\| < \delta, \\ 0 & \text{otherwise.} \end{cases}$$

Then, for any real β and positive integers N, H, such that $N^\varepsilon \ll q \ll N^{1-\varepsilon}$ for some $\varepsilon > 0$, we have

$$\begin{aligned} \sum_{n\le N} \Lambda(n)\chi(n\alpha + \beta) &= 2\delta \sum_{n\le N} \Lambda(n) + O\left(\frac{N\delta}{H}\right) \\ &\quad + O\left(N(\log H)(\log N)^7 \left(\frac{q\delta}{N} + \frac{1}{q} + \frac{1}{N^{1/2}} + \frac{\delta}{N^{1/3}}\right)^{1/2}\right). \end{aligned}$$

(Here $\Lambda(n)$ stands for the von Mangoldt function. Also $\|\theta\| := \min_{m\in\mathbb{Z}} |\theta - m|$).

From Theorem A, one can easily deduce the following.

Theorem 5.1. *Let $\varepsilon > 0$ be an arbitrarily small number. Let $\alpha \in (0, 1)$ be an irrational number such that for every real $x > x_0(\alpha)$, there exists a positive integer $q \in [x^\varepsilon, x^{1-\varepsilon}]$ for which $\|q\alpha\| < 1/q$. Then,*

$$E_\alpha(x) \ll \exp\left\{-\frac{\varepsilon}{2}\log x\right\}.$$

Remark 5.2. *Observe that the condition imposed on α in Theorem 5.1 is known to hold for almost all real numbers α and in fact for every irrational algebraic number α.*

Given $0 \le u < v < 1$, let α be an irrational number satisfying the conditions of Theorem 5.1 and set $\mathcal{B} := \{p \in \wp : \{\alpha p\} \in [u, v)\}$. Then, it follows from Theorem 5.1 that $\mathcal{B}$ is a B-set.

6. Disjoint classification of primes

Given an integer $k \ge 2$, we will now be interested in sets of prime numbers $\wp_1, \wp_2, \ldots, \wp_k$ such that

$$\wp = \wp_1 \cup \wp_2 \cup \cdots \cup \wp_k \cup \mathcal{D} \quad \text{and} \quad \wp_i \cap \wp_j = \emptyset \text{ if } i \ne j,$$

where $\mathcal{D}$ is a finite (perhaps empty) set of primes, and assume that, for each $j \in \{1, 2, \ldots, k\}$,

$$\pi_j(x) := \sum_{\substack{p\le x \\ p\in\wp_j}} 1 = \beta_j \operatorname{li}(x) + O\left(\frac{\operatorname{li}(x)}{\log^2 x}\right),$$

where each $\beta_j \in (0,1]$ and that $\sum_{j=1}^{k} \beta_j = 1$. Such a collection of subsets of primes is called a *disjoint classification of primes*, a notion we introduced in [2] a decade ago in order to create large families of normal numbers. Here, we use this concept for a different purpose.

For each $j = 1, 2, \ldots, k$, let E_j be a subinterval of $[0,1]$. Given a large integer n, we will now count those k-tuples of primes $(q_1, q_2, \ldots, q_k)$ which satisfy the conditions

$$
\begin{aligned}
&q_1 q_2 \cdots q_k \mid n, \qquad \left(\frac{n}{q_1 q_2 \cdots q_k}, Q(q_1, q_k) \right) = 1, \\
&q_j \in \wp_{\ell_j},\ q_{j+1} \in \wp_{\ell_{j+1}},\ \log q_j / \log q_{j+1} \in E_j \quad (j = 1, 2, \ldots, k-1), \qquad (6.1)
\end{aligned}
$$

and in fact we shall let $E(n)$ be the number of these primes q_1, namely those primes q_1 for which there exist primes $q_2, \ldots, q_k$ satisfying the conditions listed in (6.1).

By using our usual technique and writing $\lambda(I)$ for the length of the interval I, we can prove that, if we set

$$
T(\ell_1, \ell_2, \ldots, \ell_k) := \beta_{\ell_1} \beta_{\ell_2} \cdots \beta_{\ell_k} \lambda(E_1) \lambda(E_2) \cdots \lambda(E_{k-1}),
$$

then,

$$
\frac{1}{x} \sum_{n \le x} E(n) = T(\ell_1, \ell_2, \ldots, \ell_k) \log\log x + O(\log\log\log x).
$$

and moreover that, given any small number $\varepsilon > 0$,

$$
\lim_{x \to \infty} \frac{1}{x} \# \left\{ n \le x : \left| \frac{E(n)}{\omega(n)} - T(\ell_1, \ell_2, \ldots, \ell_k) \right| \ge \varepsilon \right\} = 0.
$$

An analog result for shifted primes can also be proved, namely that, for any fixed integer $a \neq 0$,

$$
\lim_{x \to \infty} \frac{1}{\pi(x)} \# \left\{ p \le x : \left| \frac{E(p+a)}{\omega(p+a)} - T(\ell_1, \ell_2, \ldots, \ell_k) \right| \ge \varepsilon \right\} = 0.
$$

7. The sequence $n^2 + 1$

It is known that if $p \mid n^2 + 1$, then either $p = 2$ or $p \equiv 1 \pmod 4$ (since $\left(\frac{-1}{p}\right) = 1$ if and only if $p \equiv 1 \pmod 4$). On the one hand, it is known that

$$
\sum_{n \le x} \omega(n^2 + 1) = x \log\log x + O(x). \qquad (7.1)
$$

For the sake of completeness, it can be proved by first observing that

$$
\sum_{n \le x} \omega(n^2 + 1) = \left\lfloor \frac{x+1}{2} \right\rfloor + \sum_{\substack{p \le x \\ p \equiv 1 \pmod 4}} \sum_{\substack{n \le x \\ n^2 + 1 \equiv 0 \pmod p}} 1 + O(x), \qquad (7.2)
$$

where the last term accounts for the contribution of those primes $p \in (x, x^2 + 1]$. Since for each prime $p \equiv 1 \pmod 4$, the congruence $n^2 + 1 \equiv 0 \pmod p$ has exactly two solutions, the number of positive integers $n \le x$ for which $n^2+1 \equiv 0 \pmod p$ is equal to $2x/p + O(1)$. Hence,

$$\begin{aligned}\sum_{\substack{p\le x\\ p\equiv 1 \pmod 4}} \sum_{\substack{n\le x\\ n^2+1\equiv 0 \pmod p}} 1 &= \sum_{\substack{p\le x\\ p\equiv 1 \pmod 4}} \left(\frac{2x}{p}+O(1)\right) = 2x \sum_{\substack{p\le x\\ p\equiv 1 \pmod 4}} \frac{1}{p} + O\left(\frac{x}{\log x}\right)\\ &= 2x\left(\frac{1}{2}\log\log x + O(1)\right) + O\left(\frac{x}{\log x}\right) = x\log\log x + O(x),\end{aligned}$$

which combined with (7.2) proves (7.1).

On the other hand, using estimate (7.1), it follows from the Turán-Kubilius inequality that

$$\sum_{n\le x}\left(\omega(n^2+1) - \log\log n\right)^2 \ll x\log\log x.$$

For convenience, we will now be using the notation

$$\Lambda(p,q) := \frac{\log p}{\log q} \quad \text{and} \quad Q^{(1)}(p,q) = \prod_{\substack{p<\pi<q\\ \pi\equiv 1 \pmod 4}} \pi.$$

Given a small number $\varepsilon > 0$ and $\lambda \in (\varepsilon, 1)$, let us count those pairs of primes $p < q$ such that

$$pq \mid n^2+1, \quad \varepsilon \le \Lambda(p,q) < \lambda, \quad p \equiv q \equiv 1 \pmod 4, \quad \left(\frac{n^2+1}{pq}, Q^{(1)}(p,q)\right) = 1. \quad (7.3)$$

For each such a pair of primes p, q, let $E_{p,q}$ be the number of those positive integers $n \le x$ for which conditions (7.3) hold. Using Lemma A, we may write that

$$\begin{aligned}\sideset{}{'}\sum_{Y_1<p<q<Y_2} E_{p,q} &= \sideset{}{'}\sum_{Y_1<p<q<Y_2} \frac{x}{pq} \prod_{\substack{p<\pi<q\\ \pi\equiv 1 \pmod 4}} \left(1-\frac{1}{\pi}\right) + O(x)\\ &= \sum_{\substack{Y_1<q<Y_2\\ q\equiv 1 \pmod 4}} \frac{1}{q\log q} \sum_{\substack{q^\varepsilon<p<q^\lambda\\ p\equiv 1 \pmod 4}} \frac{\log p}{p} + O(x), \qquad (7.4)\end{aligned}$$

where the apostrophe (′) on the above sums indicates that the summation runs over those pairs of primes p, q which satisfy the conditions listed in (7.3).

It is easily established that

$$\sum_{\substack{q^\varepsilon<p<q^\lambda\\ p\equiv 1 \pmod 4}} \frac{\log p}{p} = \frac{\lambda-\varepsilon}{2}\log q + O\left(\frac{1}{\log q}\right), \qquad (7.5)$$

Using (7.5) in (7.4), we obtain that

$$\frac{1}{x}\sideset{}{'}\sum_{Y_1<p<q<Y_2} E_{p,q} = \frac{\lambda-\varepsilon}{2} \sum_{\substack{Y_1<q<Y_2\\ q\equiv 1 \pmod 4}} \frac{1}{q} = \frac{\lambda-\varepsilon}{4}\log\log x + O(\log\log\log x). \qquad (7.6)$$

Denote by $K_{\varepsilon,\lambda}(n)$ the number of those primes p for which there exists a prime $q > p$ satisfying the list of conditions (7.3). Since the contribution of those primes $p < Y_1$ and $q > Y_2$ to the above sums is relatively small, it follows from (7.6) that

$$\frac{1}{x \log\log x} \sum_{n\le x} K_{\varepsilon,\lambda}(n) = \frac{\lambda-\varepsilon}{4} + O\left(\frac{\log\log\log x}{\log\log x}\right). \tag{7.7}$$

Since ε can be chosen arbitrarily small, it follows from (7.7) that

$$\lim_{x\to\infty} \frac{1}{x\log\log x} \sum_{n\le x} K_{0,\lambda}(n) = \frac{\lambda}{4}.$$

Moreover, one can show that, given any arbitrarily small number $\delta > 0$,

$$\lim_{x\to\infty} \frac{1}{x} \# \left\{ n \le x : \left| \frac{K_{0,\lambda}(n)}{\log\log n} - \frac{\lambda}{4} \right| \ge \delta \right\} = 0.$$

We can also prove an analog of Theorem 5 in [3]. Indeed, let $\delta_0, \delta_1, \ldots, \delta_{k-1} \in (0,1)$ and set $H := \delta_0\delta_1\cdots\delta_{k-1}$. Further set $\mathcal{F}$ be the set of all $k+1$-tuples of primes $(p_0, p_1, \ldots, p_k)$ where $p_0 < p_1 < \ldots < p_k$ satisfy

$$1 - \delta_j < \Lambda(p_j, p_{j+1}) < 1 \qquad (j = 0, 1, \ldots, k-1).$$

Further let $V_{\delta_0,\delta_1,\ldots,\delta_{k-1}}(n)$ stand for the number of those prime divisors p_0 of n^2+1 for which

$$p_1 p_2 \cdots p_k \mid n^2 + 1 \quad \text{and} \quad \left(\frac{n^2+1}{p_0 p_1 \cdots p_k}, Q(p_0, p_k) \right) = 1 \text{ for } (p_0, p_1, \ldots, p_k) \in \mathcal{F}.$$

Then, we have the following.

Theorem 7.1. *Let $\delta_0, \delta_1, \ldots, \delta_{k-1}$, H and $\mathcal{F}$ be as above. Then, as $x \to \infty$,*

$$\begin{aligned} \sum_{n\le x} V_{\delta_0,\delta_1,\ldots,\delta_{k-1}}(n) &= (H + o(1)) x \log\log x, \\ \sum_{n\le x} V^2_{\delta_0,\delta_1,\ldots,\delta_{k-1}}(n) &= (H^2 + o(1)) x (\log\log x)^2. \end{aligned}$$

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