Digit patterns in real numbers created from permutations

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Abstract

Given a positive integer k, we construct a binary number $0.a_1a_2a_3\ldots$ having the property that any sequence $a_{m+1}\ldots a_{m+k}$ of k consecutive digits from its binary expansion appears with a frequency directly related to the various permutations of the set $\{1,2,\ldots,k+1\}$.

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1 Introduction

Given a positive integer k, let Π_k be the set of the permutations of the set $\{1,2,\ldots,k+1\}$. Various interesting aspects of this set Π_k can be studied; see for instance the book of Pemmaraju [1]. Here, we use this set to construct real numbers with an interesting property, as follows. Given $\pi \in \Pi_k$, let $j_1, j_2, \ldots, j_{k+1}$ be defined by $\pi(i) = j_i$. Further set, for each $h = 1, 2, \ldots, k$,

$$\rho(j_h, j_{h+1}) = \begin{cases} 1 & \text{if} \quad j_{h+1} > j_h, \\ 0 & \text{if} \quad j_{h+1} < j_h. \end{cases}$$

Moreover, given $(\delta_1, \delta_2, \ldots, \delta_k) \in \{0,1\}^k$, set

$$D(\delta_1, \delta_2, \ldots, \delta_k) := \#\{\pi \in \Pi_k : \rho(\pi(i), \pi(i+1)) = \delta_i \text{ for } i = 1, 2, \ldots, k\}$$

and

$$\kappa(\delta_1, \delta_2, \ldots, \delta_k) := \frac{D(\delta_1, \delta_2, \ldots, \delta_k)}{(k+1)!}.$$

As we will see in Section 4,

$$\kappa(\delta_1, \delta_2, \ldots, \delta_k) \geq \frac{1}{(k+1)!} \qquad (k = 1, 2, \ldots).$$

To illustrate the function $\kappa(\delta_1, \delta_2, \ldots, \delta_k)$, if we choose the case $k = 4$, we obtain the following table.

$(\delta_1,\delta_2,\delta_3,\delta_4)$	$D(\delta_1,\delta_2,\delta_3,\delta_4)$	$\kappa(\delta_1,\delta_2,\delta_3,\delta_4)$
(0,0,0,0)	1	1/120
(0,0,0,1)	4	1/30
(0,0,1,0)	9	3/40
(0,0,1,1)	6	1/20
(0,1,0,0)	9	3/40
(0,1,0,1)	16	2/15
(0,1,1,0)	11	11/120
(0,1,1,1)	4	1/30
(1,0,0,0)	4	1/30
(1,0,0,1)	11	11/120
(1,0,1,0)	16	2/15
(1,0,1,1)	9	3/40
(1,1,0,0)	6	1/20
(1,1,0,1)	9	3/40
(1,1,1,0)	4	1/30
(1,1,1,1)	1	1/120

Our purpose in this short paper is to construct some binary number

$$\alpha = 0.a_1a_2a_3\ldots,$$

that is, where each digit $a_i \in \{0,1\}$, and such that

$$\lim_{N\to\infty} \frac{1}{N}\#\{m \le N : a_{m+1}\ldots a_{m+k} = \delta_1\ldots\delta_k\} = \kappa(\delta_1,\ldots,\delta_k).$$

To construct α, we proceed as follows. First we set

$$\mathcal{F}_N = [e^N, e^{N+1}) \quad \text{and} \quad \mathcal{L}_N = [\log N, N] \qquad (N = 1,2,\ldots).$$

Let $p(n) = p_N(n)$ stand for the smallest prime divisor of n which is located in the interval $\mathcal{L}_N$. Observe that the number of those $n \in \mathcal{F}_N$ which do not contain any prime divisors in $\mathcal{L}_N$ is bounded by

$$ce^N \prod_{\substack{p\in\mathcal{L}_N\\ p\in\wp}} \left(1 - \frac{1}{p}\right) \le ce^N \frac{\log\log N}{\log N}.$$

To each number $n \in \mathcal{F}_N$, we associate the number

$$\epsilon_n = \begin{cases} 1 & \text{if } p(n+1) > p(n) \text{ and } n+1 \in \mathcal{F}_N, \\ 0 & \text{otherwise} \end{cases}$$

for some absolute constant $c > 0$, where $\wp$ stands for the set of all primes. Thus, $\epsilon_n = 0$ if $p(n+1) < p(n)$ or if $n < e^{N+1} < n+1$ or if either $p(n)$ or $p(n+1)$ does not exist. Then, to each $N \in \mathbb{N}$, we associate the number

$$\xi_N = \text{Concat}(\epsilon_n : n \in \mathcal{F}_N),$$

and we then define

$$\alpha = 0.\xi_2\xi_3\xi_4\ldots \tag{1.1}$$

2 The distribution function of $(\{2^n\alpha\})_{n\ge1}$

With α as in (1.1), let $0<u<1$ written as

$$u=\frac{t_1}{2}+\frac{t_2}{2^2}+\frac{t_3}{2^3}+\cdots \tag{2.1}$$

Here, we may assume that $t_n=0$ for infinitely many $n\in\mathbb{N}$. We can prove that

$$\lim_{N\to\infty}\frac{1}{N}\#\{n\le N:\{2^n\alpha\}\le u\}=F(u)\quad\text{exists.} \tag{2.2}$$

To see this, we proceed as follows. Let $r_1<r_2<\cdots$ be a sequence of integers such that $t_{r_j}=0$ for some $j\in\mathbb{N}$ and then set $u_j:=\sum_{\nu=1}^{r_{j-1}}\frac{t_\nu}{2^\nu}$ and further define $\widetilde{u_j}:=\frac{1}{2^{r_j}}+u_j$. It is clear that

$$u_j\le u<\widetilde{u_j}\qquad(j\in\mathbb{N}).$$

We then introduce the two functions

$$\begin{aligned}F_1(u)&=\liminf_{N\to\infty}\frac{1}{N}\#\{n\le N:\{2^n\alpha\}<u\},\\ F_2(u)&=\limsup_{N\to\infty}\frac{1}{N}\#\{n\le N:\{2^n\alpha\}<u\}.\end{aligned}$$

With these definitions, we easily see that

$$F(u_j)=\sum_{\frac{a_1}{2}+\frac{a_2}{2^2}+\cdots+\frac{a_{r_j}}{2^{r_j}}\le u_j}\kappa(a_1,\ldots,a_{r_j})\le F_1(u), \tag{2.3}$$

$$F(\widetilde{u_j})=\sum_{\frac{a_1}{2}+\frac{a_2}{2^2}+\cdots+\frac{a_{r_j}}{2^{r_j}}\le\widetilde{u_j}}\kappa(a_1,\ldots,a_{r_j})\ge F_2(u). \tag{2.4}$$

Moreover,

$$F(\widetilde{u_j})-F(u_j)=\kappa(t_1,\ldots,t_{r_{j-1}},1). \tag{2.5}$$

Also, observe that it will follow from Theorem 4.1 below that

$$\lim_{m\to\infty}\max_{\delta_1,\ldots,\delta_m\in\{0,1\}^m}\kappa(\delta_1,\ldots,\delta_m)=0. \tag{2.6}$$

It then follows from (2.3), (2.4), (2.5) and (2.6) that $F_1(u)=F_2(u)$ and therefore that $F(u)$ exists, as claimed.

We can even prove that $F(u)$ is a continuous function. To show this, we first fix u and choose two sequences of numbers $(u_M)_{M\ge1}$ and $(v_M)_{M\ge1}$ such that $u_M<u<v_M$ for each $M\ge1$, and such that $u_M\to u$ and $v_M\to u$ as $M\to\infty$. Then, let s be an integer such that $\lfloor u\cdot 2^M\rfloor=s$ and choose $u_M=\frac{s}{2^M}$ and $v_M=\frac{s+1}{2^M}$. We then have

$$F(u_M)=\sum_{\frac{a_1}{2}+\cdots+\frac{a_M}{2^M}\le s/2^M}\kappa(a_1,\ldots,a_M),$$

$$F(v_M) \quad = \sum_{\frac{a_1}{2}+\cdots+\frac{a_M}{2^M}\le (s+1)/2^M} \kappa(a_1,\ldots,a_M),$$

with

$$\frac{s+1}{2^M} = \frac{b_1}{2} + \cdots + \frac{b_M}{2^M}.$$

Since $F(v_M)-F(u_M) = \kappa(b_1,\ldots,b_M) \to 0$ as $M \to \infty$, we therefore have that $\lim_{M\to\infty} F(u_M) = F(u)$ and $\lim_{M\to\infty} F(v_M) = F(u)$. Thus we proved that $F(u)$ is continuous at the points $u \in \mathbb{R}\setminus\mathbb{Q}$ and also continuous from the right at those points $u \in \mathbb{Q}$ of the form $u = s/(2^R)$, where s is an odd integer. Moreover, assuming that K is an integer larger than R and setting $u_K = u - 1/2^K$, we have $F(u) - F(u_K) = \kappa(t_1, t_2, \ldots, t_{R-1}, 0, 1, 1, \ldots, 1)$ which tends to 0 as K tends to infinity, thereby establishing that $F(u)$ is continuous from the left, as well.

3 Main theorem

Theorem 3.1. *Given an integer $k \ge 2$ and an arbitrary k-tuple $(\delta_1,\ldots,\delta_k) \in \{0,1\}^k$, we have*

$$\lim_{M\to\infty} \frac{1}{M} \# \left\{ m \le M : \{2^m\alpha\} \in \left[\frac{\delta_1}{2} + \cdots + \frac{\delta_k}{2^k}, \frac{\delta_1}{2} + \cdots + \frac{\delta_k}{2^k} + \frac{1}{2^k} \right) \right\} = \kappa(\delta_1,\ldots,\delta_k).$$

Proof. Let $p_1, p_2, \ldots, p_{k+1}$ be distinct primes located in the interval $\mathcal{L}_N$. Let us count those $n \in \mathcal{F}_N$ for which $p(n+j) = p_j$. Also, let $\{i_1,\ldots,i_{k+1}\}$ be that particular permutation of the set $\{1,2,\ldots,k+1\}$ for which $p_{i_1} < p_{i_2} < \cdots < p_{i_{k+1}}$. Then, set

$$Q_U := \prod_{\substack{\log N < p < U \\ p \in \wp}} p.$$

Since $n + j \equiv 0 \pmod{p_j}$ for $j = 1, 2, \ldots, k+1$, it follows that $n = m \cdot p_1 p_2 \cdots p_{k+1} + r$ with $(r, p_1 p_2 \cdots p_{k+1}) = 1$. Moreover, $(n+i_1, Q_{p_{i_1}}) = 1$, $(n+i_2, Q_{p_{i_2}}) = 1$, …, $(n+i_{k+1}, Q_{p_{i_{k+1}}}) = 1$. Using standard asymptotic sieve techniques, we can write these conditions in the form

$$\prod_{\ell=1}^{k+1} (n+i_\ell, Q_{p_{i_1}}) = 1,\ \prod_{\ell=2}^{k+1} \left(n+i_\ell, \frac{Q_{p_{i_2}}}{Q_{p_{i_1}}} \right) = 1, \ldots, \left(n+i_{k+1}, \frac{Q_{p_{i_{k+1}}}}{Q_{p_{i_k}}} \right) = 1,$$
$$n \equiv r \pmod{p_1 p_2 \cdots p_{k+1}}.$$

Thus the number of such numbers $n \in \mathcal{F}_N$ is, as $N \to \infty$,

$$(1+o(1)) \frac{\#\mathcal{F}_N}{p_1 p_2 \cdots p_{k+1}} \cdot \prod_{\substack{\log N < q < p_{i_1} \\ q \in \wp}} \left(1 - \frac{k+1}{q}\right) \cdot \prod_{\substack{p_{i_1} < q < p_{i_2} \\ q \in \wp}} \left(1 - \frac{k}{q}\right) \cdots \prod_{\substack{p_{i_k} < q < p_{i_{k+1}} \\ q \in \wp}} \left(1 - \frac{1}{q}\right)$$
$$= (1+o(1)) \#\mathcal{F}_N \cdot \log\log\log N \cdot \prod_{i=1}^{k+1} \frac{1}{p_i \log\log p_i}.$$

The important observation here is that this asymptotic behavior does not depend on the particular permutation of the primes $p_1, p_2, \ldots, p_{k+1}$ we choose. We may therefore conclude that, as $N \to \infty$,

$$\frac{1}{\#\mathcal{F}_N}\#\left\{m \in \mathcal{F}_N : \{2^m\alpha\} \in \left[\frac{\delta_1}{2}+\cdots+\frac{\delta_k}{2^k}, \frac{\delta_1}{2}+\cdots+\frac{\delta_k}{2^k}+\frac{1}{2^k}\right)\right\} = (1+o(1))\kappa(\delta_1,\ldots,\delta_k).$$

Now, we need to count those $\{2^m\alpha\}$ ($m = 1, 2, \ldots, \lfloor x\rfloor$) not only for the particular values $x = e^N$, but also for the more general values $x \in (e^N, e^{N+1})$.

So, let $\varepsilon > 0$ be an arbitrarily small number and set $x = e^{N+\theta}$ with $0 < \theta < 1$. We now examine two separate cases. If $\theta < \varepsilon$, then

$$\#\{n : e^N \le n < x\} < e^N(e^\varepsilon - 1) < 2\varepsilon e^N.$$

On the other hand, if $\theta > \varepsilon$, setting $S := [e^N, e^{N+\theta})$, we may then repeat the above argument for the interval S instead of $\mathcal{F}_N$ and obtain the same result. Therefore, in both cases, the proof is complete. □

4 The size of $\kappa(\delta_1, \ldots, \delta_k)$

Theorem 4.1. *Let $k \ge 2$ and let $a_1, a_2, \ldots, a_k \in \{0, 1\}$ be given. Then,*

$$\frac{1}{(k+1)!} \le \kappa(a_1, a_2, \ldots, a_k) < \frac{1}{2^{\lfloor k/2\rfloor}}.$$

Proof. First, we prove the first inequality, namely

$$\kappa(a_1, a_2, \ldots, a_k) \ge \frac{1}{(k+1)!}. \tag{4.1}$$

To do so, we let $j_1, \ldots, j_r$ be the indices of those a_i's for which $a_{j_\nu} = 0$ ($\nu = 1, \ldots, r$) and let $t_1, \ldots, t_s$ be the indices of those a_i's for which $a_{t_\mu} = 1$ ($\mu = 1, \ldots, s$). The case where one of the two sets $\{j_1, \ldots, j_r\}$ or $\{t_1, \ldots, t_s\}$ is empty is much more simple. So, let $S = \{1, \ldots, r\}$ and $M = \{r+1, \ldots, k+1\}$. Now, let $\{u(1), \ldots, u(k+1)\}$ be a permutation of $\{1, 2, \ldots, k+1\}$ satisfying

1. $\{u(j_1+1), u(j_2+1), \ldots, u(j_r+1)\}$ is a permutation of S satisfying the condition

   $$\text{If } j_{\ell+1} = j_\ell + 1 \text{ for some } \ell \in \{1, \ldots, r\}, \quad \text{then} \quad u(j_\ell+1) > u(j_{\ell+1}+1).$$

   Observe that such a permutation clearly exists.

2. $\{u(t_1+1), u(t_2+1), \ldots, u(t_{k-r}+1)\}$ is a permutation of M satisfying the condition

   $$\text{If } t_{\nu+1} = t_\nu + 1 \text{ for some } \nu \in \{1, \ldots, s\}, \quad \text{then} \quad u(t_\nu+1) < u(t_{\nu+1}+1).$$

   Such a permutation also clearly exists.

For such a permutation $\{u(1), \ldots, u(k+1)\}$, we have that $\rho(u(\ell), u(\ell+1)) = a_\ell$ for $\ell = 1, \ldots, k$. The special case $S = \emptyset$ is very simple, because in this case, $u(j) = j$ for each $j = 1, \ldots, k+1$. On the other hand if $M = \emptyset$, then

$$u(1) = k+1, \; u(2) = k, \; \ldots, \; u(k+1) = 1.$$

This completes the proof of (4.1).

We will now prove the second inequality in Theorem 4.1, namely

$$\kappa(a_1, a_2, \ldots, a_k) < \frac{1}{2^{\lfloor k/2 \rfloor}}. \tag{4.2}$$

Assume that $\{j_1, j_2, \ldots, j_{k+1}\}$ is a permutation of $\{1, 2, \ldots, k+1\}$ satisfying $\rho(j_\ell, j_{\ell+1}) = a_\ell$ for $\ell = 1, \ldots, k$. Assume first that $k+1$ is even and consider the pairs

$$(j_1, j_2), \quad (j_3, j_4), \quad \ldots, \quad (j_k, j_{k+1}).$$

If $a_1 = 1$, then $j_1 < j_2$; if $a_1 = 0$, then $j_2 > j_1$; if $a_3 = 1$, then $j_3 < j_4$; if $a_3 = 0$, then $j_3 > j_4$, and so on, up to if $a_k = 1$, then $j_k < j_{k+1}$; if $a_k = 0$, then $j_{k+1} < j_k$.

To sum up, this means that the number of associated permutations is no larger than $\dfrac{(k+1)!}{2^{(k+1)/2}}$.

The case where $k+1$ is odd can be treated in a similar manner, since we then have that k is even, in which case we consider the $k/2+1$ numbers

$$(j_1, j_2), \quad (j_3, j_4), \quad \ldots, \quad (j_{k-1}, j_k), \quad j_{k+1},$$

which allows us in the end to conclude that the number of associated permutations is no larger than $\dfrac{(k+1)!}{2^{k/2}}$.

In both cases, we have proved (4.2) and the proof of Theorem 4.1 is complete.

□

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