New approaches in the construction of normal numbers

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Plan

1. Normal numbers created using the prime factorization of integers
2. Normal numbers created using the distribution of the digits of prime numbers
3. The concept of sharp normality
4. Open problems and conjectures

1. Normal numbers created using the prime factorization of integers

The concept of disjoint classification of primes

Let $q \geq 2$ be a fixed integer.

Let $\wp =$ the set of all primes.

Let $\wp_0, \wp_1, \ldots, \wp_{q-1}$ be disjoint sets of primes of respective densities δ_i such that $\sum_{i=0}^{q-1} \delta_i = 1$ and

$$\wp = \mathcal{R} \cup \wp_0 \cup \wp_1 \cup \cdots \cup \wp_{q-1},$$

where $\mathcal{R}$ is a given finite (perhaps empty) set of primes.

We call $\mathcal{R}, \wp_0, \wp_1, \ldots, \wp_{q-1}$ a *disjoint classification of primes*.

Example of a disjoint classification of primes

Example of a disjoint classification of primes :

$$\begin{aligned} \mathcal{R} &= \{2\}, \\ \wp_0 &= \{p : p \equiv 1 \pmod 4\}, \\ \wp_1 &= \{p : p \equiv 3 \pmod 4\} \end{aligned}$$

The general idea

Given an integer $n \geq 2$, we associate to each prime factor appearing in its prime factorization the index i of the set of primes $\wp_i$ to which this prime factor belongs, that is :

$$n = p_1^{a_1} \cdots p_r^{a_r} \quad \longmapsto \quad \ell_1 \ldots \ell_r,$$

where each ℓ_j is such that $p_j \in \wp_{\ell_j}$.

Definition

For each integer $q \geq 2$, let $\mathcal{A}_q := \{0, 1, \ldots, q-1\}$.

Given an integer $t \geq 1$, an expression of the form

$$i_1 i_2 \ldots i_t, \qquad \text{where each } i_j \in \mathcal{A}_q$$

is called a *word* of length t.

Let $\nu_\beta(\alpha)$ stand for the number of occurrences of β in α.

Let Λ stand for the empty word.

Theorem

Let $q \geq 2$ be an integer and $\mathcal{R}, \wp_0, \wp_1, \ldots, \wp_{q-1}$ a disjoint classification of primes. Assume that, for a certain constant $c \geq 5$,

$$\pi([u, u+v] \cap \wp_j) = \frac{1}{q}\pi([u, u+v]) + O\left(\frac{u}{\log^c u}\right)$$

uniformly for $2 \leq v \leq u$, $j = 0, 1, \ldots, q-1$, as $u \to \infty$. Further, let T be defined on $\mathbb{N}$ by $T(1) = \Lambda$ and for $n \geq 2$ by

$$T(n) = T(p_1^{a_1} \cdots p_r^{a_r}) = H(p_1) \ldots H(p_r),$$

where

$$H(p) = \begin{cases} j & \text{if } p \in \wp_j \text{ for some } j \in \mathcal{A}_q, \\ \Lambda & \text{if } p \in \mathcal{R}. \end{cases}$$

Then, the number $0.T(1)T(2)T(3)T(4)\ldots$ is a q-normal number.

Application

Let $q = 2$,

$$\mathcal{R} = \{2\}, \quad \wp_0 = \{p : p \equiv 1 \pmod 4\}, \quad \wp_1 = \{p : p \equiv 3 \pmod 4\}$$

Then,

$$\{T(1), T(2), \ldots, T(15)\} = \{\Lambda, \Lambda, 1, \Lambda, 0, 1, 1, \Lambda, 1, 0, 1, 1, 0, 1, 10\}$$

and

$$0.T(1)T(2)T(3)T(4)\ldots = 0.101110110110\ldots$$

is a binary normal number.

Main tool

(JMDK & IK, Acta Arith., 1995) Let $\mathcal{R}, \wp_0, \wp_1, \ldots, \wp_{q-1}$ be a disjoint classification of primes such that, for each $i = 0, 1, \ldots, q-1$,

$$\pi([u, u+v] \cap \wp_i) = \delta_i \pi([u, u+v]) + O\left(\frac{u}{\log^5 u}\right)$$

holds uniformly for $2 \le v \le u$, where $\delta_0, \delta_1, \ldots, \delta_{q-1}$ are positive constants such that $\sum_{i=0}^{q-1} \delta_i = 1$. Let $\lim_{x\to\infty} w_x = +\infty$, $w_x = O(\log\log\log x)$, $A \le \log\log x$ with $P(A) \le w_x$. Then, for $\sqrt{x} \le Y \le x$ and $1 \le k \le c \log\log x$, we have

$$\#\{n = An_1 \le Y : p(n_1) > w_x,\ \omega(n_1) = k,\ H(n_1) = i_1 \ldots i_k\}$$

$$= (1+o(1))\delta_{i_1} \cdots \delta_{i_k} \frac{Y}{A \log Y} \frac{(\log\log x)^{k-1}}{(k-1)!} \varphi_{w_x}\left(\frac{k-1}{\log\log x}\right) F\left(\frac{k-1}{\log\log x}\right),$$

where

$$\varphi_{w_x}(z) := \prod_{p \le w_x} \left(1 + \frac{z}{p}\right)^{-1} \text{ and } F(z) := \frac{1}{\Gamma(z+1)} \prod_p \left(1 + \frac{z}{p}\right)\left(1 - \frac{1}{p}\right)^z.$$

Further generalization

Let $\mathcal{R}, \wp_0, \ldots, \wp_{q-1}$ be disjoint sets of primes, whose union is $\wp$.

Assume that there exists $\delta \in (0,1)$ such that, for $i = 0, 1, \ldots, q-1$,

$$\pi([u, u+v] \cap \wp_i) = \delta \pi([u, u+v]) + O\left(\frac{u}{\log^5 u}\right)$$

holds uniformly for $2 \leq v \leq u$, and similarly

$$\pi([u, u+v] \cap \mathcal{R}) = (1 - q\delta)\pi([u, u+v]) + O\left(\frac{u}{\log^5 u}\right).$$

Let $H(p) = \begin{cases} \Lambda & \text{if } p \in \mathcal{R}, \\ \ell & \text{if } p \in \wp_\ell \text{ for some } \ell \in \mathcal{A}_q \end{cases}$

and let T be defined on $\mathbb{N}$ by $T(1) = \Lambda$ and for $n \geq 2$ by

$$T(n) = T(p_1^{a_1} \cdots p_r^{a_r}) = H(p_1) \ldots H(p_r).$$

Then, the number $\xi = 0.T(1)T(2)T(3)\ldots$ is q-normal.

Applications

1. Let $\wp_0 = \{p : p \equiv 1 \pmod 8\}$, $\wp_1 = \{p : p \equiv 7 \pmod 8\}$ and $\mathcal{R} = \{2\} \cup \{p : p \equiv 3, 5 \pmod 8\}$. Let

$$H(p) = \begin{cases} \Lambda & \text{if } p = 2 \text{ or } p \equiv 3, 5 \pmod 8, \\ 0 & \text{if } p \equiv 1 \pmod 8, \\ 1 & \text{if } p \equiv 7 \pmod 8) \end{cases}$$

and let T be defined by $T(1) = \Lambda$ and for $n \geq 2$ by

$$T(n) = T(p_1^{a_1} \cdots p_r^{a_r}) = H(p_1) \ldots H(p_r).$$

Then, $0.T(1)T(2)T(3)\ldots$ is a binary normal number.

Applications

2. Let $F(x) = e_k x^k + \cdots + e_1 x \in \mathbb{R}[x]$ be a polynomial with at least one irrational coefficient.

Let I_0 and I_1 be two disjoint intervals in $[0,1)$ of equal length.

Consider the three sets of primes

$$\begin{aligned} \wp_0 &= \{p : \{F(p)\} \in I_0\}, \\ \wp_1 &= \{p : \{F(p)\} \in I_1\}, \\ \mathcal{R} &= \wp \setminus (\wp_0 \cup \wp_1). \end{aligned}$$

Let $H(p) = \begin{cases} \Lambda & \text{if } p \in \mathcal{R}, \\ \ell & \text{if } p \in \wp_\ell \text{ for some } \ell \in \mathcal{A}_2 \end{cases}$
and let T be defined by $T(1) = \Lambda$ and for $n \geq 2$ by

$$T(n) = T(p_1^{a_1} \cdots p_r^{a_r}) = H(p_1) \ldots H(p_r).$$

Then, $0.T(1)T(2)T(3)\ldots$ is a binary normal number.

Using reduced residue classes

Fix an integer $D \geq 3$ and let $h_0, h_1, \ldots, h_{\varphi(D)-1}$ be those positive integers $< D$ which are relatively prime with D.
Let

$$H(p^a) = H(p) = \begin{cases} j & \text{if } p \equiv h_j \pmod{D}, \\ \Lambda & \text{if } p | D \end{cases}$$

and

$$T(n) = T(p_1^{a_1} \cdots p_r^{a_r}) = H(p_1) \ldots H(p_r).$$

Then, given a positive integer a with $(a, D) = 1$, the real number ξ whose $\varphi(D)$-ary expansion is given by

$$\xi = 0.T(2+a)T(3+a)T(5+a)\ldots T(p+a)\ldots$$

is $\varphi(D)$-normal.

Using the prime factors of k consecutive integers

Fix an integer $k \geq 2$ and set $E(n) := n(n+1)\cdots(n+k-1)$.
For each positive integer n, set $e(n) = \prod_{\substack{q^\beta \| E(n) \\ q \leq k-1}} q^\beta$.

Define the sequence ρ_n on the prime powers q^a of $E(n)$ by

$$\rho_n(q^a) = \rho_n(q) = \begin{cases} \Lambda & \text{if } q | e(n), \\ \ell & \text{if } q | n+\ell,\ \gcd(q, e(n)) = 1,\ 0 \leq \ell \leq k-1. \end{cases}$$

Write $E(n) = q_1^{a_1} q_2^{a_2} \cdots q_r^{a_r}$ and set

$$S(E(n)) = \rho_n(q_1)\rho_n(q_2)\ldots\rho_n(q_r).$$

Then, the number $0.S(E(1))S(E(2))\ldots S(E(n))\ldots$ is k-normal.

Similar results with shifted primes

Fix an integer $k \geq 2$ and set $E(n) := n(n+1)\cdots(n+k-1)$.
For each positive integer n, set $e(n) = \prod_{\substack{q^\beta \| E(n) \\ q \leq k-1}} q^\beta$.

Define the sequence ρ_n on the prime powers q^a of $E(n)$ by

$$\rho_n(q^a) = \rho_n(q) = \begin{cases} \Lambda & \text{if } q | e(n), \\ \ell & \text{if } q | n+\ell,\ \gcd(q, e(n)) = 1,\ 0 \leq \ell \leq k-1. \end{cases}$$

Write $E(n) = q_1^{a_1} q_2^{a_2} \cdots q_r^{a_r}$ and set

$$S(E(n)) = \rho_n(q_1)\rho_n(q_2)\ldots\rho_n(q_r).$$

Let $p_1 < p_2 < \cdots$ be the sequence of all primes.
Then, the number $0.S(E(p_1+1))S(E(p_2+1))\ldots$ is k-normal.

Using the relative size of the prime factors of an integer

Let $q \geq 2$ be a fixed integer. Given $n = p_1^{e_1} \cdots p_{k+1}^{e_{k+1}}$, let

$$c_j(n) := \left\lfloor \frac{q \log p_j}{\log p_{j+1}} \right\rfloor \in \mathcal{A}_q \qquad (j = 1, \ldots, k)$$

and consider the arithmetic function H defined by

$$H(n) = H(p_1^{e_1} \cdots p_{k+1}^{e_{k+1}}) = \begin{cases} c_1(n) \ldots c_k(n) & \text{if } \omega(n) \geq 2, \\ \Lambda & \text{if } \omega(n) \leq 1. \end{cases}$$

Then, the number $0.H(1)H(2)H(3)\ldots$ is q-normal.

Generalization

Let $q \geq 2$ be a fixed integer. Given $n = p_1^{e_1} \cdots p_{k+1}^{e_{k+1}}$, let

$$c_j(n) := \left\lfloor \frac{q \log p_j}{\log p_{j+1}} \right\rfloor \in \mathcal{A}_q \qquad (j = 1, \ldots, k)$$

and consider the arithmetic function H defined by

$$H(n) = H(p_1^{e_1} \cdots p_{k+1}^{e_{k+1}}) = \begin{cases} c_1(n) \ldots c_k(n) & \text{if } \omega(n) \geq 2, \\ \Lambda & \text{if } \omega(n) \leq 1. \end{cases}$$

Let $F \in \mathbb{Z}[x]$ be of positive degree and such that $F(m) > 0$ for all $m \geq m_0$ and set $\xi = 0.H(F(m_0))H(F(m_0+1))H(F(m_0+2))\ldots$
Also, let $m_0 \leq p_1 < p_2 < \cdots$ be the sequence of all primes no smaller than m_0 and set $\eta = 0.H(F(p_1))H(F(p_2))H(F(p_3))\ldots$
Then, the numbers ξ and η are q-normal.

Concatenating the prime factors of an integer

Let $q \geq 2$ be a fixed integer.

Let $S(x) \in \mathbb{Z}[x]$ of positive degree r_0 such that $S(n) > 0$ for all integers $n \geq 1$.

Given an integer $n \geq 2$, write its prime factorization as $n = p_1 p_2 \cdots p_r$, where $p_1 \leq p_2 \leq \cdots \leq p_r$ are all its prime factors.

Set $\ell(n) := \overline{S(p_1)}\,\overline{S(p_2)}\,\ldots\,\overline{S(p_r)}$, where each $S(p_i)$ is expressed in base q. For convenience, we set $\ell(1) = \Lambda$.

Then, the real number

$$\xi := 0.\ell(1)\ell(2)\ell(3)\ell(4)\ldots$$

is q-normal.

The typical approach

Set $\xi^{(x)} := \ell(1)\ell(2)\ell(3)\ldots\ell(\lfloor x \rfloor)$

- $\lambda(\xi^{(x)}) = \sum_{n \le x} \left(\left\lfloor \frac{\log \ell(n)}{\log q} \right\rfloor + 1 \right) \asymp x \log x$
- Given $\beta, \gamma \in \mathcal{A}_q^k$, we prove $\nu_\beta(\xi^{(x)}) - \nu_\gamma(\xi^{(x)}) = o(x \log x)$
- Observing that $\sum_{\gamma \in \mathcal{A}_q^k} \nu_\gamma(\xi^{(x)}) = \lambda(\xi^{(x)}) - k + 1$, we obtain

$$\begin{aligned}
\nu_\beta(\xi^{(x)}) - \frac{\lambda(\xi^{(x)})}{q^k} &= \frac{\nu_\beta(\xi^{(x)})q^k - \sum_{\gamma \in \mathcal{A}_q^k} \nu_\gamma(\xi^{(x)}) + O(1)}{q^k} \\
&= \frac{1}{q^k} \sum_{\gamma \in \mathcal{A}_q^k} \left(\nu_\beta(\xi^{(x)}) - \nu_\gamma(\xi^{(x)}) \right) + O(1) \\
&= \frac{1}{q^k} \cdot q^k \cdot o(\lambda(\xi^{(x)})) = o(\lambda(\xi^{(x)})).
\end{aligned}$$

2. Normal numbers created using the distribution of the digits of primes

Question raised by Igor Shparlinski

(Shparlinski, 2010) Let $P(n)$ stand for the largest prime factor of $n \geq 2$. Is the number

$$0.P(2)\,P(3)\,P(4)\,\ldots$$

a normal number ?

Notation

Given a positive integer n, write its q-ary expansion as

$$n = \varepsilon_0(n) + \varepsilon_1(n)q + \cdots + \varepsilon_t(n)q^t,$$

where each $\varepsilon_i(n) \in \mathcal{A}_q$ and $\varepsilon_t(n) \neq 0$. Then write

$$\begin{aligned} \overline{n} &= \varepsilon_0(n)\varepsilon_1(n)\ldots\varepsilon_t(n) \\ \overline{\overline{n}} &= \varepsilon_t(n)\varepsilon_{t-1}(n)\ldots\varepsilon_0(n) \end{aligned}$$

Theorem

Let $F \in \mathbb{Z}[x]$ be a polynomial of positive degree such that $F(x) > 0$ if $x > 0$. Then the numbers

$$0.\overline{F(P(2))}\,\overline{F(P(3))}\ldots\overline{F(P(n))}\ldots$$

and

$$0.\overline{\overline{F(P(2))}}\,\overline{\overline{F(P(3))}}\ldots\overline{\overline{F(P(n))}}\ldots$$

are q-normal.

Main tool used in the proof that $0.F(P(2))F(P(3))F(P(4))\ldots$ is q-normal

(Bassily-Kátai, 1996) Fix an integer $q \geq 2$ and let $L(n) = \left\lfloor \frac{\log n}{\log q} \right\rfloor + 1$.
Let also $F \in \mathbb{Z}[x]$ be a polynomial of positive degree r such that $F(x) > 0$ for $x > 0$.
Assume that κ_u is a function of u such that $\kappa_u > 1$ for all $u > e^e$.
Let $\nu_\beta(\alpha)$ stand for the number of occurrences of β in α.
Then, given a word $\beta \in \mathcal{A}_q^k$, there exists a constant $c > 0$ such that

$$\#\left\{p \in [u, 2u] : \left|\nu_\beta(\overline{F(p)}) - \frac{L(u^r)}{q^k}\right| > \kappa_u \sqrt{L(u^r)}\right\} \leq \frac{cu}{(\log u)\kappa_u^2}.$$

Main tool for the proof that $0.F(P(2))F(P(3))F(P(4))\ldots$ is q-normal (Equivalent form)

(Bassily-Kátai, 1996) Fix an integer $q \geq 2$ and let $L(n) = \left\lfloor \frac{\log n}{\log q} \right\rfloor + 1$.
Let also $F \in \mathbb{Z}[x]$ be a polynomial of positive degree r such that $F(x) > 0$ for $x > 0$.
Assume that κ_u is a function of u such that $\kappa_u > 1$ for all $u > e^e$.
Let $\nu_\beta(\alpha)$ stand for the number of occurrences of β in α.
Then, given distinct words $\beta_1, \beta_2 \in \mathcal{A}_q^k$, there exists a constant $c > 0$ such that

$$\#\left\{p \in [u, 2u] : \left|\nu_{\beta_1}(\overline{F(p)}) - \nu_{\beta_2}(\overline{F(p)})\right| > \kappa_u \sqrt{L(u^r)}\right\} \leq \frac{cu}{(\log u)\kappa_u^2}.$$

Similar result using shifted primes

Let $F \in \mathbb{Z}[x]$ be a polynomial of positive degree such that $F(x) > 0$ if $x > 0$. Then the numbers

$$0.\overline{F(P(2+1))}\,\overline{F(P(3+1))}\ldots\overline{F(P(p+1))}\ldots$$

and

$$0.\overline{\overline{F(P(2+1))}}\,\overline{\overline{F(P(3+1))}}\ldots\overline{\overline{F(P(p+1))}}\ldots$$

are q-normal.

Additional tool used to prove that $0.P(2+1)\,P(3+1)\,P(5+1)\,\ldots\,P(p+1)\,\ldots$ is q-normal

There exists a constant $c > 0$ such that, given an arbitrary $\delta \in (0, 1/2)$, for all $x \geq 2$,

$$\#\{p \in [x, 2x] : P(p+1) \notin [x^{\delta}, x^{1-\delta}]\} < c\,\delta\,\pi(x).$$

Using the smallest prime factor

Fix an integer $q \geq 2$. Let $p(n)$ stand for the smallest prime factor of $n \geq 2$.

1. The number $0.p(2)p(3)p(4)\ldots$ is q-normal.

2. Let $F \in \mathbb{Z}[x]$ be a polynomial such that $F(x) > 0$ for all $x > 0$ and satisfying $\lim_{x\to\infty} F(x) = \infty$. Then, the number $0.F(p(2))\,F(p(3))\,F(p(4))\,\ldots$ is q-normal.

3. Let $a \geq 0$ be an even integer. Then, the number $0.p(2+a)\,p(3+a)\,p(5+a)\,\ldots\,p(\pi+a)\,\ldots$ is q-normal. (The case $a = 0$ was proved by Davenport and Erdős in 1952.)

4. Let $a \geq 0$ be an even integer and let F be as above. Then, the number $0.F(p(2+a))\,F(p(3+a))\,F(p(5+a))\,\ldots\,F(p(\pi+a))\,\ldots$ is q-normal.

Using the large prime factors of an integer

Notation :

A function $\eta(x)$ *is said to be slowly increasing if, for every real* $c > 0$*, we have*

$$\lim_{x\to\infty} \frac{\eta(cx)}{\eta(x)} = 1.$$

Examples :

$$\log x, \quad \log\log x, \quad e^{\sqrt{\log x}}$$

Using the large prime factors of an integer

Given a positive integer n, write its q-ary expansion as

$$n = \varepsilon_0(n) + \varepsilon_1(n)q + \cdots + \varepsilon_t(n)q^t,$$

where $\varepsilon_i(n) \in \mathcal{A}_q$ for $0 \le i \le t$ and $\varepsilon_t(n) \neq 0$. To this representation, associate the word

$$\overline{n} = \varepsilon_0(n)\varepsilon_1(n)\ldots\varepsilon_t(n) = \varepsilon_0\varepsilon_1\ldots\varepsilon_t \in \mathcal{A}_q^{t+1}.$$

Let $Q(n) = \begin{cases} \min\{p|n : p \ge \eta(n)\} & \text{if } P(n) \ge \eta(n), \\ 1 & \text{if } P(n) < \eta(n). \end{cases}$

Then, the number $0.\overline{Q(1)}\,\overline{Q(2)}\,\overline{Q(3)}\ldots$ is q-normal.

Further generalization

Let $\wp$ stand for the set of all primes. Given an integer $q \geq 2$, let $\mathcal{R}, \wp_0, \wp_1, \ldots, \wp_{q-1}$ be disjoint sets of prime numbers such that

$$\wp = \mathcal{R} \cup \wp_0 \cup \wp_1 \cup \cdots \cup \wp_{q-1},$$

and such that, uniformly for $2 \leq v \leq u$ as $u \to \infty$,

$$\pi([u, u+v] \cap \wp_j) = \frac{1}{q}\pi([u, u+v]) + O\left(\frac{u}{\log^5 u}\right) \qquad (j = 0, 1, \ldots, q-1),$$

so that, in particular,

$$\pi([u, u+v] \cap \mathcal{R}) = O\left(\frac{u}{\log^5 u}\right).$$

Theorem

Consider the function κ defined on $\wp$ by

$$\kappa(p) = \begin{cases} \ell & \text{if } p \in \wp_\ell, \\ \Lambda & \text{if } p \in \mathcal{R}. \end{cases}$$

Then,

(i) the number $0.\kappa(Q(1))\kappa(Q(2))\kappa(Q(3))\ldots$ is q-normal ;

(ii) letting $a > 0$ be a fixed integer, the number

$$0.\kappa(Q(2+a))\kappa(Q(3+a))\kappa(Q(5+a))\ldots\kappa(Q(p+a))\ldots,$$

where p runs through the set of primes, is q-normal.

Theorem

Define $\wp^*$ as the set of all the prime numbers $p \equiv 1 \pmod 4$. Then, let $\mathcal{R}^*, \wp_0^*, \wp_1^*, \ldots, \wp_{q-1}^*$ be disjoint sets of prime numbers such that

$$\wp^* = \mathcal{R}^* \cup \wp_0^* \cup \wp_1^* \cup \cdots \cup \wp_{q-1}^*,$$

and such that, uniformly for $2 \leq v \leq u$ as $u \to \infty$,

$$\pi([u, u+v] \cap \wp_j^*) = \frac{1}{q}\pi([u, u+v] \cap \wp^*) + O\left(\frac{u}{\log^5 u}\right)$$

$$(j = 0, 1, \ldots, q-1),$$

so that, in particular,

$$\pi([u, u+v] \cap \mathcal{R}^*) = O\left(\frac{u}{\log^5 u}\right).$$

Theorem

Consider the function defined on $\wp$ by

$$\nu(p) = \begin{cases} \ell & \text{if } p \in \wp_\ell^*, \\ \Lambda & \text{if } p \notin \bigcup_{\ell=0}^{q-1} \wp_\ell^*. \end{cases}$$

Then, the number $0.\nu(Q(1))\nu(Q(2))\nu(Q(3))\ldots$ is q-normal.

Theorem

Let $f(n) = n^2 + 1$. Consider the two numbers

$$\begin{aligned}\xi_1 &= 0.\kappa(Q(f(1)))\kappa(Q(f(2)))\kappa(Q(f(3)))\ldots,\\ \xi_2 &= 0.\kappa(Q(f(2)))\kappa(Q(f(3)))\kappa(Q(f(5)))\ldots\kappa(Q(f(p)))\ldots,\end{aligned}$$

where p runs through the set of primes.

Then, ξ_1 and ξ_2 are q-normal numbers.

The middle prime factor

$p_m(n)$:= the *middle prime factor* of n

De Koninck and Luca (2014) : As $x \to \infty$,

$$\sum_{n \le x} \frac{1}{p_m(n)} = \frac{x}{\log x} \exp\left((1 + o(1)) \sqrt{2 \log\log x \ \log\log\log x} \right)$$

The size of $\log p_m(n)$

Let $g(x)$ tend to infinity arbitrarily slowly as $x \to \infty$.

Then, as $x \to \infty$,

$$\frac{1}{x}\#\left\{n \le x : e^{-\sqrt{\log\log x}\, g(x)} \le \frac{\log p_m(n)}{\sqrt{\log x}} \le e^{\sqrt{\log\log x}\, g(x)}\right\} \to 1.$$

Moreover, as $x \to \infty$,

$$\frac{1}{x}\#\left\{n \le x : \left|\log\log p_m(n) - \frac{1}{2}\log\log x\right| \le \sqrt{\log\log x}\, g(x)\right\} \to 1.$$

Theorem

Let $q \geq 2$ be a fixed integer. Given an integer $n \geq 2$, let $\overline{p_m(n)}$ stand for the concatenation of the base q digits of $p_m(n)$.

Then, the real number

$$0.\overline{p_m(2)}\,\overline{p_m(3)}\,\overline{p_m(4)}\,\ldots$$

is q-normal.

The number of prime factors of shifted primes

Let $\omega(n) = \sum_{p|n} 1$.

The number $\xi := 0.\overline{\omega(2)}\,\overline{\omega(3)}\,\overline{\omega(4)}\,\overline{\omega(5)}\ldots$, where $\overline{\omega(n)}$ stands for the base q digits of $\omega(n)$, does not yield a normal number.

Indeed, since the interval $I := [e^{e^{r-1}}, e^{e^r}]$, where $r := \lfloor \log\log x \rfloor$, covers most of the interval $[1, x]$ and since $\left| \dfrac{\omega(n)}{r} - 1 \right| < \dfrac{1}{r^{1/4}}$, say, with the exception of a negligible number of integers $n \in I$, it follows that ξ cannot be normal in base q.

The number of prime factors of shifted primes

For each integer $n \geq 3$, let $a(n) := |\omega(n) - \lfloor \log\log n \rfloor|$.

Fix an integer $q \geq 2$.

Let $\overline{a(n)}$ stand for the concatenation of the base q digits of $a(n)$.

Then,

- $0.\overline{a(3)}\,\overline{a(4)}\,\overline{a(5)}\ldots$ is normal in base q.
- $0.\overline{a(2+1)}\,\overline{a(3+1)}\,\overline{a(5+1)}\ldots$ is normal in base q.

3. Sharp normality

Uniform distribution mod 1 and discrepancy

1. A sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers is said to be *uniformly distributed mod 1* if for every interval $[a,b)\subseteq[0,1)$,

$$\lim_{N\to\infty}\frac{1}{N}\#\{n\le N:\{x_n\}\in[a,b)\}=b-a.$$

2. Given a set of N real numbers $x_1,\ldots,x_N$, the *discrepancy* of this set is defined as the quantity

$$D(x_1,\ldots,x_N):=\sup_{[a,b)\subseteq[0,1)}\left|\frac{1}{N}\sum_{\substack{n\le N\\ \{x_n\}\in[a,b)}}1-(b-a)\right|.$$

Known results

1. **Theorem.** A sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers is uniformly distributed mod 1 if and only if $D(x_1,\ldots,x_N)\to 0$ as $N\to\infty$.

2. **Theorem.** Fix an integer $q\geq 2$. A real number α is normal in base q if and only if the sequence $(\{q^n\alpha\})_{n\in\mathbb{N}}$ is uniformly distributed mod 1.

Sharp uniform distribution mod 1

For each positive integer N, let

(∗)

$$M = M_N := \lfloor \delta_N \sqrt{N} \rfloor, \text{ where } \delta_N \to 0 \text{ and } \delta_N \log N \to \infty \text{ as } N \to \infty.$$

We shall say that an infinite sequence of real numbers $(x_n)_{n\geq 1}$ is *sharply uniformly distributed* mod 1 if

$$D(x_{N+1}, \ldots, x_{N+M}) \to 0 \quad \text{as} \quad N \to \infty$$

for every choice of δ_N satisfying (∗).

Definition of sharp normality

We say that an irrational number α is a *sharp normal number* in base q (or a *sharp q-normal number*) if the sequence $(x_n)_{n\in\mathbb{N}}$, where $x_n = \{q^n\alpha\}$, is sharply uniformly distributed mod 1.

Two remarks

1. $(x_n)_{n\in\mathbb{N}}$ sharply uniformly distributed mod 1

$$\implies \quad (x_n)_{n\in\mathbb{N}} \text{ is uniformly distributed mod 1}$$

2. α sharply normal $\quad\implies\quad$ α is normal

The Champernowne number is not sharply normal

The Champernowne number

$$0.1\,10\,11\,100\,101\,110\,111\,1000\,1001\,1010\,1011\,1100\,1101\,1110\,\ldots$$

is normal in base 2 but **not sharply normal** in base 2.

How to prove that : show that the sequence

$$2^{2n}+1, 2^{2n}+2, 2^{2n}+3, \ldots, 2^{2n}+S_n, \text{ where } S_n = \lfloor 2^n/(\sqrt{n}\log n)\rfloor$$

contains too many zeros (in binary), and in fact that out of the $(2n+1)S_n$ digits in the above sequence, roughly 75% of them are zeros, so that setting $M = M_N = (2n+1)S_n \approx \sqrt{N}/\log\log N$, we find that the discrepancy of the sequence of numbers $x_{N+1}, \ldots, x_{N+M}$ is $\geq 1/4$ and therefore does not tend to 0 as $N \to \infty$, thereby implying that the Champernowne number is not sharply normal.

A simple criteria for sharp normality

Consider a positive real number $\alpha < 1$ whose q-ary expansion is written as $\alpha = 0.\epsilon_1\epsilon_2\ldots$, where each $\epsilon_j \in \mathcal{A}_q := \{0, 1, \ldots, q-1\}$. For an arbitrary word $\beta = \delta_1 \ldots \delta_k \in \mathcal{A}_q^k$, let $R_{N,M}(\beta)$ stand for the number of times that the word β appears as a subword of the word $\epsilon_{N+1} \cdots \epsilon_{N+M}$.

Then, a positive real number $\alpha < 1$ is sharply q-normal if and only if, given an arbitrary word $\beta = \delta_1 \ldots \delta_k \in \mathcal{A}_q^k$ and $M = M_N$ as in $(*)$,

$$\lim_{N\to\infty} \frac{R_{N,M}(\beta)}{M} = \frac{1}{q^k}.$$

The construction of sharp normal numbers

Let $\alpha = 0.\epsilon_1\epsilon_2\dots$ be a normal number in base $q \geq 2$.

For each $T \in \mathbb{N}$, consider the corresponding word $\alpha_T = \epsilon_1\epsilon_2\dots\epsilon_T$.

Then, if $T_1 < T_2 < \cdots$ and $m_1 < m_2 < \cdots$ are chosen appropriately, the number

$$\beta = 0.\alpha_{T_1}^{m_1}\alpha_{T_2}^{m_2}\dots$$

is a sharp normal number in base q.
(Here $\gamma^m = \underbrace{\gamma\dots\gamma}_{m\ \text{times}}$, the concatenation of m times the word γ)

Theorem. Let α be a q-normal number. Then, the number $\beta = 0.\alpha_1^1\alpha_2^2\alpha_3^3\dots$ is a sharp normal number in base q.

Most numbers are sharply normal

Theorem.

Fix an integer $\mathbf{q} \geq 2$. *The Lebesgue measure of the set of all those real numbers* $\alpha \in [0, 1]$ *which are not sharply* $\mathbf{q}$*-normal is equal to 0.*

Variations on the definition of sharp normality

Instead of choosing $M_N = \lfloor \delta_N \sqrt{N} \rfloor$ in

$$M = M_N := \lfloor \delta_N \sqrt{N} \rfloor, \text{ where } \delta_N \to 0 \text{ and } \delta_N \log N \to \infty \text{ as } N \to \infty,$$

we could have set $M_N = \lfloor \delta_N N^\gamma \rfloor$ for a fixed $\gamma \in (0,1)$, and then introduce the corresponding concept of a γ-sharply uniformly distributed sequence mod 1, with corresponding γ-sharp normal numbers.

Lemma. If $0 < \gamma_1 < \gamma_2 < 1$, then any γ_1-sharp normal number is also a γ_2-sharp normal number.

Variations on the definition of sharp normality

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we could have chosen $M = \lfloor \log^2 N \rfloor$.

It is possible to show that the choice $M = \lfloor \log N \rfloor$ is not appropriate.

(Indeed, it is possible to exhibit a normal number α containing sequences of digits covering intervals of the form $[N+1, N+M]$, with $M \approx \log N$, and made up only of zeros)

4. Open problems and conjectures

Open problems

1. Let $\mathcal{M}$ be the semi-group generated by the integers 2 and 3. Let $m_1 < m_2 < \cdots$ be the list of all the elements of $\mathcal{M}$.
Is it possible to construct a real number α such that the sequence $(y_n)_{n\in\mathbb{N}}$, where $y_n = \{m_n\alpha\}$, is uniformly distributed in the interval $[0,1)$?

2. Is it possible to construct a real number β for which the corresponding sequence $(s_n)_{n\in\mathbb{N}}$, where $s_n = \{(\sqrt{2})^n\beta\}$, is uniformly distributed in the interval $[0,1)$?

Conjectures

3. Fix an integer $q \geq 3$ and let $1 = \ell_0 < \ell_1 < \cdots < \ell_{\varphi(q)-1}$ be the list of reduced residues modulo q.
Let $\wp_q = \{p \in \wp : p \nmid q\} = \{p_1, p_2, \ldots\}$.
For each $p \in \wp_q$, let $h(p) = \nu$ if $p \equiv \ell_\nu \pmod{q}$.
Let $\alpha = 0.h(p_1)\, h(p_2)\, h(p_3) \ldots$ ($\varphi(q)$-ary expansion).

- `Conjecture 1` : α is a $\varphi(q)$-ary normal number.
- `Conjecture 2` : α is a $\varphi(q)$-ary normal number with weight $1/n$, that is, for every positive integer k, given $e_1 \ldots e_k$, an arbitrary block of k digits in $\{0, 1, \ldots, \varphi(q)-1\}$, we have

$$\lim_{N\to\infty} \frac{1}{\log N} \sum_{\substack{n \leq N \\ h(p_{n+1})\ldots h(p_{n+k}) = e_1 \ldots e_k}} \frac{1}{n} = \frac{1}{\varphi(q)^k}.$$

- `Conjecture 3` : The sequence $(\{\varphi(q)^n \alpha\})_{n\in\mathbb{N}}$ is everywhere dense in the interval $[0, 1)$.

Conjectures

4. Fix an integer $q \geq 2$.
Consider the two sequences $(\varepsilon_n)_{n\in\mathbb{N}}$ and $(\delta_n)_{n\in\mathbb{N}}$ defined by

$$\varepsilon_n = \omega(n) \pmod q \quad \text{and} \quad \delta_n = \Omega(n) \pmod q.$$

Then, let

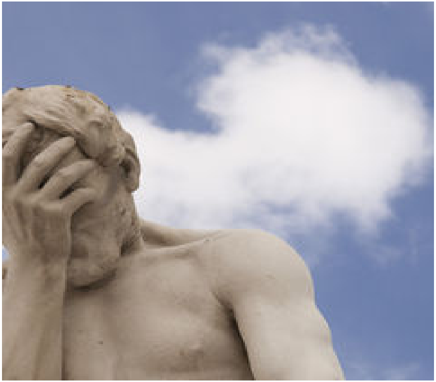
$$\alpha_q := 0.\varepsilon_1\varepsilon_2\ldots \quad \text{and} \quad \beta_q := 0.\delta_1\delta_2\ldots$$

Consider the sequence $(\kappa_n)_{n\in\mathbb{N}}$ defined by $\kappa_n = \Omega(p_n+1) \pmod q$, where p_n stands for the n-th prime, and let $\gamma_q := 0.\kappa_1\kappa_2\ldots$

`Conjecture:` The numbers α_q, β_q and γ_q are all q-normal numbers.

In the case $q = 2$, this is the Chowla conjecture.

Food for thought...



Thank you !

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