On the consecutive prime divisors of an integer

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Abstract

Paul Erdős, Janos Galambos and others have studied the relative size of the consecutive prime divisors of an integer. Here, we further extend this study by examining the distribution of the consecutive neighbour spacings between the prime divisors $p_1(n) < p_2(n) < \cdots < p_r(n)$ of a typical integer $n \geq 2$. In particular, setting $\gamma_j(n) := \log p_j(n)/\log p_{j+1}(n)$ for $j = 1, 2, \ldots, r-1$ and, for any $\lambda \in (0, 1]$, introducing $U_\lambda(n) := \#\{j \in \{1, 2, \ldots, r-1\} : \gamma_j(n) < \lambda\}$, we establish the mean value of $U_\lambda(n)$ and prove that $U_\lambda(n)/r \sim \lambda$ for almost all integers $n \geq 2$. We also examine the shifted prime version of these two results and study other related functions.

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1 Introduction

Given an integer $n \geq 2$, let $\omega(n)$ stand for the number of distinct prime divisors of $n \geq 2$ (setting $\omega(1) = 0$) and let

$$p_1(n) < p_2(n) < \cdots < p_{\omega(n)}(n) \quad \text{or for short} \quad p_1 < p_2 < \cdots < p_{\omega(n)} \tag{1.1}$$

be these prime divisors. Many have shown interest for the relative size of these prime factors $p_j(n)$.

Let $\xi(n) \to \infty$ as $n \to \infty$. In 1946, P. Erdős [4] proved that given any small number $\varepsilon > 0$,

$$e^{e^{k(1-\varepsilon)}} < p_k(n) < e^{e^{k(1+\varepsilon)}} \quad (\xi(n) \leq k \leq \omega(n)) \quad \text{for almost all } n \leq x.$$

In 1976, J. Galambos [5] strengthened this result by showing that, given any small $\varepsilon > 0$ and a function $k = k(x)$ which tends to infinity with x in such a manner that $k(x) = o(\log\log x)$ as $x \to \infty$, then

$$\lim_{x\to\infty} \frac{1}{x}\#\{n \leq x : e^{-(1+\varepsilon)k}\log x < \log p_{\omega(n)-k}(n) < e^{-(1-\varepsilon)k}\log x\} = 1.$$

Galambos also established that, given any small $\varepsilon > 0$ and a function $j = j(x)$ which tends to infinity with x in such a manner that $j(x) \leq (1-\varepsilon)\log\log x$, then, for any fixed real number $z > 1$,

$$\lim_{x\to\infty} \frac{1}{x}\#\left\{n \leq x : \frac{\log p_{j+1}(n)}{\log p_j(n)} < z\right\} = 1 - \frac{1}{z}.$$

In 1987, De Koninck and Galambos [2] showed that for almost all n (with the prime factors of n written as in (1.1)) and for any fixed positive integer k, the corresponding

$$(\log\log p_{j+1} - \log\log p_j, \log\log p_{j+2} - \log\log p_{j+1}, \ldots, \log\log p_{j+k} - \log\log p_{j+k-1}))$$

is distributed as a k-tuple of independent exponential random variables with parameter 1.

Finally, in 2007, Granville [6], [7] proved that if $S_k(x)$ stands for the number of positive integers $n \le x$ such that $\omega(n) = k$, then for all but $o(S_\ell(x))$ of the integers $n \in S_\ell(x)$, the sets

$$\left\{ \frac{\log\log p}{\frac{1}{\ell}\log\log(n^{1/\ell})} : p \mid n,\ p \le n^{1/\ell} \right\}$$

are Poisson distributed.

Here, we further expand on the above results by examining the distribution of the consecutive neighbour spacings between the prime divisors of a typical integer n.

2 Setting the table

Writing the distinct prime factors of an integer n as in (1.1), we introduce the functions

$$\gamma_j(n) := \frac{\log p_j(n)}{\log p_{j+1}(n)} \qquad (j = 1, \dots, r-1). \tag{2.1}$$

From here on, the letters p, q, π and at times P will denote primes. Given positive real numbers $u < v$, we set

$$Q(u, v) := \prod_{u<p<v} p.$$

Also, given an integer $n > 1$ and a prime divisor p of n which is smaller than $P(n)$, the largest prime divisor of n, we set

$$\nu_p = \nu_p(n) := \min\{q \mid n : q > p\}. \tag{2.2}$$

In light of these notation, we have

$$\mathrm{GCD}\left(\frac{n}{p\,\nu_p}, Q(p, \nu_p)\right) = 1.$$

When the context is clear, we shall write (a, b) instead of $\mathrm{GCD}(a, b)$. Given a real number $\lambda \in (0, 1]$, we introduce the function

$$U_\lambda(n) := \sum_{\substack{p \mid n \\ \frac{\log p}{\log \nu_p(n)} < \lambda}} 1,$$

or equivalently

$$U_\lambda(n) := \sum_{\substack{p \mid n \\ \log\log \nu_p(n) - \log\log p > -\log\lambda}} 1.$$

One expects an exponential distribution for the gaps $\log\log \nu_p(n) - \log\log p$, meaning that for a typical integer n, one should expect to have

$$U_\lambda(n) \approx \log\log n \int_{-\log\lambda}^{\infty} e^{-s}\, ds = \lambda \log\log n,$$

which is essentially what we obtain in the first part of Theorem 1 below.

In what follows, ϕ and μ will denote the Euler totient function and the Möbius function. We also let $\phi_0(n) := \phi(n)/n$ and introduce the strongly multiplicative function $\widetilde{\phi}_0(n)$ defined on primes by

$$\widetilde{\phi}_0(p) = 1 - \frac{1}{p-1},$$

so that in particular

$$\widetilde{\phi}_0(n) = \phi_0(n) \prod_{p|n} \left(1 - \frac{1}{(p-1)^2}\right). \tag{2.3}$$

Also, at times, given primes $p < q$, we will use the notation $\Lambda(p,q) := \dfrac{\log p}{\log q}$.

We shall use the letters c and C with or without subscripts to denote positive constants, not necessarily the same at each occurrence. As usual we write $\pi(x)$ for the number of primes not exceeding x and we will be using the logarithmic integral $\mathrm{li}(x) := \displaystyle\int_2^x \frac{dt}{\log t}$.

3 Main results

Theorem 1. *Given an arbitrary real number $\lambda \in (0,1]$,*

$$\sum_{n \le x} U_\lambda(n) = (1+o(1))\lambda\, x \log\log x \qquad (x \to \infty). \tag{3.1}$$

Moreover, for every $\varepsilon > 0$,

$$\lim_{x\to\infty} \frac{1}{x} \# \left\{ n \le x : \left| \frac{U_\lambda(n)}{\omega(n)} - \lambda \right| > \varepsilon \right\} = 0. \tag{3.2}$$

Remark. *Although we will provide complete proofs of both statements in Theorem 1, we should mention that the second statement does follow from Theorem 1 in Granville's paper [6], as he showed that for almost all n and for all $L > 0$, one has*

$$\frac{1}{\log\log n} \int_{\substack{t=0 \\ \#\{p|n : p\in[t,t+L)\}=0}}^{\log\log n} 1\, dt = e^{-L}(1+o(1)),$$

from which the distribution function for $U_\lambda(n)$ can be obtained by taking differences of the left-hand side of the above expression at $L+\delta$ and L, and taking the limit as $\delta \to 0$ suitably slowly.

The following is essentially the analogue of Theorem 1 for the shifted primes.

Theorem 2. *Given an arbitrary real number $\lambda \in (0,1]$,*

$$\sum_{p \le x} U_\lambda(p+1) = (1+o(1))\lambda\, \mathrm{li}(x) \log\log x \qquad (x \to \infty). \tag{3.3}$$

Moreover, for every $\varepsilon > 0$,

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \# \left\{ p \le x : \left| \frac{U_\lambda(p+1)}{\omega(p+1)} - \lambda \right| > \varepsilon \right\} = 0. \tag{3.4}$$

Let $k_1, k_2, \ldots, k_r$ be non-negative integers and recall the definition of $\nu_p(n)$ given in (2.2). Consider the linear expressions $\ell_j(n) := a_j n + b_j$ ($j = 0, 1, \ldots, r$), where $a_j, b_j \in \mathbb{Z}$ satisfy $a_i b_j \neq a_j b_i$ whenever $i \neq j$. Further let $U_{k_1,\ldots,k_r}(\ell_0(n))$ be the number of those prime divisors p of $\ell_0(n)$ for which each $\ell_j(n)$, $j = 1, \ldots, r$, has exactly k_j prime divisors in the interval $[p, \nu_p(\ell_0(n))]$. Finally, let

$$C(k_1, \ldots, k_r) := \frac{s!}{(r+1)^{s+1}\, k_1! \cdots k_r!}, \text{ where } s = k_1 + \cdots + k_r.$$

We then have the following two results.

Theorem 3. *As $x \to \infty$,*

$$\sum_{n \le x} U_{k_1,\ldots,k_r}(\ell_0(n)) = (1 + o(1)) C(k_1, \ldots, k_r) x \log\log x \tag{3.5}$$

and, given any small number $\varepsilon > 0$,

$$\lim_{x \to \infty} \frac{1}{x} \# \left\{ n \le x : \left| \frac{U_{k_1,\ldots,k_r}(\ell_0(n))}{\omega(n)} - C(k_1, \ldots, k_r) \right| > \varepsilon \right\} = 0. \tag{3.6}$$

Theorem 4. *As $x \to \infty$,*

$$\sum_{p \le x} U_{k_1,\ldots,k_r}(\ell_0(p+1)) = (1 + o(1)) C(k_1, \ldots, k_r) \mathrm{li}(x) \log\log x \tag{3.7}$$

and, given any small number $\varepsilon > 0$,

$$\lim_{x \to \infty} \frac{1}{\pi(x)} \# \left\{ p \le x : \left| \frac{U_{k_1,\ldots,k_r}(\ell_0(p+1))}{\omega(p+1)} - C(k_1, \ldots, k_r) \right| > \varepsilon \right\} = 0. \tag{3.8}$$

Now, let $\delta_0, \ldots, \delta_{k-1} \in (0, 1)$ and set $H := \delta_0 \cdots \delta_{k-1}$. Let $\mathcal{F}$ be the set of prime $k+1$-tuples $(p_0, p_1, \ldots, p_k)$ which satisfy

$$p_0 < p_1 < \cdots < p_k \quad \text{and} \quad 1 - \delta_j < \Lambda(p_j, p_{j+1}) < 1 \text{ for } j = 0, 1, \ldots, k-1.$$

Given an integer $n \ge 2$, let $V_{\delta_0,\ldots,\delta_{k-1}}(n)$ be the number of those prime divisors p_0 of n for which

$$\nu_{p_0}(n) = p_1, \ldots, \nu_{p_{k-1}}(n) = p_k$$

for some element $(p_0, p_1, \ldots, p_k) \in \mathcal{F}$.

We then have the following two results.

Theorem 5. *As $x \to \infty$,*

$$S_1(x) := \sum_{n \le x} V_{\delta_0,\ldots,\delta_{k-1}}(n) = (H + o(1)) x \log\log x \tag{3.9}$$

and

$$S_2(x) := \sum_{n\le x} V_{\delta_0,\ldots,\delta_{k-1}}(n)^2 = (H^2 + o(1))x\,(\log\log x)^2. \tag{3.10}$$

Consequently, given any small number $\varepsilon > 0$,

$$\lim_{x\to\infty} \frac{1}{x}\#\left\{n \le x : \left|\frac{V_{\delta_0,\ldots,\delta_{k-1}}(n)}{\omega(n)} - H\right| > \varepsilon\right\} = 0. \tag{3.11}$$

Theorem 6. *As* $x \to \infty$, *we have*

$$S_3(x) := \sum_{p\le x} V_{\delta_0,\ldots,\delta_{k-1}}(p+1) = (H + o(1))\mathrm{li}(x)\,\log\log x \tag{3.12}$$

and

$$S_4(x) := \sum_{p\le x} V_{\delta_0,\ldots,\delta_{k-1}}(p+1)^2 = (H^2 + o(1))\mathrm{li}(x)\,(\log\log x)^2. \tag{3.13}$$

Consequently, given any small number $\varepsilon > 0$,

$$\lim_{x\to\infty} \frac{1}{\pi(x)}\#\left\{p \le x : \left|\frac{V_{\delta_0,\ldots,\delta_{k-1}}(p+1)}{\omega(p+1)} - H\right| > \varepsilon\right\} = 0. \tag{3.14}$$

4 Preliminary results

We first recall the Brun-Titchmarsh inequality and the Bombieri-Vinogradov theorem which we state as follows.

Theorem A. (Brun-Titchmarsh) *Given* $\ell \in \mathbb{Z}$ *and a positive integer* D *such that* $(\ell, D) = 1$, *there exists an absolute constant* $C > 0$ *such that for all* $x \ge 2$,

$$\pi(x; D, \ell) < C\frac{\mathrm{li}(x)}{\phi(D)}$$

uniformly for $D \le \sqrt{x}$.

For a proof of Theorem A, see Iwaniec [8].

Let us now define

$$K(x \mid k) := \max_{(k,\ell)=1} \max_{y\le x} \left|\pi(y; k, \ell) - \frac{\mathrm{li}(y)}{\phi(k)}\right|. \tag{4.1}$$

We then have the following.

Theorem B. (Bombieri-Vinogradov) *Given any fixed number* $A > 0$, *there exists a number* $B = B(A) > 0$ *such that*

$$\sum_{k\le\sqrt{x}/(\log^B x)} K(x \mid k) = O\left(\frac{x}{\log^A x}\right).$$

For a proof of Theorem B, see Theorem 17.1 in the book of Iwaniec and Kowalski [9].

We now move to state and prove eight lemmas.

Lemma 1. *Given real numbers $u < v$, then, for any fixed number $A > 1$,*

$$\sum_{u<p<v} \frac{\log p}{p} = \log v - \log u + O\left(\frac{1}{\log^A u}\right).$$

Proof. This is an immediate consequence of the prime number theorem. $\square$

From here on, we will be using the function

$$\varepsilon(x) := \frac{1}{2}\frac{\log\log\log x}{\log\log x}. \tag{4.2}$$

Lemma 2. *Let $\varepsilon(x)$ be as above. Given a large number x, set*

$$Y_1 := Y_1(x) = \exp\left\{(\log x)^{\varepsilon(x)}\right\} \quad \text{and} \quad Y_2 := Y_2(x) = \exp\left\{(\log x)^{1-\varepsilon(x)}\right\}.$$

Then,

$$\sum_{p<Y_1} \frac{1}{p} + \sum_{Y_2<p\le\sqrt{x}} \frac{1}{p} = O\left(\varepsilon(x)\log\log x\right), \tag{4.3}$$

so that

$$\sum_{\substack{p|n\\ p<Y_1}} 1 + \sum_{\substack{p|n\\ p>Y_2}} 1 = o(\log\log x) \quad \text{for almost all } n \le x. \tag{4.4}$$

Proof. Using the well known estimate

$$\sum_{p\le x} \frac{1}{p} = \log\log x + C + o(1) \quad \text{for some absolute constant } C \qquad (x\to\infty),$$

we obtain

$$\begin{aligned}\sum_{p<Y_1} \frac{1}{p} + \sum_{Y_2<p\le\sqrt{x}} \frac{1}{p} &= \varepsilon(x)\log\log x + \log\log x - (1-\varepsilon(x))\log\log x + O(1)\\ &\ll \varepsilon(x)\log\log x,\end{aligned}$$

from which (4.3) follows immediately. Estimate (4.4) is an obvious consequence of (4.3). $\square$

Setting $\pi(x;k,\ell) := \#\{p \le x : p \equiv \ell \pmod{k}\}$, we also have the following.

Lemma 3. *There exists a positive constant C such that*

$$\begin{aligned}\sum_{p<Y_1} \pi(x;p,-1) + \sum_{Y_2<p\le\sqrt{x}} \pi(x;p,-1) &\le C\,\mathrm{li}(x)\left(\sum_{p<Y_1}\frac{1}{p} + \sum_{Y_2<p\le\sqrt{x}}\frac{1}{p}\right)\\ &\ll \varepsilon(x)\,\mathrm{li}(x)\,\log\log x,\end{aligned} \tag{4.5}$$

so that

$$\sum_{\substack{p|q+1\\ p\notin(Y_1,Y_2)}} 1 = o(\log\log x) \quad \text{for almost all primes } q \le x. \tag{4.6}$$

Proof. Estimates (4.5) and (4.6) follow from the Brun-Titchmarsh inequality (see Theorem A) and Lemma 2. □

The following provides an easily obtained upper bound for $U_\varepsilon(n)$.

Lemma 4. *For every $\varepsilon > 0$ and every integer $n > e^2$,*

$$U_\varepsilon(n) \leq \frac{\log\log n}{\log(1/\varepsilon)}.$$

Proof. Let $p_1(n) < \cdots < p_{\omega(n)}(n)$ be the distinct prime factors of n and let $\gamma_j(n)$ be the function defined in (2.1). Further, let $j_1, \ldots, j_k$ be those indices for which $\dfrac{1}{\gamma_j(n)} > \dfrac{1}{\varepsilon}$. Since

$$\frac{\log p_{j_k+1}}{\log p_1} \geq \left(\frac{1}{\varepsilon}\right)^k,$$

it follows that

$$\log\log n \geq k\log(1/\varepsilon),$$

thereby completing the proof of Lemma 4. □

The following two lemmas can be derived from Lemma 2.1 in the book of Elliott [3]. Nevertheless, for the sake of completeness, we do provide here a proof for each of these lemmas.

Lemma 5. *Let $x \geq e^{e^{100}}$ and let $Y_1 = Y_1(x)$ and $Y_2 = Y_2(x)$ be as in Lemma 2. Let $\pi_1 < \cdots < \pi_s$ be s primes located in the interval (Y_1, Y_2). Write their product as $B = \pi_1 \cdots \pi_s$. Further set*

$$\eta := \sum_{i=1}^{s} \frac{1}{\pi_i}$$

and

$$S_B(x) := \sum_{\substack{n \leq x \\ (n,B)=1}} 1.$$

Assume that $\eta \leq K$, where K is an arbitrary number, and let h be a positive integer satisfying $h \geq 3e^2K$. Then,

$$|S_B(x) - \phi_0(B)x| \leq x\,(3e)^{-h} + 2Y_2^h, \tag{4.7}$$

so that in particular, choosing $h = \lfloor \log\log\log x \rfloor$, there exists a positive constant c such that

$$|S_B(x) - \phi_0(B)x| \leq \frac{c\,x}{(\log\log x)^2}. \tag{4.8}$$

Proof. First set

$$T_B^{(h)}(x) := \sum_{\substack{d|B \\ \omega(d)\le h}} \mu(d) \left\lfloor \frac{x}{d} \right\rfloor. \tag{4.9}$$

As is well known,

$$S_B(x) - T_B^{(h)}(x) \text{ is } \begin{cases} \ge 0 & \text{if } h \text{ is odd,} \\ \le 0 & \text{if } h \text{ is even.} \end{cases} \tag{4.10}$$

We have from (4.9) that

$$T_B^{(h)}(x) = x \sum_{\substack{d|B \\ \omega(d)\le h}} \frac{\mu(d)}{d} + \theta_h, \tag{4.11}$$

where

$$|\theta_h| \le \sum_{\substack{d|B \\ \omega(d)\le h}} 1 \le 2\omega(B)^h = 2s^h < 2Y_2^h. \tag{4.12}$$

On the other hand, since $\dfrac{\eta}{h+1} < \dfrac{1}{2}$, one can easily establish that

$$\left| \phi_0(B) - \sum_{\substack{d|B \\ \omega(d)\le h}} \frac{\mu(d)}{d} \right| \le \frac{\eta^{h+1}}{(h+1)!} + \frac{\eta^{h+2}}{(h+2)!} + \cdots < \frac{\eta^h}{h!}. \tag{4.13}$$

Moreover, using the inequality $h! > h^h e^{-h}$ valid for all integers $h \ge 1$, we have

$$\frac{\eta^h}{h!} < \frac{\eta^h}{h^h e^{-h}} = \left(\frac{e\eta}{h}\right)^h < \left(\frac{1}{3e}\right)^h, \tag{4.14}$$

where we used the fact that $\eta \le K \le h/(3e^2)$.

Gathering estimates (4.10) to (4.14) completes the proof of (4.7).

To see why (4.8) holds, we proceed as follows. First observe that

$$(3e)^h = \exp\{\lfloor \log\log\log x \rfloor \cdot \log(3e)\} > \exp\{2\log\log\log x\} = (\log\log x)^2. \tag{4.15}$$

On the other hand, we can show that, given any arbitrarily small number $\delta > 0$,

$$Y_2^h < x^\delta \text{ for } x \text{ sufficiently large.} \tag{4.16}$$

This inequality follows from the fact that, recalling the definition of $\varepsilon(x)$ given in (4.2), we have

$$\log Y_2^h = h \log Y_2 = h(\log x)^{1-\varepsilon(x)} = \frac{h \cdot \log x}{(\log x)^{\varepsilon(x)}}$$

$$
\begin{aligned}
&= \exp\{\log h + \log\log x - \varepsilon(x)\log\log x\} = \exp\{\log h + \log\log x - \frac{1}{2}\log\log\log x\} \\
&\le \exp\{\log\log\log\log x + \log\log x - \frac{1}{2}\log\log\log x\} \\
&< \exp\{\log\log x - \frac{1}{4}\log\log\log x\} \\
&= \exp\left\{\log\left(\frac{\log x}{(\log\log x)^{1/4}}\right)\right\} = \frac{\log x}{(\log\log x)^{1/4}} < \delta\log x,
\end{aligned}
$$

provided x is sufficiently large.

Using (4.15) and (4.16) in (4.7) completes the proof of (4.8). □

Lemma 6. *Let x, B, h, η and K be as in Lemma 5. Let D be a positive integer $\le Y_2^{c_0}$, where c_0 is an arbitrary positive constant, and assume that $(B, D) = 1$. Consider the sum*

$$
\overline{S}_{B,D}(x) := \sum_{\substack{p\le x \\ p+1\equiv 0 \pmod D \\ (p+1,B)=1}} 1. \tag{4.17}
$$

Then, for some positive constant c_1, we have

$$
\left|\overline{S}_{B,D}(x) - \widetilde{\phi}_0(B)\frac{\mathrm{li}(x)}{\phi(D)}\right| \le c_1\frac{\mathrm{li}(x)\,(3e)^{-h}}{\phi(D)} + Y_2^h, \tag{4.18}
$$

so that in particular, by choosing $h = \lfloor \log\log\log x\rfloor$, we have that for some positive constant c_2,

$$
\left|\overline{S}_{B,D}(x) - \widetilde{\phi}_0(B)\frac{\mathrm{li}(x)}{\phi(D)}\right| \le c_2\frac{\mathrm{li}(x)}{\phi(D)\,(\log\log x)^2}. \tag{4.19}
$$

Proof. Making use of Theorem A and proceeding as in the proof of Lemma 5, estimates (4.18) and (4.19) are easily obtained. □

Lemma 7. *Given real numbers $1 < A < B$ and $\ell > 0$ satisfying $\ell/A < 1/2$, then*

$$
\int_A^B t^\ell e^{-t}\,dt < 2A^\ell e^{-A}.
$$

Proof. Using integration by parts, we have

$$
\int_A^B t^\ell e^{-t}\,dt = -t^\ell e^{-t}\Big|_A^B + \ell\int_A^B t^{\ell-1}e^{-t}\,dt \le A^\ell e^{-A} + \ell\int_A^B t^{\ell-1}e^{-t}\,dt. \tag{4.20}
$$

Using integration by parts repetitively in this last integral, one can deduct from (4.20) that

$$
\int_A^B t^\ell e^{-t}\,dt \le A^\ell e^{-A}\left(1 + \frac{\ell}{A} + \frac{\ell(\ell-1)}{A^2} + \cdots\right) < 2A^\ell e^{-A}, \tag{4.21}
$$

where, for this last inequality, we used the hypothesis $\ell/A < 1/2$. □

Lemma 8. *Given positive real numbers $A < B$ and non-negative integers s and r, set*

$$L_s(A,B) := \int_A^B (B-t)^s e^{t(r+1)}\, dt.$$

Then,

$$L_0(A,B) = \frac{e^{B(r+1)} - e^{A(r+1)}}{r+1}$$

and for each $s \geq 1$,

$$L_s(A,B) = -\frac{(B-A)^s}{s+1} e^{A(r+1)} + \frac{s}{r+1} L_{s-1}(A,B).$$

Consequently, for all integers $s \geq 0$, we have

$$L_s(A,B) = \frac{s!}{(r+1)^{s+1}} e^{B(r+1)} + e^{A(r+1)} P_s(B-A),$$

where $P_s(y)$ is some polynomial of degree s.

Proof. The proof is quite straightforward using repetitive integration by parts. □

5 Proof of Theorem 1

Set

$$S_\lambda(x) := \sum_{n \leq x} U_\lambda(n).$$

Instead of working directly with $U_\lambda(n)$, it will be more convenient to first study the function

$$U_\lambda^{\varepsilon_0}(n) := \sum_{\substack{p \mid n \\ Y_1 < p < Y_2 \\ \varepsilon_0 \leq \frac{\log p}{\log \nu_p(n)} < \lambda}} 1,$$

where $\varepsilon_0 > 0$ is a fixed small number, and set

$$S_\lambda^{\varepsilon_0}(x) := \sum_{n \leq x} U_\lambda^{\varepsilon_0}(n).$$

Using estimate (4.8) of Lemma 5, we have

$$\begin{aligned}
S_\lambda^{\varepsilon_0}(x) &= \sum_{\substack{Y_1 < p < q < Y_2 \\ \varepsilon_0 \leq \Lambda(p,q) < \lambda}} \sum_{\substack{n = pqm \leq x \\ (m, Q(p,q)) = 1}} 1 \\
&= \sum_{\substack{Y_1 < p < q < Y_2 \\ \varepsilon_0 \leq \Lambda(p,q) < \lambda}} S_{Q(p,q)}\left(\frac{x}{pq}\right) \\
&= \sum_{\substack{Y_1 < p < q < Y_2 \\ \varepsilon_0 \leq \Lambda(p,q) < \lambda}} \left(\frac{x}{pq} \phi_0(Q(p,q)) + O\left(\frac{x}{pq} \frac{1}{(\log\log x)^2} \right) \right)
\end{aligned}$$

$$= x \sum_{\substack{Y_1<p<q<Y_2 \\ \varepsilon_0 \le \Lambda(p,q)<\lambda}} \frac{1}{pq}\Lambda(p,q)\left(1+O\left(\frac{1}{\log p}\right)\right) + O\left(\frac{x}{(\log\log x)^2}\sum_{p<q<x}\frac{1}{pq}\right)$$

$$= x \sum_{Y_1<q<Y_2} \frac{1}{q\log q} \sum_{\varepsilon_0 \le \Lambda(p,q)<\lambda} \frac{\log p}{p} + O\left(x \sum_{p<q<x}\frac{1}{pq\log q}\right)$$

$$+O\left(\frac{x}{(\log\log x)^2}\sum_{p<q<x}\frac{1}{pq}\right). \tag{5.1}$$

Since

$$\sum_{p<q<x}\frac{1}{pq\log q} = \sum_{q<x}\frac{1}{q\log q}\sum_{p<q}\frac{1}{p} \ll \sum_{q<x}\frac{1}{q\log q}\log\log q \ll \int_2^x \frac{\log\log t}{t\log^2 t}dt = O(1)$$

and

$$\sum_{p<q<x}\frac{1}{pq} \ll (\log\log x)^2,$$

it follows from (5.1) that

$$S_\lambda^{\varepsilon_0}(x) = x\sum_{Y_1<q<Y_2}\frac{1}{q\log q}(\lambda-\varepsilon_0)\log q + O(x) = (\lambda-\varepsilon_0)x\ \log\log x + O(x), \tag{5.2}$$

where we used Lemma 2 and the definition of $\varepsilon(x)$ given in (4.2).

We now need to estimate how much $S_\lambda(x)$ differs from $S_\lambda^{\varepsilon_0}(x)$. For this, we set

$$\xi(n) := U_\lambda(n) - U_\lambda^{\varepsilon_0}(n). \tag{5.3}$$

Then, using the same development as the one used to obtain (5.1) and (5.2), we find that

$$S_\lambda(x) - S_\lambda^{\varepsilon_0}(x) = \sum_{n\le x}\xi(n) = \sum_{n\le x}\sum_{\substack{p\mid n \\ Y_1<p<Y_2 \\ \frac{\log p}{\log \nu_p(n)}<\varepsilon_0}} 1 \ll x\sum_{Y_1<q<Y_2}\frac{1}{q\log q}\sum_{p<q^{\varepsilon_0}}\frac{\log p}{p}$$

$$\ll x\varepsilon_0 \sum_{Y_1<q<Y_2}\frac{1}{q} = O(\varepsilon_0 x\log\log x). \tag{5.4}$$

Hence, (5.4) allows us to replace (5.2) by

$$S_\lambda(x) = (\lambda-\varepsilon_0)x\ \log\log x + O(x) + O(\varepsilon_0 x\log\log x) = (\lambda-\varepsilon_0)x\ \log\log x + O(\varepsilon_0 x\log\log x).$$

Since ε_0 can be chosen arbitrarily small, estimate (3.1) follows.

It remains to prove (3.2). We have

$$\sum_{n\le x} U_\lambda^{\varepsilon_0}(n)^2 = S_\lambda^{\varepsilon_0}(x) + 2\sum_{\substack{Y_1<p_1<q_1<p_2<q_2<Y_2 \\ \varepsilon_0\le\Lambda(p_i,q_i)<\lambda\ \ (i=1,2) \\ \Lambda(q_1,q_2)<\varepsilon_0}} S_{Q(p_1,q_1)Q(p_2,q_2)}\left(\frac{x}{p_1q_1p_2q_2}\right)$$

$$
\begin{aligned}
& +2 \sum_{\substack{Y_1<p_1<q_1=p_2<q_2<Y_2\\ \varepsilon_0\le\Lambda(p_i,q_i)<\lambda\ \ (i=1,2)}} S_{Q(p_1,q_2)}\left(\frac{x}{p_1p_2q_2}\right)\\
& +2 \sum_{\substack{Y_1<p_1<q_1<p_2<q_2<Y_2\\ \varepsilon_0\le\Lambda(p_i,q_i)<\lambda\ \ (i=1,2)\\ \varepsilon_0\le\Lambda(q_1,q_2)<1}} S_{Q(p_1,q_2)}\left(\frac{x}{p_1q_1p_2q_2}\right)\\
= \;& S_\lambda^{\varepsilon_0}(x)+2\Sigma_1+2\Sigma_2+2\Sigma_3, \qquad (5.5)
\end{aligned}
$$

say.

On the one hand, using estimate (4.8) of Lemma 5 as we did in the first part of the proof and making repetitive use of the prime number theorem in the form $\sum_{p\le y}\frac{\log p}{p}=\log y+O(1)$, we have

$$
\begin{aligned}
\Sigma_1 &= \sum_{\substack{Y_1<p_1<q_1<p_2<q_2<Y_2\\ \varepsilon_0\le\Lambda(p_i,q_i)<\lambda\ \ (i=1,2)\\ \Lambda(q_1,q_2)<\varepsilon_0}} \frac{x}{p_1q_1p_2q_2}\frac{\log p_1\log p_2}{\log q_1\log q_2}\\
&\qquad +O\left(\sum_{\substack{Y_1<p_1<q_1<p_2<q_2<Y_2\\ \varepsilon_0\le\Lambda(p_i,q_i)<\lambda\ \ (i=1,2)}} \frac{1}{p_1q_1p_2q_2}\cdot\frac{x}{(\log\log x)^2}\right)\\
&= x\sum_{\substack{Y_1<q_1<q_2<Y_2\\ \Lambda(q_1,q_2)<\varepsilon_0}} \frac{(\lambda-\varepsilon_0)^2}{q_1q_2}\left(1+O\left(\frac{1}{\log q_1}\right)\right)\left(1+O\left(\frac{1}{\log q_2}\right)\right)\\
&\qquad +O\left(\frac{x}{(\log\log x)^2}\log\log x\log(1/\varepsilon_0)\right)\\
&= (\lambda-\varepsilon_0)^2x\sum_{\substack{Y_1<q_1<q_2<Y_2\\ \Lambda(q_1,q_2)<\varepsilon_0}}\frac{1}{q_1q_2}+O(x\ \log\log x)\\
&= (\lambda-\varepsilon_0)^2\frac{x}{2}\left(\sum_{q<x}\frac{1}{q}\right)^2+O(x\ \log\log x)+O\left(x\sum_{\substack{\varepsilon_0\le\Lambda(q_1,q_2)<1\\ q_1,q_2<x}}\frac{1}{q_1q_2}\right). \qquad (5.6)
\end{aligned}
$$

Since

$$
\sum_{\substack{\varepsilon_0\le\Lambda(q_1,q_2)<1\\ q_1,q_2<x}}\frac{1}{q_1q_2}\ll\log\log x\log(1/\varepsilon_0),
$$

we can replace (5.6) by

$$
\begin{aligned}
\Sigma_1 &= (\lambda-\varepsilon_0)^2\frac{x}{2}(\log\log x)^2+O(x\log\log x\log(1/\varepsilon_0))\\
&= (\lambda-\varepsilon_0)^2\frac{x}{2}(\log\log x)^2+O(x\log\log x\log\log\log x), \qquad (5.7)
\end{aligned}
$$

provided $x\ge e^{e^{1/\varepsilon_0}}$.

On the other hand, using repetitively the estimate $\sum_{p\le y}\frac{\log p}{p}\ll \log y$, we have

$$
\begin{aligned}
\Sigma_2 &\ll x \sum_{Y_1<p_1<q_1<q_2<Y_2} \frac{\log p_1}{p_1}\frac{1}{q_1q_2\log q_2}\\
&\ll x \sum_{Y_1<q_1<q_2<Y_2} \frac{\log q_1}{q_1}\frac{1}{q_2\log q_2}\\
&< x\sum_{q_2<Y_2}\frac{1}{q_2} \ll x\log\log x. \qquad (5.8)
\end{aligned}
$$

Finally,

$$
\Sigma_3 \ll x \sum_{\varepsilon_0\le\Lambda(q_1,q_2)<1}\frac{1}{q_1q_2} \ll x\log\log x\log(1/\varepsilon_0) = O(x\log\log x\log\log\log x), \qquad (5.9)
$$

since we assumed that $x\ge e^{e^{1/\varepsilon_0}}$.

Substituting (5.2), (5.7), (5.8) and (5.9) in (5.5), we obtain

$$
\sum_{n\le x} U_\lambda^{\varepsilon_0}(n)^2 = (\lambda-\varepsilon_0)^2 x(\log\log x)^2 + O(x\log\log x\log\log\log x).
$$

Using this last formula along with estimate (3.1), we obtain

$$
\sum_{n\le x}\left(U_\lambda^{\varepsilon_0}(n)-(\lambda-\varepsilon_0)\log\log x\right)^2 = o\left(x(\log\log x)^2\right) \qquad (x\to\infty)
$$

and therefore

$$
\sum_{n\le x}\left(U_\lambda^{\varepsilon_0}(n)-\lambda\log\log x\right)^2 = O\left(\varepsilon_0^2 x(\log\log x)^2\right) + o\left(x(\log\log x)^2\right) \qquad (x\to\infty). \qquad (5.10)
$$

Recall that through (5.4), we could show that the error caused by replacing $\sum_{n\le x}U_\lambda^{\varepsilon_0}(n)$ by $\sum_{n\le x}U_\lambda(n)$ was "small". Using essentially the same technique, one can show that (5.10) can be replaced by

$$
\sum_{n\le x}\left(U_\lambda(n)-\lambda\log\log x\right)^2 = O\left(\varepsilon_0^2 x(\log\log x)^2\right) + o\left(x(\log\log x)^2\right) \qquad (x\to\infty),
$$

which in turn implies that there exists a positive constant C such that

$$
\frac{1}{x}\sum_{n\le x}\left(\frac{U_\lambda(n)}{\log\log x}-\lambda\right)^2 \le C\varepsilon_0^2 + o(1) \qquad (x\to\infty),
$$

so that since ε_0 can be chosen arbitrarily small, we may conclude that

$$
\lim_{x\to\infty}\frac{1}{x}\sum_{n\le x}\left(\frac{U_\lambda(n)}{\log\log x}-\lambda\right)^2 = 0,
$$

from which (3.2) follows, thus concluding the proof of Theorem 1.

6 Proof of Theorem 2

The proof of Theorem 2 follows essentially along the lines of the proof of Theorem 1. Therefore, we will not provide all the details of its proof.

Recall the notation used in the proof of Theorem 1 and let

$$\mathcal{K}_\lambda(x) := \sum_{P\le x} U_\lambda(P+1) \quad \text{and} \quad \mathcal{K}_\lambda^{\varepsilon_0}(x) := \sum_{P\le x} U_\lambda^{\varepsilon_0}(P+1).$$

Using estimate (4.19) of Lemma 6 with $D = pq$ and $B = Q(p,q)$, we have

$$\begin{aligned}
\mathcal{K}_\lambda^{\varepsilon_0}(x) &= \sum_{P\le x} \sum_{\substack{pq\mid P+1\\ \varepsilon_0\le\Lambda(p,q)<\lambda\\ (P+1,Q(p,q))=1}} 1 = \sum_{\substack{Y_1<p<q<Y_2\\ \varepsilon_0\le\Lambda(p,q)<\lambda}} \overline{S}_{Q(p,q)}\left(\frac{x}{pq}\right)\\
&= \operatorname{li}(x) \sum_{\substack{Y_1<p<q<Y_2\\ \varepsilon_0\le\Lambda(p,q)<\lambda}} \frac{\widetilde{\phi}_0(Q(p,q))}{\phi(pq)} + O\left(\frac{\operatorname{li}(x)}{(\log\log x)^2} \sum_{\substack{Y_1<p<q<Y_2\\ \varepsilon_0\le\Lambda(p,q)<\lambda}} \frac{1}{\phi(pq)}\right)\\
&= \operatorname{li}(x) \sum_{\substack{Y_1<p<q<Y_2\\ \varepsilon_0\le\Lambda(p,q)<\lambda}} \frac{\widetilde{\phi}_0(Q(p,q))}{\phi(pq)} + O\left(\operatorname{li}(x)\right). \qquad (6.1)
\end{aligned}$$

Regarding the main sum on the right-hand side of (6.1), recall that, as already mentioned in identity (2.3), we may write that

$$\widetilde{\phi}_0(Q(p,q)) = \phi_0(Q(p,q)) \prod_{\pi\mid Q(p,q)} \left(1 - \frac{1}{(\pi-1)^2}\right) = \phi_0(Q(p,q))\left(1 - O\left(\frac{1}{p}\right)\right).$$

Using this last estimate in the sum appearing on the right-hand side of (6.1) and proceeding as in the proof of Theorem 1, we can replace (6.1) by

$$\mathcal{K}_\lambda^{\varepsilon_0}(x) = (\lambda-\varepsilon_0)\operatorname{li}(x)\,\log\log x + O(\operatorname{li}(x)). \qquad (6.2)$$

Recalling the definition of $\xi(n)$ given in (5.3), we need to show that $\sum_{p\le x}\xi(p+1)$ is so small that it allows us to replace $\mathcal{K}_\lambda^{\varepsilon_0}(x)$ by $\mathcal{K}_\lambda(x)$ in estimate (6.2). In fact, proceeding as we did to obtain (5.4), one can prove that

$$\sum_{p\le x}\xi(p+1) = O(\varepsilon_0\operatorname{li}(x)\,\log\log x).$$

Using this last estimate we can replace (6.2) by

$$\mathcal{K}_\lambda(x) = (\lambda-\varepsilon_0)\operatorname{li}(x)\,\log\log x + O(\operatorname{li}(x)) + O(\varepsilon_0\operatorname{li}(x)\,\log\log x) \qquad (x\to\infty). \qquad (6.3)$$

Since ε_0 can be chosen arbitrarily small, estimate (6.3) proves (3.3).

We now move on to estimate the sum of $U_\lambda^{\varepsilon_0}(P+1)^2$ as P runs over the primes not exceeding x. We separate this sum in four smaller sums, essentially as we did in (5.5), as we write

$$\sum_{P\le x} U_\lambda^{\varepsilon_0}(P+1)^2 = \mathcal{K}_\lambda^{\varepsilon_0}(x) + 2\Sigma_1 + 2\Sigma_2 + 2\Sigma_3, \qquad (6.4)$$

where

- in Σ_1, recalling the definition of $\overline{S}_{B,D}(x)$ given in (4.17), we sum $\overline{S}_{B,D}(x)$ over those pairs $(B, D) = (Q(p_1, q_1)Q(p_2, q_2), p_1q_1p_2q_2)$ satisfying the three conditions

  (i) $Y_1 < p_1 < q_1 < p_2 < q_2 < Y_2$,

  (ii) $\Lambda(p_i, q_i) \in (\varepsilon_0, \lambda)$ for $i = 1, 2$,

  (iii) $\Lambda(q_1, q_2) \le \varepsilon_0$;

- in Σ_3, we sum $\overline{S}_{B,D}(x)$ over those pairs $(B, D) = (Q(p_1, q_1)Q(p_2, q_2), p_1q_1p_2q_2)$ satisfying the above conditions (i) and (ii) as well as the condition $\varepsilon_0 < \Lambda(q_1, q_2) < 1$;

- in Σ_2, we sum $\overline{S}_{B,D}(x)$ over those pairs $(B, D) = (Q(p_1, q_1)Q(p_2, q_2), p_1q_1p_2q_2)$ for which $q_1 = p_2$ and $\Lambda(q_1, q_2) \le \varepsilon_0$.

In Σ_1 amd Σ_3, we have, using estimate (4.19) of Lemma 6,

$$\left| \overline{S}_{B,D}(x) - \widetilde{\phi}_0(B) \frac{\mathrm{li}(x)}{\phi(D)} \right| \le c_2 \frac{\mathrm{li}(x)}{\phi(D)(\log\log x)^2}. \tag{6.5}$$

First observe that

$$\sum_{D \le x} \frac{\mathrm{li}(x)}{\phi(D)(\log\log x)^2} \ll \mathrm{li}(x). \tag{6.6}$$

Moreover, observe that given any odd integer $n \ge 3$ and letting r_n stand for the smallest prime divisor of n, we have

$$1 \ge \frac{\widetilde{\phi}_0(n)}{\phi(n)} = \prod_{\pi | n} \frac{\pi^2 - 2\pi}{\pi^2 - 2\pi + 1} \ge 1 - \frac{1}{r_n}.$$

A consequence of this is that

$$\widetilde{\phi}_0(B) = \frac{\phi(B)}{B} \left(1 + O\left(\frac{1}{p_1} \right) \right).$$

Using this relation, we may write that

$$\begin{aligned} \widetilde{\phi}_0(B) &= \phi_0(Q(p_1, q_1))\phi_0(Q(p_2, q_2)) \left(1 + O\left(\frac{1}{p_1} \right) \right) \\ &= \Lambda(p_1, q_1)\Lambda(p_2, q_2) \left(1 + O\left(\frac{1}{\log p_1} \right) \right) \left(1 + O\left(\frac{1}{\log p_2} \right) \right) \left(1 + O\left(\frac{1}{p_1} \right) \right). \end{aligned}$$

Using this relation and repeating the argument used in the proof of Theorem 1, taking into account estimates (6.5) and (6.6), we obtain that

$$\Sigma_1 = \mathrm{li}(x) {\sum}' \frac{\Lambda(p_1, q_1)\Lambda(p_2, q_2)}{\phi(p_1q_1)\phi(p_2q_2)} + O\left(\frac{\mathrm{li}(x)}{(\log\log x)^2} {\sum}' \frac{1}{p_1q_1p_2q_2} \right) + O\left(\mathrm{li}(x) \right),$$

where the dash on each of the above two sums indicates that the sums run over those primes p_1, q_1, p_2, q_2 satisfying the conditions described earlier for the sum Σ_1.

Proceeding in a similar way to estimate Σ_2 and Σ_3, and gathering the corresponding estimates for Σ_1, Σ_2 and Σ_3, we finally obtain from (6.4) that

$$\sum_{P\le x} U_\lambda^{\varepsilon_0}(P+1)^2 = (\lambda-\varepsilon_0)^2\frac{\mathrm{li}(x)}{2}(\log\log x)^2 + O\left(\mathrm{li}(x)\log\log x\right). \tag{6.7}$$

Reasoning as above, we can replace $U_\lambda^{\varepsilon_0}(P+1)^2$ by $U_\lambda(P+1)^2$ in relation (6.7) and conclude that

$$\sum_{P\le x} U_\lambda(P+1)^2 = (\lambda-\varepsilon_0)^2\frac{\mathrm{li}(x)}{2}(\log\log x)^2 + O\left(\mathrm{li}(x)\log\log x\right). \tag{6.8}$$

Combining estimates (6.3) and (6.8), we easily obtain that

$$\frac{1}{\pi(x)}\sum_{P\le x}\left(\frac{U_\lambda(P+1)}{\log\log x}-\lambda\right)^2 \to 0 \qquad \text{as } x\to\infty,$$

from which (3.4) follows, thus completing the proof of Theorem 2.

7 The proofs of Theorems 3 and 4

Let us write

$$\sum_{n\le x} U_{k_1,\dots,k_r}(\ell_0(n)) = \sum_{p<q} W(x;p,q), \tag{7.1}$$

where $W(x;p,q)$ stands for the number of positive integers $n\le x$ such that $pq \mid \ell_0(n)$, $\left(\frac{\ell_0(n)}{pq}, Q(p,q)\right)=1$ and for which there exist

$$R_j \mid Q(p,q),\ \omega(R_j)=k_j \text{ and } \left(\ell_j(n), \frac{Q(p,q)}{R_j}\right)=1 \text{ for } j=1,\dots,r.$$

Observe that the contribution to the sum (7.1) of those primes p,q for which $p<Y_1$ or $q>Y_2$ is $o(x\log\log x)$ and can therefore be ignored.

So for now, let $p<q$ (with $p,q\in(Y_1,Y_2)$) be fixed. Given fixed $R_1,\dots,R_r$ with $(R_i,R_j)=1$ for $i\ne j$, let $\Sigma_{p,q}(R_1,\dots,R_r)$ be the number of those positive integers $n\le x$ satisfying the above conditions. Using standard sieve techniques similar to what we did in the proof of Lemma 5 and due to the coprimality of the numbers R_i, we then have

$$\Sigma_{p,q}(R_1,\dots,R_r) = (1+o(1))\frac{x}{pqR_1\cdots R_r}\prod_{p<\pi<q}\left(1-\frac{\kappa(\pi)}{\pi}\right) \qquad (x\to\infty), \tag{7.2}$$

where

$$\kappa(\pi) = \begin{cases} r & \text{if } \pi \mid R_1\cdots R_r, \\ r+1 & \text{if } \pi\nmid R_1\cdots R_r. \end{cases}$$

Since

$$\sum_{p<\pi<q} \log\left(1-\frac{\kappa(\pi)}{\pi}\right) = -\sum_{p<\pi<q} \frac{r+1}{\pi} + O\left(\sum_{\pi \mid R_1\cdots R_r} \frac{1}{\pi}\right) + O\left(\frac{1}{p}\right)$$
$$= (r+1)\log \Lambda(p,q) + O\left(\frac{1}{p}\right),$$

we have that

$$\prod_{p<\pi<q}\left(1-\frac{\kappa(\pi)}{\pi}\right) = (1+o(1))\Lambda(p,q)^{r+1}. \tag{7.3}$$

We now need to estimate how many ways we can choose $R_1,\ldots,R_r$ such that

$$R_j \mid \ell_j(n), \quad R_j \mid Q(p,q) \text{ for } j=1,\ldots,r, \text{ with } (R_i,R_j)=1 \text{ for } i\neq j,$$

or written differently, how many solutions are there of

$$R_1\cdots R_r = m \text{ with } m \mid Q(p,q) \text{ and } \omega(m) = k_1+\cdots+k_r.$$

One can see that this number of solutions is equal to $\dfrac{s!}{k_1!\cdots k_r!}$, where $s = k_1+\cdots+k_r$.

It follows from this that

$$T_{p,q} := \sum_{\substack{R_1\cdots R_r = m\\ m\mid Q(p,q)\\ \omega(m)=s}} \frac{1}{R_1\cdots R_r} = \frac{s!}{k_1!\cdots k_r!}\sum_{\substack{m\mid Q(p,q)\\ \omega(m)=s}}\frac{1}{m}. \tag{7.4}$$

Now, as $x\to\infty$,

$$\sum_{\substack{m\mid Q(p,q)\\ \omega(m)=s}}\frac{1}{m} = (1+o(1))\frac{1}{s!}\left(\sum_{p<\pi<q}\frac{1}{\pi}\right)^s = (1+o(1))\frac{1}{s!}\left(\log\frac{1}{\Lambda(p,q)}\right)^s. \tag{7.5}$$

Using (7.5) in (7.4), we obtain

$$T_{p,q} = \frac{(1+o(1))}{k_1!\cdots k_r!}\left(\log\frac{1}{\Lambda(p,q)}\right)^s. \tag{7.6}$$

Combining (7.3) and (7.6) in (7.2), we obtain that

$$W(x;p,q) = \frac{(1+o(1))}{pqk_1!\cdots k_r!}x\left(\log\frac{1}{\Lambda(p,q)}\right)^s \Lambda(p,q)^{r+1}. \tag{7.7}$$

On the other hand, we have

$$\Phi(Y_1,Y_2) := \sum_{Y_1<p<q<Y_2}\frac{1}{pq}\left(\log\frac{\log q}{\log p}\right)^s \Lambda(p,q)^{r+1}$$

$$= \sum_{Y_1<q<Y_2} \frac{1}{q(\log q)^{r+1}} \sum_{Y_1<p<q} \frac{1}{p} (\log\log q - \log\log p)^s (\log p)^{r+1}$$

$$= \sum_{Y_1<q<Y_2} \frac{1}{q(\log q)^{r+1}} \cdot Z_q, \tag{7.8}$$

say. In order to evaluate Z_q, we will make use of the prime number theorem with a "good error term", namely in the form

$$\pi(x) = \mathrm{li}(x) + R(x), \text{ with } |R(x)| \le x \exp\{-\sqrt{\log x}\} \text{ provided } x \text{ is sufficiently large} \tag{7.9}$$

(see for instance pages 61-62 in the book of De Koninck and Doyon [1]).

We write

$$\begin{aligned} Z_q &= \int_{Y_1}^{q} \frac{(\log u)^{r+1}}{u} (\log\log q - \log\log u)^s \, d\,\pi(u) \\ &= \int_{Y_1}^{q} \frac{(\log u)^{r+1}}{u} (\log\log q - \log\log u)^s \, d\,\mathrm{li}(u) \\ &\qquad + \int_{Y_1}^{q} \frac{(\log u)^{r+1}}{u} (\log\log q - \log\log u)^s \, d\,R(u) \\ &= \int_{Y_1}^{q} H(u)\, d\,\mathrm{li}(u) + \int_{Y_1}^{q} H(u)\, d\,R(u) \\ &= I_q + E_q, \end{aligned} \tag{7.10}$$

say. We will now evaluate I_q and E_q separately. We start by bounding E_q using (7.9). More precisely, we will prove that

$$E_q \ll (\log\log q)^s \, \log^{r+3/2} Y_1 \, \exp\{-\sqrt{\log Y_1}\}. \tag{7.11}$$

In order to prove (7.11), we first use partial integration to write E_q as

$$E_q = H(q)R(q) - H(Y_1)R(Y_1) - \int_{Y_1}^{q} R(u)H'(u)\, du = J_1(q) + J_2(q), \tag{7.12}$$

say.

On the one hand, using (7.9) and since $H(q) = 0$, we have that

$$J_1(q) = H(Y_1)R(Y_1) \ll \frac{\log^{r+1} Y_1}{Y_1} (\log\log q)^s \cdot Y_1 \exp\{-\sqrt{\log Y_1}\} \ll (\log\log q)^s, \tag{7.13}$$

provided $Y_1 = Y_1(x)$ is large enough.

On the other hand, regarding the function $H(u)$ defined implicity in (7.10), it is straigthforward that

$$\begin{aligned} H'(u) &= \frac{(r+1)(\log u)^r - (\log u)^{r+1}}{u^2} (\log\log q - \log\log u)^s \\ &\qquad + \frac{(\log u)^{r+1}}{u} s (\log\log q - \log\log u)^{s-1} \cdot \left(-\frac{1}{u\log u}\right) \end{aligned}$$

$$\ll \frac{(\log u)^{r+1}}{u^2}(\log\log q)^s. \tag{7.14}$$

Combining estimates (7.9) and (7.14), and then using the change of variables $v = \log u$ and $t = \sqrt{v}$ in the coming integrals, we obtain

$$\begin{aligned}
J_2(q) &\ll (\log\log q)^s \int_{Y_1}^{q} \frac{(\log u)^{r+1}}{u^2} \cdot u e^{-\sqrt{\log u}}\, du \\
&= (\log\log q)^s \int_{\log Y_1}^{\log q} v^{r+1} e^{-\sqrt{v}}\, dv \\
&= 2(\log\log q)^s \int_{\sqrt{\log Y_1}}^{\sqrt{\log q}} t^{\frac{r+3}{2}} e^{-t}\, dt \\
&< 4(\log\log q)^s \left(\sqrt{\log Y_1}\right)^{\frac{r+3}{2}} e^{-\sqrt{\log Y_1}}, \qquad (7.15)
\end{aligned}$$

where, in order to obtain the last inequality, we made use of Lemma 7 after having observed that the hypothesis $((r+3)/2)/Y_1 < 1/2$ is indeed satisfied, provided Y_1 is sufficiently large.

Hence, combining estimates (7.13) and (7.15) in (7.12), our claim (7.11) follows immediately.

Now, integrating and using the change of variables $v = \log u$ and thereafter $t = \log v$, we get that

$$\begin{aligned}
I_q &= \int_{Y_1}^{q} \frac{(\log u)^{r+1}}{u \log u} (\log\log q - \log\log u)^s\, du \\
&= \int_{\log Y_1}^{\log q} v^r (\log\log q - \log v)^s dv \\
&= \int_{\log\log Y_1}^{\log\log q} (\log\log q - t)^s e^{(r+1)t}\, dt \\
&= \frac{s!}{(r+1)^{s+1}} (\log q)^{r+1} + O\left((\log Y_1)^{r+1} (\log\log q)^s\right), \qquad (7.16)
\end{aligned}$$

where we used Lemma 8 with $A = \log\log Y_1$ and $B = \log\log q$.

Combining estimates (7.16) and (7.11) in (7.10), we get that

$$Z_q = \frac{s!}{(r+1)^{s+1}} (\log q)^{r+1} + O\left((\log Y_1)^{r+1} (\log\log q)^s\right). \tag{7.17}$$

Using (7.17) in (7.8), we obtain that, as $x \to \infty$,

$$\begin{aligned}
\Phi(Y_1, Y_2) &= (1+o(1)) \sum_{Y_1 < q < Y_2} \frac{s!}{(r+1)^{s+1}} (\log q)^{r+1} \cdot \frac{1}{q(\log q)^{r+1}} \\
&= (1+o(1)) \frac{s!}{(r+1)^{s+1}} \log\log x. \qquad (7.18)
\end{aligned}$$

Finally, using (7.18) and (7.7) in (7.1), we obtain that

$$\sum_{n\le x} U_{k_1,\dots,k_r}(\ell_0(n)) = (1+o(1))\frac{s!}{(r+1)^{s+1}k_1!\cdots k_r!}\, x\,\log\log x \qquad (x\to\infty),$$

thereby completing the proof of (3.5).

In order to prove (3.6), it is clear that one only needs to prove

$$\sum_{n\le x} U_{k_1,\dots,k_r}(\ell_0(n))^2 = (1+o(1))x\left(C(k_1,\dots,k_r)\,(\log\log x)\right)^2 \qquad (x\to\infty). \tag{7.19}$$

To estimate the left-hand side of (7.19), we need to count the number of those

$$p_1<q_1,\quad p_2<q_2,\quad R_1^{(1)},\dots,R_r^{(1)},\quad R_1^{(2)},\dots,R_r^{(2)} \tag{7.20}$$

satisfying the three conditions

(i) $p_jq_j \mid \ell_0(n), \quad (\ell_0(n), Q(p_j,q_j)) = 1 \quad$ for $j=1,2$,

(ii) $\omega(R_h^{(j)}) = k_h \quad (h=1,\dots,r) \quad$ for $j=1,2$,

(iii) $R_h^{(j)} \mid \ell_h(n) \quad (h=1,\dots,r) \quad$ for $j=1,2$.

Then, for each $n\le x$, we need to count those numbers in (7.20) which satisfy conditions (i), (ii) and (iii). This can be done in three steps, namely by showing that

1. The contribution of those numbers in (7.20) also satisfying $\min(p_1,q_1)<Y_1$ or $\min(p_2,q_2)>Y_2$ is $o(x(\log\log x)^2)$ as $x\to\infty$.

2. The contribution of those numbers in (7.20) for which the intervals $[p_1,q_1]$ and $[p_2,q_2]$ are not disjoint is $o(x(\log\log x)^2)$ as $x\to\infty$.

3. Finally, the number of those $n\le x$ with corresponding numbers $p_i,q_i,R_1^{(i)},\dots,R_r^{(i)}$, $i=1,2$, satisfying properties (i), (ii) and (iii) is equal to

   $$(1+o(1))\left(\frac{1}{x}\Sigma^*_{p_1,q_1}\right)\left(\frac{1}{x}\Sigma^*_{p_2,q_2}\right) \qquad (x\to\infty),$$

   where in the sum $\Sigma^*_{p_1,q_1}$, we sum over those $p_j,q_j,R_h^{(j)}$ with $j=1$ and $h=1,\dots,r$ satisfying conditions (i), (ii) and (iii), whereas in the sum $\Sigma^*_{p_2,q_2}$, we do the same but this time with $j=2$.

From steps 1, 2 and 3, we may then conclude that (7.19) holds, from which formula (3.6) follows, thus completing the proof of Theorem 3.

The proof of Theorem 4 is very similar to that of Theorem 3. The only difference is that, not only do we use standard sieve technique, but we also make use of the Bombieri-Vinogradov theorem (Theorem B). We only give the highlights of the proof. We start as we did in (7.1) but this time for shifted primes, namely by writing

$$\sum_{P\le x} U_{k_1,\dots,k_r}(\ell_0(P+1)) = \sum_{p<q} W^*(x;p,q), \tag{7.21}$$

where $W^*(x;p,q)$ stands for the number of those primes $P \le x$ such that

$$pq \mid \ell_0(P+1), \qquad \left(\frac{\ell_0(P+1)}{pq}, Q(p,q)\right) = 1$$

and for which there exist $R_j \mid Q(p,q)$, $\omega(R_j) = k_j$ and $\left(\ell_j(P+1), \frac{Q(p,q)}{R_j}\right) = 1$ for $j = 1,\dots,r$. We first observe that

(7.22)
$$\sum_{\substack{p<q\\ p<Y_1}} W^*(x;p,q) = o(x\log\log x) \quad \text{and} \quad \sum_{\substack{p<q\\ q>Y_2}} W^*(x;p,q) = o(x\log\log x) \qquad (x\to\infty).$$

We then proceed as in the proof of Theorem 3 and obtain that

$$\sum_{p<q} W^*(x;p,q) = (1+o(1))\frac{\mathrm{li}(x)}{\phi(pq)R_1\cdots R_r}\prod_{p<\pi<q}\left(1-\frac{\kappa(\pi)}{\pi-1}\right) + E_{p,q}, \tag{7.23}$$

where the error term $E_{p,q}$ is such that

$$E_{p,q} \ll \sum_{\substack{\delta\mid Q(p,q)\\ \omega(\delta)\le h}} K(x \mid pq\delta), \tag{7.24}$$

where as previously, we chose $h = \lfloor \log\log\log x \rfloor$. Using Theorem B, it follows from (7.24) that, for any given $A > 1$,

$$\sum_{Y_1<p<q<Y_2} E_{p,q} \ll \frac{x}{\log^A x}. \tag{7.25}$$

Gathering estimates (7.22), (7.23) and (7.25) in (7.21), estimate (3.7) follows, from which (3.8) is an immediate consequence. This completes the proof of Theorem 4.

8 The proof of Theorem 5

Let $\mathcal{F}$ be the set defined in Section 3, before the statement of Theorem 5.

Now, consider the subset $\mathcal{F}^*$ of $\mathcal{F}$ made up of those prime $k+1$-tuples $(p_0,p_1,\dots,p_k)$ with the additional condition $p_0,p_1,\dots,p_k \in (Y_1,Y_2)$.

It is clear that

$$S_1(x) = \sum_{(p_0,\dots,p_k)\in\mathcal{F}} \#\left\{m \le \frac{x}{p_0\cdots p_k} : \left(m, \frac{Q(p_0,p_k)}{p_1\cdots p_{k-1}}\right) = 1\right\} + o(x\log\log x), \tag{8.1}$$

where the error term accounts for the contribution of those $k+1$-tuples $(p_0,p_1,\dots,p_k) \in \mathcal{F}\setminus\mathcal{F}^*$, that is, those $k+1$-tuples $(p_0,p_1,\dots,p_k)\in\mathcal{F}$ for which $p_0 < Y_1$ or $p_k > Y_2$.

Given a fixed element $(p_0,p_1,\dots,p_k)\in\mathcal{F}$, using a standard sieve technique, one can establish that, as $x\to\infty$,

$$\#\left\{m \le \frac{x}{p_0\cdots p_k} : \left(m, \frac{Q(p_0,p_k)}{p_1\cdots p_{k-1}}\right) = 1\right\} \;=\; (1+o(1))\frac{x}{p_0\cdots p_k}\prod_{\substack{\pi\mid Q(p_0,p_k)\\ \pi\ne p_1,\dots,p_{k-1}}}\left(1-\frac{1}{\pi}\right)$$

$$(8.2) \qquad = (1+o(1))\frac{x}{p_0\cdots p_k}\frac{\log p_0}{\log p_k}.$$

On the one hand,

$$\sum_{(p_0,\ldots,p_k)\in\mathcal{F}} \frac{1}{p_0\cdots p_k}\frac{\log p_0}{\log p_k} = \sum_{p_1,\ldots,p_k}{}' \frac{1}{p_1\cdots p_k\log p_k} \sum_{p_0\in(p_1^{1-\delta_0},p_1)} \frac{\log p_0}{p_0}$$

$$(8.3) \qquad = \delta_0 \sum_{p_1,\ldots,p_k}{}' \frac{\log p_1}{p_1\cdots p_k\log p_k}\left(1+O\left(\frac{1}{\log p_1}\right)\right),$$

where the dash on the above sums indicates that the primes $p_1,\ldots,p_k$ are the last k coordinates of an element $(p_0,p_1,\ldots,p_k)\in\mathcal{F}$ and where we used Lemma 1 to obtain that

$$\sum_{p_0\in(p_1^{1-\delta_0},p_1)} \frac{\log p_0}{p_0} = \log p_1 - \log p_1^{1-\delta_0} + O\left(\frac{1}{\log p_1}\right) = \delta_0\log p_1 + O\left(\frac{1}{\log p_1}\right).$$

Repeating $k-2$ times the reasoning which led to (8.3), we obtain from (8.1), (8.2) and (8.3) that, as $x\to\infty$,

$$S_1(x) = \sum_{(p_0,\ldots,p_k)\in\mathcal{F}} \frac{1}{p_0\cdots p_k}\frac{\log p_0}{\log p_k} = (1+o(1))\delta_0\cdots\delta_{k-1}\sum_{Y_1<p_k<Y_2}\frac{\log p_k}{p_k\log p_k}$$

$$(8.4) \qquad = (H+o(1))\log\log x,$$

thereby completing the proof of (3.9).

To prove (3.10), we first observe that $V_{\delta_0,\ldots,\delta_{k-1}}(n)^2$ is equal to the number of prime $k+1$-tuples $(p_0,\ldots,p_k)$ and $(q_0,\ldots,q_k)$ that belong to $\mathcal{F}$ and satisfy the conditions

$$p_0\cdots p_k \mid n,\ q_0\cdots q_k\mid n,\ \left(\frac{n}{p_0\cdots p_k}, Q(p_0,p_k)\right)=1,\ \left(\frac{n}{q_0\cdots q_k}, Q(q_0,q_k)\right)=1.$$

As we sum over $n\le x$, two possibilities may occur regarding the prime $k+1$-tuples $(p_0,\ldots,p_k)$ and $(q_0,\ldots,q_k)$:

(i) $p_0,\ldots,p_k,q_0,\ldots,q_k$ are disjoint primes,

(ii) $(p_0,\ldots,p_k)\cap(q_0,\ldots,q_k)\neq\emptyset$.

In case (i), the contribution of these $k+1$-tuples to the sum $S_2(x)$ is equal to $\sum_{n\le x} V_{\delta_0,\ldots,\delta_{k-1}}(n)$ which, we know from (3.9) is $O(x\log\log x)$.

In case (ii), if $p_0=q_0$, then clearly $p_j=q_j$ for each $j=1,\ldots,k$, which brings us back to case (i). We can therefore assume that $p_0\neq q_0$, say with $p_0<q_0$ (the case when $p_0>q_0$ can be handled in a similar manner). Since $\nu_{p_j}(n)=p_{j+1}$ for $j=0,\ldots,k-1$, the only possibility is that $p_k=q_0$, in which case the contribution to the sum $S_2(x)$ is

$$\ll \sum_{\substack{(p_0,\ldots,p_k),(q_0,\ldots,q_k)\in\mathcal{F}^*\\(p_0,\ldots,p_k)\cap(q_0,\ldots,q_k)\neq\emptyset}} \frac{x}{p_0\ldots p_k q_1\cdots q_k}\frac{\log p_0}{\log q_k} \ll x\log\log x.$$

The contribution to $S_2(x)$ of the remaining cases is therefore

$$(1+o(1))x \sum_{\substack{(p_0,\dots,p_k),(q_0,\dots,q_k)\in\mathcal{F}^* \\ (p_0,\dots,p_k)\cap(q_0,\dots,q_k)\neq\emptyset}} \frac{\log p_0}{p_0\cdots p_k \log p_k}\cdot\frac{\log q_0}{q_0\cdots q_k\log q_k}$$

$$= (1+o(1))x\left\{\sum_{(p_0,\dots,p_k)\in\mathcal{F}^*}\frac{\log p_0}{p_0\cdots p_k\log p_k}\right\}^2 + o(x(\log\log x)^2), \tag{8.5}$$

The last sum in (8.5) is precisely the sum we estimated in (8.4), allowing us to conclude that

$$S_2(x) = (1+o(1))H^2x(\log\log x)^2 \qquad (x\to\infty),$$

thus proving (3.10). Finally, (3.11) follows as a consequence of (3.10).

9 The proof of Theorem 6

We follow essentially the same approach as the one used in Theorem 5. For this reason, we will only provide a sketch of the proof.

We first fix an element $(p_0,\dots,p_k)\in\mathcal{F}^*$. Then, using standard sieve methods, we obtain that

$$\begin{aligned}
&\#\left\{P\le x : P+1\equiv 0 \pmod{p_0\cdots p_k},\ \left(\frac{P+1}{p_0\cdots p_k},\frac{Q(p_0,p_k)}{p_1\cdots p_{k-1}}\right)=1\right\}\\
&= \frac{\mathrm{li}(x)}{(p_0-1)\cdots(p_k-1)}\prod_{\substack{\pi\mid Q(p_0,p_k)\\ \pi\neq p_1,\dots,p_{k-1}}}\left(1-\frac{1}{\pi-1}\right)\left(1+O\left(\frac{1}{\log p_1}\right)\right)\\
&\quad +O(E_{p_0,\dots,p_k}),
\end{aligned} \tag{9.1}$$

where for the error term, we have

$$E_{p_0,\dots,p_k} = \sum_{\substack{\delta\mid Q(p_0,p_k)\\ \omega(\delta)\le h}} K(x\mid \delta p_0\cdots p_k),$$

where as previously, we chose $h=\lfloor\log\log\log x\rfloor$.

Adding those error terms for each $(p_0,\dots,p_k)\in\mathcal{F}^*$, we get, using Theorem B,

$$\begin{aligned}
\sum_{(p_0,\dots,p_k)\in\mathcal{F}^*} E_{p_0,\dots,p_k} &\le \sum_{(p_0,\dots,p_k)\in\mathcal{F}^*} K(x\mid m)\#\{m=\delta p_0\cdots p_k\}\\
&\le \sum_{(p_0,\dots,p_k)\in\mathcal{F}^*} K(x\mid m)\omega(m)^{k+1}\\
&< \sum_{(p_0,\dots,p_k)\in\mathcal{F}^*} K(x\mid m)(\log x)^{k+1}\\
&\le (\log x)^{k+1}\frac{x}{\log^A x} \ll \frac{x}{\log^2 x},
\end{aligned} \tag{9.2}$$

provided we choose A sufficiently large. Now proceeding as in the proof of Theorem 2, we easily obtain that summing the main term on the right-hand side of (9.1) over all those $k+1$-tuples $(p_0,\ldots,p_k)\in\mathcal{F}^*$, we obtain, as $x\to\infty$, the main expression $(1+o(1))H\mathrm{li}(x)\log\log x$. Gathering this result with (9.1) snd (9.2), formula (3.12) follows.

We now move to prove formula (3.13). For this, first let $(p_0,\ldots,p_k)$ and $(q_0,\ldots,q_k)$ be two $k+1$-tuples in $\mathcal{F}^*$. Our goal is to count the number $N(x)$ of those primes $p\le x$ for which $p+1\equiv 0 \pmod{\mathrm{GCD}(p_0,\ldots,p_k,q_0,\ldots,q_k)}$. The contribution to $N(x)$ of the cases when $(p_0,\ldots,p_k)\cap(q_0,\ldots,q_k)\ne\emptyset$ is clearly $o(\mathrm{li}(x)(\log\log x)^2)$ as $x\to\infty$. For the remaining cases, we either have $p_k<q_0$ or $q_k<p_0$. Thus, given two fixed $k+1$-tuples $(p_0,\ldots,p_k)$ and $(q_0,\ldots,q_k)$ with $p_k<q_0$, we need to estimate

$$\mathcal{H}(x):=\#\left\{P\le x: P+1\equiv 0 \pmod{p_0\cdots p_k q_0\cdots q_k},\ \left(\frac{P+1}{p_0\cdots p_k q_0\cdots q_k}, Q(p_0,p_k)Q(q_0,q_k)\right)=1\right\}.$$

Using the prime number theorem for arithmetic progressions, we obtain that, as $x\to\infty$,

$$\begin{aligned}\mathcal{H}(x) &= (1+o(1))\frac{\mathrm{li}(x)}{(p_0-1)\cdots(p_k-1)(q_0-1)\cdots(q_k-1)}\\ &\quad\times\prod_{\substack{\pi\mid Q(p_0,p_k)\\ \pi\nmid p_1\cdots p_{k-1}}}\left(1-\frac{1}{\pi-1}\right)\prod_{\substack{\pi\mid Q(q_0,q_k)\\ \pi\nmid q_1\cdots q_{k-1}}}\left(1-\frac{1}{\pi-1}\right)\\ &\quad+E_{p_0,\ldots,p_k,q_0,\ldots,q_k},\end{aligned}\tag{9.3}$$

where the last term accounts for the error term. Proceeding as in the proof of Theorem 4, we find that

$$E_{p_0,\ldots,p_k,q_0,\ldots,q_k}=\sum_{\substack{\delta_1\mid Q(p_0,p_k),\ \delta_2\mid Q(q_0,q_k)\\ \omega(\delta_1)\le h,\ \omega(\delta_2)\le h}} K(x\mid p_0\cdots p_k q_0\cdots q_k).\tag{9.4}$$

As before, using Theorem B, we easily establish from (9.4) that, for any large number A,

$$\sum_{(p_0,\ldots,p_k),(q_0,\ldots,q_k)\in\mathcal{F}^*} E_{p_0,\ldots,p_k,q_0,\ldots,q_k}\ll\frac{x}{\log^A x}.\tag{9.5}$$

Finally, if $\widetilde{\mathcal{H}(x)}$ stands for the main term on the right-hand side of (9.3), we find that

$$\sum_{(p_0,\ldots,p_k),(q_0,\ldots,q_k)\in\mathcal{F}^*}\widetilde{\mathcal{H}(x)}=(H+o(1))\mathrm{li}(x)(\log\log x)^2.\tag{9.6}$$

Gathering estimates (9.3), (9.5) and (9.6), we obtain (3.13), from which as before (3.14) follows immediately.

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