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On the index of friability

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Given an integer $n \geq 2$, let $P(n)$ stand for its largest prime factor, setting $P(1) = 1$. We introduce the arithmetic function $\mathrm{fria}(n) := \log n / \log P(n)$ (with $\mathrm{fria}(1) = 1$) and call it the *index of friability* of the integer n. The index of friability of an integer is an absolute measure of its friability (or smoothness). We first determine the respective mean values of the functions $\mathrm{fria}(n)$ and $1/\mathrm{fria}(n)$, thereafter obtaining various estimates comparing the index of friability with the index of composition. Then, given any finite set of natural numbers, we order its members according to their index of friability and obtain results regarding their distribution. In particular, this allows us to construct arbitrarily long monotonic sequences of integers with increasing index of friability.

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1. Introduction and basic properties

Given an integer $n \geq 2$, let $P(n)$ stand for its largest prime factor and $\gamma(n) := \prod_{p|n} p$. We set $P(1) = \gamma(1) = 1$. In 2001, De Koninck and Doyon [3] introduced the function $\lambda(n) := \log n / \log \gamma(n)$ and called it the *index of composition* of n, setting for convenience $\lambda(1) = 1$. The function $\lambda(n)$ was further examined by many, including De Koninck & Kátai [5], Zhai [20], De Koninck, Kátai & Subbarao [6], De Koninck & Luca [8] and Zhang & Zhai [21], to name only a few.

We now introduce a function which measures the “friability” (or the “smoothness”[a]) of a positive integer. Given an integer $n \geq 2$, we let $\mathrm{fria}(n) := \log n / \log P(n)$ and call it the *index of friability* of the integer n. For convenience, we set $\mathrm{fria}(1) = 1$.

[a]In his recent book [17], Gérald Tenenbaum explains at length why the word “friable” is more appropriate than the word “smooth” to describe a number without large prime factors.

Here, we first determine the respective mean values of the functions $\mathrm{fria}(n)$ and $1/\mathrm{fria}(n)$, thereafter obtaining various estimates comparing the index of friability with the index of composition. Then, given any finite set of natural numbers, we order its members according to their index of friability and obtain results regarding their distribution. In particular, this allows us to construct arbitrarily long monotonic sequences of integers with increasing index of friability.

In what follows, we let $p_1, p_2, \ldots$ stand for the sequence of prime numbers and let $\pi(x)$ stand for the number of primes not exceeding x. Given an integer $n \geq 2$, we let $\omega(n)$ stand for the number of distinct prime factors of n and set $\Omega(n) := \sum_{p^\alpha \| n} \alpha$ with $\omega(1) = \Omega(1) = 0$.

The index of friability has the following basic properties:

(i) $\mathrm{fria}(n)$ is an integer if and only if n is a prime power.
(ii) If $z = \mathrm{fria}(n)$ is not an integer, then z is irrational and in fact it is a transcendental number.
(iii) For all positive integers m and n, we have $\mathrm{fria}(mn) \leq \mathrm{fria}(m) + \mathrm{fria}(n)$, with equality if and only if $P(m) = P(n)$.
(iv) For each integer $n \geq 2$, we have $\mathrm{fria}(n) \leq \Omega(n)$, with equality if and only if n is a prime power.
(v) Given any $\alpha > 1$, there exists an increasing sequence of integers $(n_i)_{i\geq 1}$ such that $\lim_{i\to\infty} \mathrm{fria}(n_i) = \alpha$.

Property (i) is obvious. The first assertion in property (ii) follows from the fact that if $z = a/b$, with $a, b \in \mathbb{Z}$ and $b \neq 0$, then

$$\frac{\log n}{\log P(n)} = \frac{a}{b}, \text{ so that } n^b = P(n)^a \text{ and } n = P(n)^{a/b},$$

which, since factorization is unique, implies that a/b is an integer, thus contradicting our hypothesis. To prove the second assertion in (ii), assume that z is algebraic. Then $n = P(n)^z$ is an algebraic number, and this is false if z is irrational by the Gelfond–Schneider theorem, according to which if a and b are complex algebraic numbers with $a \neq 0, 1$, and b not rational, then any value of a^b is a transcendental number (see Baker ([1], p. 10)). Since we just proved that z is irrational, the proof of (ii) is complete.

To prove (iii), one can proceed as follows. Without any loss in generality, we can assume that $P(m) \leq P(n)$. We then have

$$\begin{aligned}\mathrm{fria}(mn) &= \frac{\log m + \log n}{\log P(n)} = \frac{\log m}{\log P(n)} + \frac{\log n}{\log P(n)} \\ &= \frac{\log m}{\log P(m)} \cdot \frac{\log P(m)}{\log P(n)} + \frac{\log n}{\log P(n)} = \mathrm{fria}(m) \cdot \frac{\log P(m)}{\log P(n)} + \mathrm{fria}(n) \\ &\leq \mathrm{fria}(m) + \mathrm{fria}(n),\end{aligned}$$

which proves (iii).

To prove (iv), we first write n in the form $n = \prod_{i=1}^{r} q_i^{\alpha_i}$, where $q_1 < \cdots < q_r$ are the prime factors of n and $\alpha_i \in \mathbb{N}$ for $i = 1, \ldots, r$. Then, using part (iii), we obtain that

$$\text{fria}(n) = \text{fria}\left(\prod_{i=1}^{r} q_i^{\alpha_i}\right) \le \sum_{i=1}^{r} \text{fria}\left(q_i^{\alpha_i}\right) = \sum_{i=1}^{r} \alpha_i = \Omega(n),$$

thus establishing property (iv).

For (v), it suffices to prove it when α is not an integer as when α is an integer the property is obvious. Consider the sequence of integers

$$n_i = p_i^{\lfloor\alpha\rfloor} \cdot \left\lfloor p_i^{\alpha-\lfloor\alpha\rfloor} \right\rfloor \qquad (i = 1, 2, \ldots). \tag{1.1}$$

This sequence is strictly increasing because, for each integer $i \ge 2$,

$$n_i = p_i^{\lfloor\alpha\rfloor} \cdot \left\lfloor p_i^{\alpha-\lfloor\alpha\rfloor} \right\rfloor \ge p_i^{\lfloor\alpha\rfloor} \cdot \left\lfloor p_{i-1}^{\alpha-\lfloor\alpha\rfloor} \right\rfloor > p_{i-1}^{\lfloor\alpha\rfloor} \cdot \left\lfloor p_{i-1}^{\alpha-\lfloor\alpha\rfloor} \right\rfloor = n_{i-1}.$$

Now, observe that $\left\lfloor p_i^{\alpha-\lfloor\alpha\rfloor} \right\rfloor < p_i$, which implies that $P(n_i) = p_i$ and therefore that

$$\text{fria}(n_i) = \frac{\log n_i}{\log p_i}. \tag{1.2}$$

On the other hand, since

$$p_i^{\lfloor\alpha\rfloor} \cdot \left\lfloor p_i^{\alpha-\lfloor\alpha\rfloor} \right\rfloor > p_i^{\lfloor\alpha\rfloor}\left(p_i^{\alpha-\lfloor\alpha\rfloor} - 1\right) = p_i^{\alpha} - p_i^{\lfloor\alpha\rfloor} = p_i^{\alpha}\left(1 - \frac{1}{p_i^{\alpha-\lfloor\alpha\rfloor}}\right),$$

it follows from the definition of n_i given in (1.1) that

$$\begin{aligned} p_i^{\alpha}\left(1 - \frac{1}{p_i^{\alpha-\lfloor\alpha\rfloor}}\right) &< \quad n_i \quad < p_i^{\alpha}, \\ \alpha\log p_i + \log\left(1 - \frac{1}{p_i^{\alpha-\lfloor\alpha\rfloor}}\right) &< \log n_i < \alpha\log p_i. \end{aligned}$$

Dividing both sides of the above by $\log p_i$ and taking into account (1.2), we find that

$$\alpha + \frac{\log\left(1 - 1/p_i^{\alpha-\lfloor\alpha\rfloor}\right)}{\log p_i} < \text{fria}(n_i) < \alpha. \tag{1.3}$$

Now clearly, for some $i_0 = i_0(\alpha)$,

$$p_i^{\alpha-\lfloor\alpha\rfloor} > 2 \quad \text{for all } i \ge i_0.$$

Therefore, since $\log(1-y) > -2y$ for all $y \in (0, 1/2)$, we have

$$\log\left(1 - 1/p_i^{\alpha-\lfloor\alpha\rfloor}\right) > -\frac{2}{p_i^{\alpha-\lfloor\alpha\rfloor}} \quad \text{for all } i \ge i_0,$$

an inequality which combined with (1.3) gives that

$$\mathrm{fria}(n_i) = \alpha + O\left(\frac{1}{\log p_i \cdot p_i^{\alpha - \lfloor \alpha \rfloor}}\right),$$

thus proving property (v).

Remark 1.1. To illustrate property (v), let us choose to approximate the Euler number e. Using formula (1.1), we obtain that the first ten terms of the sequence $(n_i)_{i\geq 1}$ are

$$4,\ 18,\ 75,\ 196,\ 605,\ 1014,\ 2023,\ 2888,\ 4761,\ 9251,$$

with corresponding friability indexes

$$2,\ 2.63093,\ 2.68261,\ 2.71241,\ 2.67119,\ 2.69856,\ 2.68682,\ 2.70623,\ 2.70076,\ 2.71211.$$

The index of friability is clearly related to the function

$$\Psi(x, y) := \#\{n \leq x : P(n) \leq y\} \qquad (2 \leq y \leq x).$$

Indeed, for a fixed $z > 1$, we have

$$\begin{aligned} F_z(x) := \sum_{\substack{n\leq x \\ \mathrm{fria}(n)\geq z}} 1 = \sum_{\substack{n\leq x \\ \frac{\log n}{\log P(n)}\geq z}} 1 = \sum_{\substack{n\leq x \\ P(n)\leq n^{1/z}}} 1 = (1+o(1)) \sum_{\substack{n\leq x \\ P(n)\leq x^{1/z}}} 1 \quad (1.4) \\ = (1+o(1))\Psi(x, x^{1/z}) = (1+o(1))\rho(z)x \qquad (x\to\infty), \end{aligned}$$

where $\rho(u)$ stands for the *Dickman function* defined by $\rho(u) = 1$ for $0 \leq u \leq 1$ and thereafter as the solution to the differential equation with differences

$$u\rho'(u) + \rho(u-1) = 0 \qquad (u > 1).$$

The Dickman function is key in describing the asymptotic behaviour of the $\Psi(x, y)$ function. Indeed, it is known that given numbers $2 \leq y \leq x$, setting $u := \log x/\log y$, we have

$$\Psi(x, y) = x\rho(u) + O\left(\frac{x}{\log y}\right) \quad \text{uniformly for } 2 \leq y \leq x. \qquad (1.5)$$

The proof of the above can be found in our book [7]. In the sequence of estimates (1.4), all of them are clear except for the last equality on the right. Note that the symmetric difference between the sets $\{n \leq x : P(n) \leq n^{1/z}\}$ and $\{n \leq x : P(n) \leq x^{1/z}\}$ is included in the union of $\{n \leq x/\log x\}$, a set of counting function $O(x/\log x) = o(x)$ as $x \to \infty$, and $\{n \leq x : (x/\log x)^{1/z} < P(n) \leq x^{1/z}\}$. The positive integers from this last set are representable as $n = qm$, where q is a prime in $((x/\log x)^{1/z}, x^{1/z}]$. For a fixed prime q, the number of such m's is $\leq x/q$. Summing up over q we get that the cardinality of this last set is

$$\begin{aligned} x \sum_{(x/\log x)^{1/z}<q\leq x^{1/z}} \frac{1}{q} &= x\left(\log\log x^{1/z} - \log\log((x/\log x)^{1/z}) + o(1)\right) \\ &\ll \frac{x\log\log x}{\log x} = o(x) \qquad (x\to\infty), \end{aligned}$$

which together with the fact that $\Psi(x, x^{1/z})$ is a positive proportion of x justify the desired estimate.

2. The mean value of the index of friability

In [3], it was shown that the function $\lambda(n)$ has mean value 1, and similarly for the function $1/\lambda(n)$. We will show that the average value of fria(n) is e^γ, where γ stands for the Euler-Mascheroni constant. This is a consequence of the following result.

Theorem 2.1. *We have*

$$\sum_{n\le x} \mathit{fria}(n) = e^\gamma\, x + O\left(\frac{x}{\log x}\right). \tag{2.1}$$

Proof. In [4], De Koninck and Ivić proved that

$$\sum_{2\le n\le x} \frac{1}{\log P(n)} = e^\gamma \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right). \tag{2.2}$$

Then, using partial summation, (2.1) follows immediately from (2.2). □

What about the mean value of $1/\mathrm{fria}(n)$? The answer is given in the next theorem. But first, let us introduce a very interesting constant called the *Golomb-Dickman constant* ξ, which can be defined by

$$\xi := \int_0^\infty \frac{\rho(u)}{u+2}\, du = 0.6243299885\ldots.$$

The constant ξ can also be written as

$$\xi = \int_0^\infty \frac{\rho(u)}{(u+1)^2}\, du = 1 - \int_1^\infty \frac{\rho(u)}{u^2}\, du = \int_0^1 e^{\mathrm{Li}(x)}\, dx,$$

where $\mathrm{Li}(x) := \displaystyle\int_0^x \frac{dt}{\log t}\, dt$. For more on this intriguing constant, see the book of Finch [9] and the nice paper of Lagarias [11].

This brings us to a nice connection between the index of friability and the Golomb-Dickman constant, namely the following.

Theorem 2.2. *Let ξ be the Golomb-Dickman constant. Then,*

$$\sum_{2\le n\le x} \frac{1}{\mathit{fria}(n)} = \xi\, x + \xi\gamma\, \mathit{li}(x) + O\left(R(x)\right), \tag{2.3}$$

where $\mathit{li}(x) := \displaystyle\int_2^x \frac{dt}{\log t}$ *and* $R(x) = x\exp\{-(\log x)^{\frac{3}{8}-\varepsilon}\}$ *with* $\varepsilon > 0$ *being any fixed number.*

Proof. In the book of Tenenbaum ([16], p. 283), it is proven that

$$S(x) := \sum_{2\le n\le x} \log P(n) = \xi\, x\, \log x - \xi(1-\gamma)x + O\left(R(x)\right). \tag{2.4}$$

So, let us first set $\varphi(t) := 1/\log t$. Using partial summation and observing that $\varphi'(t) = -1/(t\log^2 t)$, we then obtain that

$$\sum_{2\le n\le x} \frac{1}{\mathrm{fria}(n)} = \sum_{2\le n\le x} \frac{\log P(n)}{\log n} = \frac{S(x)}{\log x} - \frac{S(2)}{\log 2} + \int_2^x \frac{S(t)}{t\log^2 t}\, dt.$$

Using (2.4) in this last formula, we get that

$$\begin{aligned}\sum_{2\le n\le x} \frac{1}{\mathrm{fria}(n)} &= \xi\, x - \xi(1-\gamma)\frac{x}{\log x} + \xi\, \mathrm{li}(x) - \xi(1-\gamma)\int_2^x \frac{dt}{\log^2 t} + O\left(\frac{R(x)}{\log x}\right)\\ &= \xi\, x - \xi\frac{x}{\log x} + \xi\gamma\frac{x}{\log x} + \xi\left(\frac{x}{\log x} + \int_2^x \frac{dt}{\log^2 t}\right)\\ &\quad -\xi\int_2^x \frac{dt}{\log^2 t} + \xi\gamma\int_2^x \frac{dt}{\log^2 t} + O\left(\frac{R(x)}{\log x}\right)\\ &= \xi\, x + \xi\gamma\, \mathrm{li}(x) + O\left(\frac{R(x)}{\log x}\right).\end{aligned}$$

Finally, since $R(x)$ depends on some number $\varepsilon > 0$, we may ignore the $\log x$ factor (by simply replacing ε by a smaller positive ε), thereby completing the proof of the theorem. □

3. Comparing the index of friability with the index of composition

It is clear that $\mathrm{fria}(n) \ge \lambda(n)$ with equality if and only if n is a prime power. What about their respective average values? In the previous section, we proved that the average value of $\mathrm{fria}(n)$ is e^γ, whereas, as shown in [3], the average value of $\lambda(n)$ is 1. Now, what about the average value of the quotient of these two key functions? We have the following.

Theorem 3.1. *On average, $\mathit{fria}(n)/\lambda(n) = \mathit{fria}(n) + o(1)$ almost everywhere. More precisely,*

$$\sum_{n\le x} \frac{\mathit{fria}(n)}{\lambda(n)} = \sum_{n\le x} \mathit{fria}(n) + O\left(\frac{x}{\log x}\right). \tag{3.1}$$

Proof. First observe that for each integer $n \ge 2$,

$$\frac{\mathrm{fria}(n)}{\lambda(n)} = \frac{\log\gamma(n)}{\log P(n)} = \frac{\log n}{\log P(n)} - \frac{\log(n/\gamma(n))}{\log P(n)} = \mathrm{fria}(n) - \frac{\log(n/\gamma(n))}{\log P(n)}.$$

It follows from this identity that in order to prove (3.1), it is sufficient in light of Theorem 2.1 to prove that

$$K(x) := \sum_{n\le x} \frac{\log(n/\gamma(n))}{\log P(n)} \ll \frac{x}{\log x}. \tag{3.2}$$

Writing each integer $n \geq 2$ as $n = km$, where $k = k(n)$ is the squarefull part of n (with therefore m squarefree and $(k, m) = 1$) and observing that $P(km) \geq P(m)$ and that $\log(n/\gamma(n)) = \log(k/\gamma(k)) \leq \log k$ for all k, we have that

$$K(x) \leq \sum_{\substack{k \leq x \\ k \text{ squarefull}}} \log k \sum_{m \leq x/k} \frac{1}{\log P(m)}. \tag{3.3}$$

Letting $y = y(x) \leq x$ be an increasing function which tends to infinity with x, it is known that

$$\sum_{\substack{k > y \\ k \text{ squarefull}}} \frac{\log k}{k} \ll \frac{\log y}{\sqrt{y}}. \tag{3.4}$$

Let us recall a quick proof of it. First recall the basic upper bound for the number $N(x)$ of squarefull numbers not exceeding x, that is

$$N(x) \leq c\sqrt{x} \quad \text{for some positive constant } c, \tag{3.5}$$

an estimate proved by Golomb [10] in 1970. This also follows from the fact that any squarefull number n can be written uniquely as $n = r^2 m^3$, where $\mu^2(m) = 1$, so that

$$\begin{aligned} N(x) &= \sum_{r^2 m^3 \leq x} \mu^2(m) = \sum_{m \leq x^{1/3}} \mu^2(m) \sum_{r^2 \leq x/m^3} 1 \\ &= \sum_{m \leq x^{1/3}} \mu^2(m) \left\lfloor \sqrt{\frac{x}{m^3}} \right\rfloor \leq \sqrt{x} \sum_{m=1}^{\infty} \frac{\mu^2(m)}{m^{3/2}}. \end{aligned}$$

Using integration by parts and thereafter estimate (3.5), we obtain

$$\begin{aligned} \sum_{\substack{k > y \\ k \text{ squarefull}}} \frac{\log k}{k} &= \int_y^\infty \frac{\log t}{t} \, d\,N(t) = \frac{\log t}{t} N(t) \Big|_y^\infty - \int_y^\infty N(t) \left(\frac{1 - \log t}{t^2} \right) dt \\ &\ll \frac{\log y}{\sqrt{y}} + \int_y^\infty \frac{N(t) \log t}{t^2} \, dt \ll \frac{\log y}{\sqrt{y}} + \int_y^\infty \frac{\log t}{t^{3/2}} \, dt \ll \frac{\log y}{\sqrt{y}}, \end{aligned}$$

thus proving (3.4).

Splitting the right-hand side of (3.3), we can rewrite inequality (3.3) as follows.

$$\begin{aligned} K(x) &\leq \sum_{\substack{k \leq y \\ k \text{ squarefull}}} \log k \sum_{m \leq x/k} \frac{1}{\log P(m)} + \sum_{\substack{y < k \leq x \\ k \text{ squarefull}}} \log k \sum_{m \leq x/k} \frac{1}{\log P(m)} \\ &=: K_1(x) + K_2(x), \end{aligned} \tag{3.6}$$

say.

On the one hand, using estimate (2.2) and in light of the easily proven fact that $\sum_{k \text{ squarefull}} \frac{\log k}{k} < \infty$, we have that, choosing $y := (\log x)^3$,

$$\begin{aligned} K_1(x) &= \sum_{\substack{k\le y\\ k \text{ squarefull}}} \log k \left(\frac{e^\gamma x}{k}\frac{1}{\log(x/k)} + O\left(\frac{x}{k}\frac{1}{\log^2(x/k)}\right)\right) \\ &\ll \frac{x}{\log x - \log y} \sum_{\substack{k\le y\\ k \text{ squarefull}}} \frac{\log k}{k} + O\left(\frac{x}{(\log x)^2}\sum_{\substack{k\le y\\ k \text{ squarefull}}} \frac{\log k}{k}\right) \ll \frac{x}{\log x} \end{aligned} \tag{3.7}$$

On the other hand, using inequality (3.4), we have that

$$K_2(x) \le x \sum_{\substack{k>y\\ k \text{ squarefull}}} \frac{\log k}{k} \ll \frac{x\log y}{\sqrt{y}} \ll \frac{x\log\log x}{(\log x)^{3/2}} \ll \frac{x}{\log x}. \tag{3.8}$$

Using estimates (3.7) and (3.8) in (3.6), estimate (3.2) follows, thus completing the proof of Theorem 3.1. □

One can ask the same question about the average value of $\lambda(n)/\text{fria}(n)$. This is answered in the following result.

Theorem 3.2. *Letting ξ stand for the Golomb-Dickman constant, we have*

$$\sum_{2\le n\le x} \frac{\lambda(n)}{\mathit{fria}(n)} = \sum_{2\le n\le x} \frac{1}{\mathit{fria}(n)} + O\left(\frac{x}{\log x}\right) = \xi\, x + O\left(\frac{x}{\log x}\right).$$

Proof. First observe that, for $n \ge 2$, we have

$$\begin{aligned} \frac{\lambda(n)}{\text{fria}(n)} &= \frac{\log P(n)}{\log\gamma(n)} = \frac{\log P(n)}{\log n + \log(\gamma(n)/n)} \\ &= \frac{\log P(n)}{\log n}\left(1 - \frac{\log(\gamma(n)/n)}{\log n} + \left(\frac{\log(\gamma(n)/n)}{\log n}\right)^2 - \cdots\right) \\ &= \frac{\log P(n)}{\log n}\left(1 + \frac{\log(n/\gamma(n))}{\log n} + \left(\frac{\log(n/\gamma(n))}{\log n}\right)^2 + \cdots\right) \\ &= \frac{1}{\text{fria}(n)}\left(1 + O\left(\frac{\log(n/\gamma(n))}{\log n}\right)\right). \end{aligned} \tag{3.9}$$

Now, recalling the definition of $K(x)$ and its bound given in (3.2), we have that

$$\begin{aligned} \sum_{2\le n\le x} \frac{1}{\text{fria}(n)}\frac{\log(\gamma(n)/n)}{\log n} < \sum_{2\le n\le x} \frac{\log(\gamma(n)/n)}{\log n} < \sum_{2\le n\le x}\frac{\log(\gamma(n)/n)}{\log P(n)} \\ = K(x) \ll \frac{x}{\log x}. \end{aligned} \tag{3.10}$$

Using (3.10) in (3.9), it follows that

$$\sum_{2\le n\le x} \frac{\lambda(n)}{\mathrm{fria}(n)} = \sum_{2\le n\le x} \frac{1}{\mathrm{fria}(n)} + O\left(\frac{x}{\log x}\right).$$

Combining this last estimate with that of Theorem 2.2, the proof of Theorem 3.2 is complete. □

4. The index of friability at consecutive integers

The larger the index friability of an integer, the more friable it is. Can two consecutive integers both have a high index of friability ? How high ? What about three consecutive integers ? The short answer is "YES, all the way". Indeed, in 1998, Balog and Wooley [2] proved that given any arbitrarily small $\varepsilon > 0$ and any integer $k \ge 2$, there exist infinitely many positive integers n such that

$$P((n+1)(n+2)\cdots(n+k)) < n^{\varepsilon}.$$

In terms of the index of friability, Balog and Wooley's result can be stated as:

> *Given a fixed integer $k \ge 2$ and an arbitrarily large number K, there exist infinitely many positive integers n such that*
>
> $$\min(\mathit{fria}(n+1), \mathit{fria}(n+2), \ldots, \mathit{fria}(n+k)) > K.$$

For example, in the particular case $k = 4$ and $K = 3$, one can check that the number $n = 3\,678\,722$ is the smallest positive integer such that

$$\min(\mathrm{fria}(n+1), \mathrm{fria}(n+2), \mathrm{fria}(n+3), \mathrm{fria}(n+4)) > 3,$$

and according to Balog and Wooley's result, there exist infinitely many such positive integers n.

5. Integers with the same index of friability

It seems reasonable to claim that if two distinct positive integers have the same index of friability, then they are both prime powers. However, this statement is only true under the *Four exponentials conjecture*, which we now recall.

Conjecture 5.1. (Four exponentials conjecture) *Assume $\lambda_{i,j}$ for $1 \le i, j \le 2$ are positive algebraic numbers and consider the matrix*

$$M = \begin{pmatrix} \log\lambda_{1,1} & \log\lambda_{1,2} \\ \log\lambda_{2,1} & \log\lambda_{2,2} \end{pmatrix}.$$

Assume that the rows of M are linearly independent over $\mathbb{Q}$ and the columns of M are linearly independent over $\mathbb{Q}$. Then the determinant is nonzero.

For us, assume that n_1, n_2 are such that $\text{fria}(n_1) = \text{fria}(n_2) \notin \mathbb{N}$. Consider the matrix

$$M = \begin{pmatrix} \log P(n_1) & \log P(n_2) \\ \log n_1 & \log n_2 \end{pmatrix}.$$

Let us check that the rows are linearly independent over $\mathbb{Q}$. Since $\log n_i$, $\log P(n_i)$ are all positive, if they were linearly dependent then for some integers a, b not both zero we would have

$$a(\log P(n_1), \log P(n_2)) = b(\log n_1, \log n_2).$$

We may assume that both a, b are positive. And we get $\log n_1 = (a/b) \log P(n_1)$, so that $n_1 = P(n_1)^{a/b}$, which is impossible unless a/b is an integer and n_1 and n_2 are both a prime powers which we assume that it does not hold. If the columns of M are linearly dependent over $\mathbb{Q}$, then we get that for some integers a, b not both zero we have

$$a(\log P(n_1), \log n_1) = b(\log P(n_2), \log n_2).$$

Thus, $P(n_1) = P(n_2)^{b/a}$. This makes $a = b$ and then since also $n_2 = n_1^{a/b}$, we get $n_2 = n_1$, which we assume it does not happen. So, indeed M fulfils the conditions from the Four exponentials conjecture, so under this conjecture the determinant of M is nonzero. Therefore, $\text{fria}(n_1) \neq \text{fria}(n_2)$.

One may ask what can be said unconditionally about the equation

$$\text{fria}(n_1) = \text{fria}(n_2)$$

without imposing the condition that this common value is not in $\mathbb{N}$. In this case putting

$$\mathcal{E} := \{(n_1, n_2) : n_1 < n_2 \leq x,\ \text{fria}(n_1) = \text{fria}(n_2)\},$$

the referee observed that one may take $n_1 = p^2$, $n_2 = q^2$, where $p < q \leq x^{1/2}$ are primes (for which $\text{fria}(n_1) = \text{fria}(n_2) = 2$), which gives right away that

$$\#\mathcal{E} \gg \pi(\sqrt{x})^2 \gg x/(\log x)^2$$

and asked for an upper bound. Fixing n_1, and writing $n_2 = Pm$, where $P = P(n_2)$, we see that the equation $\text{fria}(n_1) = \text{fria}(n_2) = 1 + \log m / \log P$ shows that fixing one of m, P the other one is uniquely determined. Since $Pm \leq x$, either $m \leq (x/\log x)^{1/2}$, or $P \leq (x \log x)^{1/2}$ so the number of choices for P is $\leq \pi((x \log x)^{1/2}) \ll (x/\log x)^{1/2}$, and we get that $\#\mathcal{E} \ll x^{3/2}/(\log x)^{1/2}$. We leave it to the reader to find unconditional better upper bounds for the quantity $\#\mathcal{E}$.

But we can use the *Six exponentials theorem* to conclude that there do not exist three distinct positive integers n_1, n_2, n_3 such that

$$\text{fria}(n_1) = \text{fria}(n_2) = \text{fria}(n_3) \notin \mathbb{N}. \tag{5.1}$$

The *Six exponentials theorem* can be stated as follows.

Theorem 5.2. (Six exponentials theorem) *Assume that $\lambda_{i,j}$ for $1 \le i \le 2$, $1 \le j \le 3$ are positive algebraic numbers such that the matrix*

$$M = \begin{pmatrix} \log \lambda_{1,1} & \log \lambda_{1,2} & \log \lambda_{1,3} \\ \log \lambda_{2,1} & \log \lambda_{2,2} & \log \lambda_{2,3} \end{pmatrix}$$

satisfies:

(i) $\log \lambda_{1,1}, \log \lambda_{1,2}, \log \lambda_{1,3}$ *are linearly independent over* $\mathbb{Q}$*;*
(ii) $\log \lambda_{1,1}$ *and* $\log \lambda_{2,1}$ *are also linearly independent over* $\mathbb{Q}$*.*

Then M has rank 2 over $\mathbb{R}$.

So, let

$$M = \begin{pmatrix} \log P(n_1) & \log P(n_2) & \log P(n_3) \\ \log n_1 & \log n_2 & \log n_3 \end{pmatrix}.$$

For us, $P(n_1), P(n_2), P(n_3)$ are distinct primes (indeed if $P(n_i) = P(n_j)$, then since also $\text{fria}(n_i) = \text{fria}(n_j)$, we would get $n_i = n_j$, which is false). Hence, $\log P(n_1)$, $\log P(n_2)$, $\log P(n_3)$ are linearly independent over $\mathbb{Q}$. Also, $\log P(n_1)$ and $\log n_1$ are linearly independent over $\mathbb{Q}$, since otherwise by what we have seen above we would get $n_1 = P(n_1)^r$, for some rational number r, so that r is a positive integer and n_1 is a power of $P(n_1)$, which is not our case. Hence, M fulfils the hypothesis of the Six exponentials theorem, implying that the rank of M is 2, which contradicts assumption (5.1).

We have thus proved the following result.

Theorem 5.3. *There are no three distinct positive integers n_1, n_2, n_3 such that $\text{fria}(n_1) = \text{fria}(n_2) = \text{fria}(n_3) \notin \mathbb{N}$. Under the Four exponentials conjecture, there are no two distinct positive integers n_1, n_2 such that $\text{fria}(n_1) = \text{fria}(n_2) \notin \mathbb{N}$.*

The reader will find more details on the Four exponentials conjecture and the Six exponentials theorem in the book of Lang [12] and the paper of Ramachandra [14].

6. Ordering the integers according to their index of friability

Given an integer $N \ge 10$, consider the subset of $\{6, 7, \ldots, N\}$ consisting of those integers which are not prime powers and order its members according to their index of friability. More formally, letting $A_N := \{n \le N : \omega(n) \ge 2\}$ and given any two integers $m, n \in A_N$, we will write

$$m \lhd n \quad \text{if} \quad \text{fria}(m) \le \text{fria}(n). \tag{6.1}$$

For instance, from the set of integers $\{6, 7, \ldots, 20\}$, remove the prime powers and sort the remaining set of integers according to their index of friability as follows:

$$\begin{array}{ccccccccccccc} 14 & \lhd & 10 & \lhd & 6 & \lhd & 15 & \lhd & 20 & \lhd & 12 & \lhd & 18 \\ 1.3562 & < & 1.4307 & < & 1.6309 & < & 1.6826 & < & 1.8614 & < & 2.2619 & < & 2.6309 \end{array}$$

Observe that in light of Theorem 5.3 and because the Four exponentials conjecture is most likely true, in practice, the inequality appearing in (6.1) will always be a strict inequality.

For a given integer $N \geq 6$ with corresponding set A_N with $a = \#A_N$, one will notice, at times, that for some integers $k \geq 2$, the sequence

$$n_1 \lhd n_2 \lhd \cdots \lhd n_a$$

contains strings $n_i \lhd n_{i+1} \lhd \cdots \lhd n_{i+k-1}$ of k elements of A_N which fall into one of the following two categories:

(i) $n_i > n_{i+1} > \cdots > n_{i+k-1}$,
(ii) $n_i < n_{i+1} < \cdots < n_{i+k-1}$.

Numerical experiments show that as we order A_N with the order relation $\lhd$ and list its element from smallest to largest, this order relation starts with a long string as in (i) and ends with a long string as in (ii). This we prove to be the case in the next two theorems. We also show that for large N the maximum possible lengths of a decreasing string as in (i) is given by the beginning string. We conjecture that this is true for large N for the largest increasing string as in (ii) (namely that is given by the end of the ordering of A_N with the order relation $\lhd$) and we leave that as a problem to the reader.

Let us now state the last two theorems of this paper.

Theorem 6.1. *Given an integer $N \geq 6$, let $A_N := \{n \leq N : \omega(n) \geq 2\}$ with $a = \#A_N$. Set $s := \pi(N/2)$, $t := \pi(N/3)$ and consider the set*

$$\mathcal{B} := \{2p_s, 2p_{s-1}, \ldots, 2p_r\},$$

where p_r is the smallest prime number satisfying

$$\log p_t \cdot \frac{\log 2}{\log 3} < \log p_r. \tag{6.2}$$

Then,

(A) The list of elements in $\mathcal{B}$ is decreasing, whereas the sequence formed by their corresponding index of friability is increasing, that is,

$$2p_s > 2p_{s-1} > \cdots > 2p_r \tag{6.3}$$

and

$$\mathit{fria}(2p_s) < \mathit{fria}(2p_{s-1}) < \cdots < \mathit{fria}(2p_r).$$

(B) The cardinality of $\mathcal{B}$ is $k := s - r + 1$.

(C) The list of elements in $\mathcal{B}$ are the first k elements of the entire sequence

$$n_1 \lhd n_2 \lhd \cdots \lhd n_a. \tag{6.4}$$

(D) The string formed by the elements in $\mathcal{B}$ is the longest decreasing subsequence in (6.4).

For example, with $N = 100$, we find that $a = \#A_N = 64$ and that the reordering of the elements of set A_N according to their index of friability gives the sequence

$$94, 86, 82, 74, 62, 58, 46, 38, 34, 26, 22, 93, 87, 69, 14, 57, 51, 39, 10, 92, 33, 76,$$
$$68, 52, 95, 21, 85, 44, 65, 6, 55, 15, 78, 28, 66, 91, 77, 35, 20, 88, 99, 42, 56, 30,$$
$$63, 70, 12, 84, 40, 98, 45, 50, 60, 18, 75, 80, 90, 100, 24, 36, 48, 54, 72, 96.$$

In this case we have $s = \pi(N/2) = 15$, $t = \pi(N/3) = 11$, $r = 5$, $p_r = 11$, $p_t = 31$ so the set

$$\mathcal{B} = \{94, 86, 82, 74, 62, 58, 46, 38, 34, 26, 22\}$$

has $k = s - r + 1 = 15 - 5 + 1 = 11$ elements.

Theorem 6.2. *Given an integer $N \geq 10$, let $A_N := \{n \leq N : \omega(n) \geq 2\}$ and let N_0 be the number*

$$N_0 := \max\{n \in A_N : P(n) = 5\}.$$

Then, letting ℓ be the cardinality of the set

$$\mathcal{C} := \{m \leq N : \gamma(m) = 6 \text{ and } \mathit{fria}(m) > \mathit{fria}(N_0)\} = \{b_1, b_2, \ldots, b_\ell\},$$

where $b_1 < b_2 < \cdots < b_\ell$, we have

$$\ell = \ell(N) = (c + o(1))(\log N)^2 \qquad (N \to \infty),$$

where

$$c := \frac{1}{2 \log 2 \log 3}\left(1 - \left(\frac{\log 3}{\log 5}\right)^2\right).$$

Moreover, provided N is sufficiently large, the string of the last ℓ largest elements in A_N (ordered by $\lhd$) is precisely the string composed of the elements of $\mathcal{C}$.

Conjecture 6.3. *For large N, the longest string of elements $n_1 < n_2 < \cdots < n_k$, with $n_i \in A_N$, such that*

$$n_1 \lhd n_2 \lhd \cdots \lhd n_k,$$

that is, such that

$$\mathit{fria}(n_1) < \mathit{fria}(n_2) < \cdots < \mathit{fria}(n_k),$$

is precisely the string composed of the elements of $\mathcal{C}$ (so that also $k = \ell$).

We leave this as a challenge to the reader. For example, with $N = 167$, we obtain $N_0 = 160$ and easily compute that $\mathrm{fria}(N_0) = \mathrm{fria}(160) = 3.15338$. Therefore, in this case, we have the following as the longest possible sequence made of increasing numbers ≤ 167 with increasing index of friability:

$$36, 48, 54, 72, 96, 108, 144, 162,$$

whose corresponding index of friability are:

$$3.26186,\ 3.52372,\ 3.63093,\ 3.89279,\ 4.15465,\ 4.26186,\ 4.52372,\ 4.63093,$$

so $k = \ell = 8$.

Remark 6.4. For some "small" integers N, the set $\mathcal{C}$ mentioned in Theorem 6.2 may not be the only provider of the longest string of increasing integers $n_1 < n_2 < \cdots < n_k$, where each $n_i \in A_N$ and $\text{fria}(n_1) < \text{fria}(n_2) < \cdots < \text{fria}(n_k)$. Indeed, choose for instance $N = 67$. In this case, $\#A_N = 38$ and in fact the list of elements of A_N rearranged according to their index of friability is as follows:
62,58,46,38,34,26,22,14,57,51,39,10,33,52,21,44,65,6,55,
15,28,66,35,20,42,56,30,63,12,40,45,50,60,18,24,36,48,54.

In this case there exist two longest increasing subsequences $n_1 < n_2 < \cdots < n_k$ (here with $k = 5$), namely

$$12,\ 40,\ 45,\ 50,\ 60 \quad \text{and} \quad 18,\ 24,\ 36,\ 48,\ 54.$$

Before we move on with the proofs of Theorems 6.1 and 6.2, we establish a few lemmas.

Lemma 6.5. *Let $N \geq 10$ and $A_N := \{n \leq N : \omega(n) \geq 2\}$ with $a = \#A_N$. Let $n_1 \lhd n_2 \lhd \cdots \lhd n_a$ be the elements of A_N ordered according to their index of friability.*

(i) If for some $i < a$, we have $n_i < n_{i+1}$, then $P(n_i) \leq P(n_{i+1})$.
(ii) If for some $i < a$, we have $n_i > n_{i+1}$, then $P(n_i) > P(n_{i+1})$.

Proof. To establish part (i), we attempt a proof by contradiction. Assume that $p := P(n_i) > q := P(n_{i+1})$. We attempt to construct $n \in A_N$ satisfying

$$\text{fria}(n_i) < \text{fria}(n) < \text{fria}(n_{i+1}), \text{ that is, } n_i \lhd n \lhd n_{i+1}, \tag{6.5}$$

thus contradicting the fact that n_i and n_{i+1} are two consecutive elements in the string $n_1 \lhd n_2 \lhd \cdots \lhd n_a$. To do so, first observe that there exist integers $m' \geq 2$, $\alpha \geq 1$ and $m_0 \geq 2$ such that $n_i = p^\alpha m'$ and $n_{i+1} = m_0 q$, where $P(m') < p$ and $P(m_0) \leq q$. Let $r := \max\{P(m'), q\} < p$. By our assumptions, we have

$$n := m' r^\alpha < p^\alpha m' = n_i < n_{i+1} = m_0 q < m_0 p, \quad \text{implying that } p^{\alpha-1} m' < m_0. \tag{6.6}$$

Assume $n \in A_N$. In light of (6.6), we have

$$\begin{aligned} \text{fria}(n_i) &= \frac{\log(p^{\alpha-1}m')}{\log p} + 1 = \frac{\log m'}{\log p} + \alpha < \frac{\log m'}{\log r} + \alpha = \frac{\log(r^{\alpha-1}m')}{\log r} + 1 \\ &= \text{fria}(n) < \frac{\log m_0}{\log q} + 1 = \text{fria}(n_{i+1}), \end{aligned}$$

thus proving (6.5) and thereby contradicting the fact that $n_i \lhd n_{i+1}$. But it is possible that $n \notin A_N$. This happens if n is a prime power, therefore $m' = r^\gamma$ for some $\gamma \geq 1$ and $r \geq q$. In this case, we attempt to take $n := (r-1)r^{\alpha-1+\gamma} < p^\alpha m' = n_i$, so $n < N$. Furthermore, since $q = P(n_{i+1}) \le r$ it follows that $r \ge 3$, therefore $\omega(n) = \omega(r-1) + 1 \ge 2$. Thus, $n \in A_N$. Now

$$\mathrm{fria}(n) = \frac{\log n}{\log r} = \frac{\log((r-1)r^{\gamma-1})}{\log r} + \alpha < \frac{\log m'}{\log r} + \alpha < \frac{\log m_0}{\log q} + 1 = \mathrm{fria}(n_{i+1}).$$

We want $\mathrm{fria}(n) > \mathrm{fria}(n_i)$. This is equivalent to

$$\frac{\log(r^\gamma)}{\log p} + \alpha < \frac{\log((r-1)r^{\alpha-1+\gamma-1})}{\log r} + 1 = \frac{\log((r-1)r^{\gamma-1})}{\log r} + \alpha,$$

which is equivalent to

$$\gamma\frac{\log r}{\log p} < (\gamma-1) + \frac{\log(r-1)}{\log r} \qquad \text{or} \qquad \frac{1-\log(r-1)/\log r}{1-\log r/\log p} < \gamma.$$

The above inequality is satisfied for all $p \ge r+2$ and $r \ge 5$ with any $\gamma \ge 1$. It is also satisfied for $r = 3$ and $p \ge 7$ with any $\gamma \ge 1$. It is also satisfied for $r = 3,\ p = 5$ and any $\gamma \ge 2$. So, the only remaining case is $p = 5,\ r = 3$, so $q = r = 3$, $n_i = pq = 15$, $n_{i+1} = 3m_0$ with $P(m_0) \le 3$. Since $n_{i+1} \in A_N$, it follows that $n_{i+1} = 2^a3^b$ with $b \ge 1$ and $a \ge 1$. Since $n_i < n_{i+1}$, it follows that $n_{i+1} \ge 18$ (so, either $a \ge 3$, or $b \ge 2$). In both cases $n_i = 15 \lhd 12 \lhd n_{i+1}$, a contradiction.

The proof of (ii) is somewhat different and in fact simpler. To seek a contradiction, we will assume that $p := P(n_i) \le q := P(n_{i+1})$. Set again $n_i = mp$ and $n_{i+1} = m_0q$. In this case, we then have

$$mp = n_i > n_{i+1} = m_0q \ge m_0p, \quad \text{implying that } m > m_0. \tag{6.7}$$

Using our assumption that $p \le q$ and (6.7), we then have

$$\mathrm{fria}(n_{i+1}) = \frac{\log m_0}{\log q} + 1 < \frac{\log m}{\log q} + 1 \le \frac{\log m}{\log p} + 1 = \mathrm{fria}(n_i),$$

thus contradicting the fact that $n_i \lhd n_{i+1}$, and completing the proof of Lemma 6.5. □

Lemma 6.6. *Let $N \ge 10$ be an integer and assume that $n \in A_N$ is such that $P(n) = p_r$ for some $r \ge 4$. Then there exists an integer $n_0 \in A_N$ such that $P(n_0) \le p_{r-1}$ and fria$(n) <$ fria(n_0).*

Proof. For that particular n given in the statement of the lemma, there exist positive integers m and α such that $n = mp_r^\alpha$ with $P(m) \le p_{r-1}$. Then, consider the number $n_0 := mp_{r-1}^\alpha$. Observe that $n_0 < n \le N$. First assume that $n_0 \in A_N$. Then

$$\mathrm{fria}(n) = \frac{\log m}{\log p_r} + \alpha < \frac{\log m}{\log p_{r-1}} + \alpha = \mathrm{fria}(mp_{r-1}^\alpha) = \mathrm{fria}(n_0),$$

thus obtaining the desired conclusion. But it can be that $n_0 \notin A_N$. This happens exactly if $m = p_{r-1}^\beta$ for some positive integer β. We then consider the number

$n_0 := 2\cdot 3^{\alpha+\beta}$. We have $n_0 = 3^{\alpha+\beta-1}\cdot 6 \le p_{r-1}^{\alpha+\beta-1} p_r \le n$, where we used that $r \ge 4$, so $p_r \ge p_4 = 7 > 6$ and $p_{r-1} \ge p_3 = 5 > 3$. Furthermore,

$$\mathrm{fria}(n) = \beta \frac{\log p_{r-1}}{\log p_r} + \alpha < \alpha + \beta < \mathrm{fria}(n_0),$$

completing the proof of Lemma 6.6. □

Lemma 6.7. *Let $N \ge 6$ and assume that some two distinct elements $m, n \in A_N$ have the same largest prime factor. Then $m < n \Longleftrightarrow m \lhd n$.*

Proof. First, assume that $m < n$ and set $p := P(m) = P(n)$. Then

$$\mathrm{fria}(m) = \frac{\log m}{\log p} < \frac{\log n}{\log p} = \mathrm{fria}(n),$$

implying that $m \lhd n$. Now assume that $m \lhd n$. Then,

$$\frac{\log m}{\log p} < \frac{\log n}{\log p},$$

implying that $m < n$. □

Let $q_1 < \cdots < q_r$ be fixed primes and let

$$\mathcal{S} = \{q_1^{\alpha_1} \cdots q_r^{\alpha_r} : \alpha_i \ge 0,\ i = 1, \ldots, r\}$$

be the set of positive integers whose prime factors are among $q_1, \ldots, q_r$. We enumerate $1 = n_1 < n_2 < \cdots$ the $\mathcal{S}$ integers ordered increasingly. The next result is due to Tijdeman [18], [19].

Lemma 6.8. *There exist effective constants C_1, C_2 such that*

$$\frac{n_i}{(\log n_i)^{C_1}} \ll n_{i+1} - n_i \ll \frac{n_i}{(\log n_i)^{C_2}}$$

for all $i \ge 2$.

In the particular case when $r = 2$, all constants in Tijdeman's results have been made explicit in [13].

Lemma 6.9. *Assuming $r = 2$, we have for $n_i \ge 3$*

$$C_3 \frac{n_i}{(\log n_i)^{C_1}} < n_{i+1} - n_i < C_4 \frac{n_i}{(\log n_i)^{C_2}},$$

where $C_1 = 2\cdot 10^9 \log q_1 \log q_2,\ C_2 = C_1^{-1},\ C_3 = (\log q_1)^{C_1},\ C_4 = 8q_2$.

Lemma 6.10. *Given an integer $N \ge 6$, let*

$$\mathcal{D} := \{n \in \mathbb{N} : \gamma(n) = 6\} \quad \textit{and} \quad \mathcal{D}(N) := \#\{n \le N : n \in \mathcal{D}\}.$$

Then,

$$\mathcal{D}(N) = \frac{1}{2} \frac{\log^2 N}{\log 2 \log 3} + O(\log N). \tag{6.8}$$

Moreover, letting ℓ be as in Theorem 6.2, we have

$$\ell = (c_0 + o(1))\mathcal{D}(N), \tag{6.9}$$

where

$$c_0 = 1 - \left(\frac{\log 3}{\log 5}\right)^2.$$

Proof. Estimate (6.8) is Corollary 3.1 in [16] for $y = 2$ and $x = N$. Lemma 6.8 for the set $\{q_1, q_2, q_3\} = \{2, 3, 5\}$ tells us that for large N, we have

$$N_0 = N - O\left(\frac{N}{(\log N)^{C_2}}\right).$$

Thus,

$$\text{fria}(N_0) = \frac{\log N_0}{\log 5} = \frac{\log N}{\log 5} + O\left(\frac{1}{(\log N)^{C_2}}\right).$$

Thus, b_ℓ is some integer of the form $2^a \cdot 3^b$ with positive $a,\ b$ with the property that

$$\text{fria}(b_\ell) \in \left(\frac{\log N}{\log 5} - O\left(\frac{1}{(\log N)^{C_2}}\right), \frac{\log N}{\log 5}\right),$$

This makes

$$b_\ell \in \left(N^{\log 3/\log 5}\left(1 + O\left(\frac{1}{(\log N)^{C_2}}\right)\right), N^{\log 3/\log 5}\right).$$

Up to eventually making the constant C_2 smaller, the above interval does contain numbers of the form $2^a \cdot 3^b$ for large N thanks again to Lemma 6.8 for $\{q_1, q_2\} = \{2, 3\}$. It now follows that

$$\begin{aligned}
\ell &= \mathcal{D}(N) - \mathcal{D}(b_\ell) = \left(\frac{(\log N)^2}{2\log 2\log 3} + O(\log N)\right) - \left(\frac{(\log b_\ell)^2}{2\log 2\log 3} + O(\log b_\ell)\right) \\
&= \frac{1}{2\log 2\log 3}((\log N)^2 - (\log b_\ell)^2) + O(\log N) \\
&= \frac{1}{2\log 2\log 3}\left((\log N)^2 - \left(\frac{\log 3}{\log 5}\right)^2 (\log N)^2 \left(1 + O\left(\frac{1}{(\log N)^{C_2}}\right)\right)^2\right) \\
&\quad + O(\log N) \\
&= \frac{1}{2\log 2\log 3}\left(1 - \left(\frac{\log 3}{\log 5}\right)^2\right)(\log N)^2 + O((\log N)^{2-C_2}).
\end{aligned}$$

□

We now have the necessary tools to prove our two theorems.

Proof of Theorem 6.1. Conclusions (A) and (B) are immediate. To see that (C) holds, first observe that $2p_s$ is the element of A_N with the smallest index of friability, meaning that $n_1 = 2p_s$. This comes from the fact that p_s is the largest prime such that $2p_s \le N$ and therefore that

$$\mathrm{fria}(2p_s) = \frac{\log 2}{\log p_s} + 1 < \mathrm{fria}(n) \quad \text{for all } n \in A_N,\ n \ne 2p_s. \tag{6.10}$$

Indeed, given any $n \in A_N$ where $n \ne 2p_s$, then there exist some positive integer $i < s$ and an integer $m \ge 2$ such that $n = mp_i$ with $P(m) \le p_i$, in which case we have that

$$\mathrm{fria}(n) = \frac{\log m}{\log p_i} + 1 > \frac{\log 2}{\log p_s} + 1 = \mathrm{fria}(2p_s),$$

thus proving (6.10).

Using the same reasoning, one can establish that $n_2 = 2p_{s-1}$, $n_3 = 2p_{s-2}$, and so on. In the end, we obtain

$$n_1 = 2p_s, \quad n_2 = 2p_{s-1}, \quad \ldots, \quad n_k = 2p_r.$$

Now, according to (6.2), we have that

$$\log p_{r-1} < \log p_t \cdot \frac{\log 2}{\log 3} < \log p_r, \tag{6.11}$$

and therefore

$$\frac{\log 2}{\log p_r} + 1 < \frac{\log 3}{\log p_t} + 1 < \frac{\log 2}{\log p_{r-1}} + 1.$$

That is,

$$\mathrm{fria}(2p_r) < \mathrm{fria}(3p_t) < \mathrm{fria}(2p_{r-1}). \tag{6.12}$$

On the other hand, (6.11) also implies that

$$p_t > p_{r-1}^{\log 3/\log 2}. \tag{6.13}$$

Observe that one can easily establish that

$$p_{j-1}^{\log 3/\log 2} \ge p_j \qquad (j \ge 2). \tag{6.14}$$

This inequality can be obtained using the inequalities

$$j(\log j + \log\log j - 3/2) < p_j < j(\log j + \log\log j - 1/2) \qquad (j \ge 20)$$

which can be found in Rosser and Schoenfeld [15] and checked individually for the remaining j in $[2, 20]$.

It follows from (6.14) and (6.13) that $p_t > p_r$ and thus that

$$3p_t > 2p_r. \tag{6.15}$$

Hence, combining (6.12) and (6.15) establishes that the number $3p_t$ interrupts the descent of the numbers in (6.3) right after $2p_r$, due to the fact that $\mathrm{fria}(3p_t)$ is located in between the indexes of friability of $2p_r$ and $2p_{r-1}$ as confirmed by (6.12). This completes the proof of (C).

For (D), we use a counting argument. That is, we already established that the first $k = s - r + 1$ elements in A_N with the order $\lhd$ form the decreasing string

$$2p_s > 2p_{s-1} > \cdots > 2p_r.$$

Note that $s = \pi(N/2)$ and $p_{r-1} < p_t^{\log 2/\log 3} < (N/3)^{\log 2/\log 3}$. Thus,

$$k = s - (r-1) \geq \pi(N/2) - \pi((N/3)^{\log 2/\log 3}).$$

Let us look at any other string

$$n'_1 > n'_2 > \cdots > n'_{k'} \quad \text{all in } A_N$$

with

$$n'_1 \lhd n'_2 \lhd \cdots \lhd n'_{k'}.$$

Write $n'_i := m'_i p'_i$ with $p'_i = P(n'_i)$. If $m'_1 \geq 3$, then $p'_1 \leq N/3$ and $p'_{i+1} < p'_i$ for $i = 1, \ldots, k'-1$ thanks to Lemma 6.5 (ii). Thus, $k' \leq \pi(N/3)$ in this case. Assume next that $m'_i = 2$. Since the current increasing chain is not the initial one, it follows that $p'_i \leq p_{r-1}$. Hence, $k' \leq r - 1 \leq \pi((N/3)^{\log 2/\log 3}) \leq \pi(N/3)$. So, in both cases, the length of any other increasing string in the order $\lhd$ which is formed by a decreasing string of integers is of length at most $\pi(N/3)$. Thus, it suffices to verify that

$$\pi(N/2) - \pi((N/3)^{\log 2/\log 3}) > \pi(N/3). \tag{6.16}$$

Using Theorem 2 and Corollary 1 in Rosser and Schoenfeld [15], we have

$$\frac{x}{\log x - 1/2} < \pi(x) < \frac{x}{\log x - 3/2} \qquad \text{for} \quad x \geq 67,$$

it remains to verify that

$$\begin{aligned}\pi(N/2) > \frac{N/2}{\log(N/2) - 1/2} &> \frac{N/3}{\log(N/3) - 3/2} + (N/3)^{\log 2/\log 3}\\ &> \pi(N/3) + \pi((N/3)^{\log 2/\log 3}),\end{aligned}$$

and the middle inequality holds for all $N > 27000$. In the range $N \in [10, 27000]$, the inequality (6.16) fails sometimes but the largest value for which it fails is $N = 253$. For $N \in [10, 253]$, we checked that the longest string satisfying (i) is indeed given by the number k. This finishes the proof of Theorem 6.1. □

Proof of Theorem 6.2. Since $\omega(n) \geq 2$ in A_N, it follows that the largest elements of A_N are obtained for numbers of the form $n = 2^a \cdot 3^b$ such that $\text{fria}(n) \geq \text{fria}(N_0)$. The rest of the assertions have been verified in Lemma 6.10. □

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