On the quotient of the logarithms of the middle divisors of an integer

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Abstract

Given a positive integer n, let $\rho_1(n) := \max\{d \mid n : d \le \sqrt{n}\}$ and $\rho_2(n) := \min\{d \mid n : d \ge \sqrt{n}\}$ stand for the *middle divisors* of n. We examine the average value of the quotient $\log \rho_2(n)/\log \rho_1(n)$ as n runs through the composite integers and of the quotient $\log \rho_1(n)/\log \rho_2(n)$ as n runs through all integers $n \ge 2$.

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1 Introduction

Given a positive integer n, let

$$\rho_1(n) := \max\{d \mid n : d \le \sqrt{n}\} \qquad \text{and} \qquad \rho_2(n) := \min\{d \mid n : d \ge \sqrt{n}\}$$

stand for the *middle divisors* of n.

The mean value of $\rho_2(n)$ has been established more than 40 years ago as Tenenbaum [4] proved that

$$\sum_{n\le x}\rho_2(n)=\frac{\pi^2}{12}\frac{x^2}{\log x}\left(1+O\left(\frac{1}{\log x}\right)\right).$$

Recently, we [2] slightly improved and generalised this estimate by showing that, given any real number $a>0$ and any integer $k\ge 1$,

$$\sum_{n\le x}\rho_2(n)^a=c_0\frac{x^{a+1}}{\log x}+c_1\frac{x^{a+1}}{\log^2 x}+\cdots+c_{k-1}\frac{x^{a+1}}{\log^k x}+O\left(\frac{x^{a+1}}{\log^{k+1} x}\right),$$

where, for $\ell=0,1,\ldots,k-1$, $c_\ell=c_\ell(a)=\dfrac{\ell!}{(a+1)^{\ell+1}}\displaystyle\sum_{j=0}^{\ell}\frac{(a+1)^j(-1)^j\zeta^{(j)}(a+1)}{j!}$ with ζ standing for the Riemann zeta function.

On the other hand, finding the mean value of $\rho_1(n)$ is not an easy task. The best known result in that direction is due to Ford [3] as he showed

$$\sum_{n\le x}\rho_1(n)\asymp\frac{x^{3/2}}{(\log x)^\delta(\log\log x)^{3/2}},$$

where $\delta=1-\dfrac{1+\log\log 2}{\log 2}\approx 0.086071$.

In [2], we provided estimates for $\displaystyle\sum_{n\le x}\frac{\rho_2(n)}{\rho_1(n)}$ and $\displaystyle\sum_{n\le x}\frac{\rho_1(n)}{\rho_2(n)}$.

Here we examine the average value of the quotient $\dfrac{\log\rho_2(n)}{\log\rho_1(n)}$ as n runs through the composite integers and that of the quotient $\dfrac{\log\rho_1(n)}{\log\rho_2(n)}$ as n runs through all integers $n\ge 2$.

In what follows, given an integer $n\ge 2$, we let $P(n)$ stand for the largest prime factor of n. For convenience, we set $P(1)=1$. Moreover, for $2\le y\le x$, let $\Psi(x,y):=\#\{n\le x: P(n)\le y\}$. Also, we let $\rho(u)$ stand for the Dickman function which is defined as the unique continuous function $\rho:[0,\infty)\to(0,1]$ which is differentiable on $[1,\infty)$ and satisfies $\rho(u)=1$ for $0\le u\le 1$ and $u\rho'(u)+\rho(u-1)=0$ for $u\ge 1$.

2 Main results

Theorem 1 *Set* $S(x):=\displaystyle\sum_{\substack{4\le n\le x\\ n\ne prime}}\frac{\log\rho_2(n)}{\log\rho_1(n)}$. *Then,*

$$S(x)=x\log\log x+O(x).$$

Theorem 2 *Set* $T(x) := \sum_{2 \le n \le x} \frac{\log \rho_1(n)}{\log \rho_2(n)}$. *Then, for all* x *sufficient large,*

$$c_1\, x < T(x) < c_2\, x, \tag{1}$$

where

$$\begin{aligned} c_1 &= 1 - \log 2 + \int_1^2 \frac{1 - \log u}{u(u+1)}\, du + \int_3^\infty \frac{u-1}{u+1} \frac{\rho(u-1)}{u}\, du \approx 0.528087, \\ c_2 &= 2 - 2\log 2 \approx 0.613706. \end{aligned}$$

3 Preliminary results

The following results will be helpful in the proofs of Theorems 1 and 2.

Proposition 3 *For* $2 \le y \le x$,

$$\Psi(x,y) \ll x \exp\left\{ -\frac{1}{2} \frac{\log x}{\log y} \right\}.$$

Proof This upper bound of $\Psi(x,y)$ is due to Tenenbaum (see for instance Théorème 5.1 in his book [5]). □

Proposition 4 *Letting* $u = \log x / \log y$, *we have*

$$\Psi(x,y) = x\rho(u) + O\left(\frac{x}{\log y}\right) \qquad (2 \le y \le x).$$

Proof For a proof, see Theorem 9.14 in the book of De Koninck and Luca [1]. □

The following two lemmas will be useful in the proofs of these two theorems.

Lemma 5 *Given any integer* $n \ge 2$, $P(n) \ge \sqrt{n}$ *if and only if* $\rho_2(n) = P(n)$.

Proof If n is prime, then the result is obvious. If n is composite and $P(n) \ge \sqrt{n}$, then all other divisors of n smaller than n must not exceed $\sqrt{n}$, in which case it is clear that $\rho_2(n) = P(n)$. Conversely, if $\rho_2(n) = P(n)$, we have $\sqrt{n} \le \rho_2(n) = P(n)$, which proves our claim. □

Lemma 6 *Given any integer* $n \ge 2$, *we have*

$$\rho_1(n) \ge \sqrt{\frac{n}{P(n)}}. \tag{2}$$

Proof Clearly, inequality (2) is equivalent to

$$\rho_2(n) \le \sqrt{nP(n)}. \tag{3}$$

First consider the case where $P(n) \ge \sqrt{n}$. In this case, in light of Lemma 5, we have

$$\rho_2(n) = P(n) = \sqrt{P(n)P(n)} \le \sqrt{nP(n)},$$

thus proving (3) in this case. Let us now assume that $n^{1/3} \le P(n) < \sqrt{n}$. In this case, we have

$$\frac{n}{P(n)} > \frac{n}{\sqrt{n}} = \sqrt{n}$$

which implies that $n/P(n)$ is a divisor of n which is larger than $\sqrt{n}$, and therefore that

$$\rho_2(n) \le \frac{n}{P(n)}. \tag{4}$$

On the other hand, since in this case, $n^{1/3} \le P(n)$, we have successively

$$n \le P^3(n), \quad \frac{n}{P^2(n)} \le P(n), \quad \frac{n^2}{P^2(n)} \le nP(n),$$

which in light of (4) implies that

$$\rho_2(n) \le \frac{n}{P(n)} = \sqrt{\frac{n^2}{P^2(n)}} \le \sqrt{nP(n)},$$

thus proving (3) in this case.

It remains to consider the case where $P(n) < n^{1/3}$. In this case, we may write n as $n = p_1 p_2 \cdots p_k$, where $p_1 \ge p_2 \ge \cdots \ge p_k$ are primes and $k \ge 4$. Let r be the smallest positive integer such that $p_1 p_2 \cdots p_r > \sqrt{n}$. Then, the numbers

$$A := p_1 p_2 \cdots p_{r-1} \quad \text{and} \quad B := \frac{n}{p_1 p_2 \cdots p_r}$$

are two divisors of n no larger than $\sqrt{n}$ and therefore $\ge \rho_1(n)$. If we can show that we have either

$$A \ge \sqrt{\frac{n}{P(n)}} \quad \text{or} \quad B \ge \sqrt{\frac{n}{P(n)}}, \tag{5}$$

estimate (2) will follow.

To prove (5), we will show that assuming that $A < \sqrt{\frac{n}{P(n)}}$ implies that $B > \sqrt{\frac{n}{P(n)}}$. Indeed, if $A < \sqrt{\frac{n}{P(n)}}$, we have that

$$B = \frac{n}{Ap_r} > \frac{n\sqrt{P(n)}}{\sqrt{n}p_r} = \frac{\sqrt{n}\sqrt{P(n)}}{p_r} > \frac{\sqrt{n}\sqrt{P(n)}}{P(n)} = \sqrt{\frac{n}{P(n)}},$$

thus proving (5) and completing the proof of Lemma 6. □

4 Proof of Theorem 1

As we will see, the main contribution to the sum $S(x)$ comes from those integers n for which $P(n) > \sqrt{n}$. First we write

$$S(x) = \sum_{\substack{6 \le n \le x \\ n \ne \text{prime},\ n \ne m^2}} \frac{\log \rho_2(n)}{\log \rho_1(n)} + O(\sqrt{x})$$

$$
\begin{aligned}
&= \sum_{\substack{6\le n\le x\\ n\neq \text{prime},\ n\neq m^2\\ P(n)<\sqrt{n}}} \frac{\log\rho_2(n)}{\log\rho_1(n)} + \sum_{\substack{6\le n\le x\\ n\neq \text{prime},\ n\neq m^2\\ P(n)>\sqrt{n}}} \frac{\log\rho_2(n)}{\log\rho_1(n)} + O(\sqrt{x})\\
&= S_1(x)+S_2(x)+O(\sqrt{x}), \qquad (6)
\end{aligned}
$$

say. Clearly, in the first sum, since $P(n)<\sqrt{n}$, we have that $P(n)\le\rho_1(n)<\sqrt{n}$ and therefore,

$$
\begin{aligned}
S_1(x) &= \sum_{\substack{n\le x\\ n\neq \text{prime},\ n\neq m^2\\ P(n)<\sqrt{n}}} \frac{\log n-\log\rho_1(n)}{\log\rho_1(n)} \le \sum_{\substack{n\le x\\ n\neq \text{prime},\ n\neq m^2\\ P(n)<\sqrt{n}}} \frac{\log n-\log P(n)}{\log P(n)}\\
&< \sum_{\substack{n\le x\\ n\neq \text{prime},\ n\neq m^2\\ P(n)<\sqrt{x}}} \frac{\log n-\log P(n)}{\log P(n)} = \sum_{p<\sqrt{x}} \sum_{\substack{mp\le x,\ m\ge 2\\ P(m)\le p}} \frac{\log(mp)-\log p}{\log p}\\
&= \sum_{p<\sqrt{x}} \frac{1}{\log p} \sum_{\substack{2\le m\le x/p\\ P(m)\le p}} \log m = \sum_{p<\sqrt{x}} \frac{1}{\log p}\int_2^{x/p} \log t\cdot d\,\Psi(t,p)\\
&= \sum_{p<\sqrt{x}} \frac{1}{\log p}\left\{ \log t\cdot\Psi(t,p)\Big|_2^{x/p} - \int_2^{x/p}\Psi(t,p)\frac{dt}{t}\right\}\\
&< \sum_{p<\sqrt{x}} \frac{\log x-\log p}{\log p}\Psi\left(\frac{x}{p},p\right)\\
&< \log x\sum_{p<\sqrt{x}} \frac{1}{\log p}\Psi\left(\frac{x}{p},p\right) = \log x\cdot S_0(x), \qquad (7)
\end{aligned}
$$

say. Using Proposition 3, we find that

$$
\begin{aligned}
S_0(x) &\ll \int_2^{\sqrt{x}} \frac{1}{\log^2 t}\Psi\left(\frac{x}{t},t\right)\,dt\\
&\ll \int_2^{\sqrt{x}} \frac{1}{\log^2 t}\frac{x}{t}\exp\left\{-\frac{1}{2}\frac{\log x-\log t}{\log t}\right\}\,dt\\
&\ll x\int_{\log 2}^{\frac{1}{2}\log x} \frac{1}{u^2\exp\{\frac{1}{2}\frac{\log x}{u}\}}\,du, \qquad (8)
\end{aligned}
$$

where we used the change of variable $u=\log t$. Now, it is clear that the function $f(u):=\dfrac{1}{u^2\exp\{\frac{1}{2}\frac{\log x}{u}\}}$ is increasing at $u=\log 2$ and decreasing at $u=\frac{1}{2}\log x$, reaching its maximum value in between these two values. In order to find this maximum, we set $g(u):=\log f(u)=-2\log u-\frac{1}{2}\frac{\log x}{u}$ and search for which value of u we have $g'(u) = 0$. Since

$$g'(u) = -\frac{2}{u} + \frac{1}{2}\frac{\log x}{u^2},$$

we easily see that $g'(u) = 0$ when $u = \log x/4$. Since

$$f\left(\frac{\log x}{4}\right) = \left(\frac{e^2}{16} \cdot \log^2 x\right)^{-1},$$

we conclude from (8) that

$$\begin{aligned} S_0(x) &\ll x\left(\frac{1}{2}\log x - \log 2\right) \max_{\log 2 \le u \le \frac{1}{2}\log x} f(u) \\ &= x\left(\frac{1}{2}\log x - \log 2\right)\left(\frac{e^2}{16} \cdot \log^2 x\right)^{-1} \ll \frac{x}{\log x}. \end{aligned}$$

Using this last bound in (7), we obtain that

$$S_1(x) \ll x. \tag{9}$$

It remains to estimate $S_2(x)$. Using Lemma 5, we may write

$$\begin{aligned} S_2(x) &= \sum_{\substack{6 \le n \le x,\ n \ne \text{prime} \\ P(n) > \sqrt{n}}} \frac{\log P(n)}{\log n - \log P(n)} \\ &= \sum_{\substack{6 \le n \le x,\ n \ne \text{prime} \\ P(n) > \sqrt{x}}} \frac{\log P(n)}{\log n - \log P(n)} - E(x), \end{aligned} \tag{10}$$

where $E(x)$ accounts for the error introduced by replacing the condition $P(n) > \sqrt{n}$ by $P(n) > \sqrt{x}$, that is,

$$E(x) = \sum_{\substack{6 \le n \le x,\ n \ne \text{prime} \\ \sqrt{n} < P(n) < \sqrt{x}}} \frac{\log \rho_2(n)}{\log \rho_1(n)}.$$

Observing that $\log \rho_2(n)/\log \rho_1(n) < 2\log n$ for all composite integers $n > 1$, it follows that, as $x \to \infty$,

$$\begin{aligned} E(x) &< \sum_{\substack{6 \le n \le x/\log^2 x,\ n \ne \text{prime} \\ \sqrt{n} < P(n) < \sqrt{x}}} 2\log n + \sum_{\substack{x/\log^2 x < n \le x,\ n \ne \text{prime} \\ \sqrt{n} < P(n) < \sqrt{x}}} \frac{\log P(n)}{\log n - \log P(n)} \\ &\le 2\log x \cdot \frac{x}{\log^2 x} + \sum_{\substack{x/\log^2 x < n \le x,\ n \ne \text{prime} \\ \sqrt{x}/\log x < P(n) < \sqrt{x}}} \frac{\log P(n)}{\log n - \log P(n)} \end{aligned}$$

$$\ll \frac{x}{\log x} + \sum_{\frac{\sqrt{x}}{\log x}<p<\sqrt{x}} \sum_{\substack{\frac{x}{\log^2 x}<mp\le x \\ P(m)\le p}} \frac{\log p}{\log m}$$
$$< \frac{x}{\log x} + \sum_{\frac{\sqrt{x}}{\log x}<p<\sqrt{x}} \log p \sum_{\frac{x}{p\log^2 x}<m\le \frac{x}{p}} \frac{1}{\log m}$$
$$< \frac{x}{\log x} + \sum_{\frac{\sqrt{x}}{\log x}<p<\sqrt{x}} \frac{\log p}{\log x - \log p - 2\log\log x}\cdot\frac{x}{p}$$
$$\ll \frac{x}{\log x} + \frac{x}{\log x}\sum_{\frac{\sqrt{x}}{\log x}<p<\sqrt{x}} \frac{\log p}{p} \ll \frac{x}{\log x}\log\log x = o(x). \tag{11}$$

We now move to estimate

$$\widetilde{S_2(x)} := \sum_{\substack{n\le x,\ n\ne \text{prime} \\ P(n)>\sqrt{x}}} \frac{\log P(n)}{\log n - \log P(n)}.$$

We have

$$\widetilde{S_2(x)} = \sum_{\sqrt{x}<p\le x/2} \sum_{\substack{mp\le x,\ m\ge 2 \\ P(m)\le p}} \frac{\log p}{\log(mp)-\log p} = \sum_{\sqrt{x}<p\le x/2} \log p \sum_{\substack{2\le m\le x/p \\ P(m)\le p}} \frac{1}{\log m}$$
$$= \sum_{\sqrt{x}<p\le x/2} \log p \int_2^{x/p} \frac{1}{\log t} d\,\Psi(t,p)$$
$$= \sum_{\sqrt{x}<p\le x/2} \log p \left\{ \frac{\Psi(t,p)}{\log t}\Bigg|_2^{x/p} + \int_2^{x/p} \frac{\Psi(t,p)}{t\log^2 t}\,dt \right\}. \tag{12}$$

In the range $\sqrt{x} < p \le x/2$, it is clear that for $t \in [2, x/p]$, we have $t \le x/p < \sqrt{x}$, implying that $\Psi(t,p) = t$. Using this in (12), we find that

$$\widetilde{S_2(x)} = \sum_{\sqrt{x}<p\le x/2} \frac{x\log p}{p\log(x/p)} + \sum_{\sqrt{x}<p\le x/2} \log p \int_2^{x/p} \frac{1}{\log^2 t}\,dt$$
$$= S_3(x) + S_4(x), \tag{13}$$

say. On the one hand,

$$S_3(x) = \left(1 + O\left(\frac{1}{\log x}\right)\right) \int_{\sqrt{x}}^{x/2} \frac{x}{t}\cdot\frac{1}{\log(x/t)}\,dt$$
$$= \left(1 + O\left(\frac{1}{\log x}\right)\right) I_3(x), \tag{14}$$

say. By using the change of variable $u = x/t$, we obtain

$$I_3(x) = x \int_2^{\sqrt{x}} \frac{du}{u \log u} = x \log\log x + O(x). \tag{15}$$

Using (15) in (14), we find that

$$S_3(x) = x \log\log x + O(x). \tag{16}$$

On the other hand, it is immediate that

$$S_4(x) \ll \sum_{\sqrt{x}<p\le x/2} \log p \cdot \frac{x}{p} \cdot \frac{1}{\log^2(x/p)} \ll \int_{\sqrt{x}}^{x/2} \frac{x}{t} \cdot \frac{1}{\log^2(x/t)}\, dt.$$

Again, using the change of variable $u = x/t$, we obtain

$$S_4(x) \ll x \int_2^{\sqrt{x}} \frac{du}{u \log^2 u}\, du = O(x). \tag{17}$$

Gathering estimates (16) and (17) in (13), we get

$$\widetilde{S_2(x)} = x \log\log x + O(x). \tag{18}$$

Putting (18) and (11) in (10) yields

$$S_2(x) = x \log\log x + O(x). \tag{19}$$

Using (9) and (19) in (6) completes the proof of Theorem 1.

5 Proof of Theorem 2

We begin by splitting the sum $T(x)$ as follows.

$$\begin{aligned} T(x) &= \sum_{\substack{2\le n\le x\\ P(n)>\sqrt{x}}} \frac{\log \rho_1(n)}{\log \rho_2(n)} + \sum_{\substack{2\le n\le x\\ P(n)\le\sqrt{x}}} \frac{\log \rho_1(n)}{\log \rho_2(n)} \\ &= T_1(x) + T_2(x), \end{aligned} \tag{20}$$

say.

We first establish that

$$T_1(x) = (1 - \log 2)\, x + O\left(\frac{x}{\log x}\right). \tag{21}$$

Indeed, in light of Lemma 5, we have that

$$
\begin{aligned}
T_1(x) &= \sum_{\sqrt{x}<p\le x} \sum_{\substack{mp\le x\\ P(m)\le p}} \frac{\log(mp)-\log p}{\log p}\\
&= \sum_{\sqrt{x}<p\le x} \frac{1}{\log p} \sum_{\substack{m\le x/p\\ P(m)\le p}} \log m = \sum_{\sqrt{x}<p\le x} \frac{1}{\log p} \int_1^{x/p} \log t \cdot d\,\Psi(t,p)\\
&= \sum_{\sqrt{x}<p\le x} \frac{1}{\log p} \left\{ \log t \cdot \Psi(t,p)\Big|_1^{x/p} - \int_1^{x/p} \frac{\Psi(t,p)}{t}\,dt \right\}\\
&= \sum_{\sqrt{x}<p\le x} \frac{1}{\log p} \left\{ \log(x/p)\Psi(x/p,p) - \int_1^{x/p} dt \right\}\\
&= \sum_{\sqrt{x}<p\le x} \frac{1}{\log p} \cdot \log(x/p) \cdot \frac{x}{p} - x \sum_{\sqrt{x}<p\le x} \frac{1}{p\log p}\\
&= T_1'(x) + T_1''(x),
\end{aligned}
\tag{22}
$$

say, where once more, we used the fact that, because $\sqrt{x} < p \le x/2$ and $t \in [2, x/p]$, we have $t \le x/p < \sqrt{x}$, implying that in those ranges, $\Psi(x/p,p) = x/p$ and $\Psi(t,p) = t$. Now, it is clear that

$$
T_1''(x) \ll x \int_{\sqrt{x}}^{x} \frac{1}{t\log^2 t}\,dt = x \frac{-1}{\log t}\Big|_{\sqrt{x}}^{x} \ll \frac{x}{\log x}. \tag{23}
$$

On the other hand, letting $\pi(x)$ stand for the number of prime numbers not exceeding x and using the prime number theorem in the form $\pi(x) = x/\log x + O(x/\log^2 x)$, we have

$$
\begin{aligned}
T_1'(x) &= \int_{\sqrt{x}}^{x} \frac{1}{\log t}\frac{x}{t} \log\left(\frac{x}{t}\right) d\pi(t)\\
&= \frac{1}{\log t}\frac{x}{t}\log\left(\frac{x}{t}\right)\pi(t)\Big|_{\sqrt{x}}^{x} - \int_{\sqrt{x}}^{x} \pi(t)\frac{d}{dt}\left(\frac{1}{\log t}\cdot\frac{x}{t}\cdot\log\left(\frac{x}{t}\right)\right) dt\\
&= \frac{x}{\log^2 t}\log\left(\frac{x}{t}\right)\Big|_{\sqrt{x}}^{x} + \int_{\sqrt{x}}^{x} \pi(t)\frac{x}{t^2}\frac{1}{\log t}\log\left(\frac{x}{t}\right)\left(1+O\left(\frac{1}{\log x}\right)\right) dt\\
&= O\left(\frac{x}{\log x}\right) + \int_{\sqrt{x}}^{x} \frac{t}{\log t}\left(1+O\left(\frac{1}{\log t}\right)\right)\frac{x}{t^2}\frac{1}{\log t}\log\left(\frac{x}{t}\right) dt\\
&= O\left(\frac{x}{\log x}\right) + \left(1+O\left(\frac{1}{\log x}\right)\right)\int_{\sqrt{x}}^{x} \frac{1}{\log^2 t}\frac{x}{t}\log\left(\frac{x}{t}\right) dt.
\end{aligned}
\tag{24}
$$

Using the change of variable $u = x/t$ in this last integral, we obtain

$$I(x) := \int_{\sqrt{x}}^{x} \frac{1}{\log^2 t} \frac{x}{t} \log\left(\frac{x}{t}\right) dt = \int_1^{\sqrt{x}} \frac{\log u}{\log^2(x/u)} \frac{x}{u} \, du.$$

Doing yet another change of variable, this time setting $w = \log x - \log u$, we get

$$I(x) = x \int_{\frac{1}{2}\log x}^{\log x} \frac{\log x - w}{w^2} \, dw = x(1 - \log 2).$$

Substituting this value of $I(x)$ in (24), we obtain

$$T_1'(x) = (1 - \log 2)x + O\left(\frac{x}{\log x}\right). \tag{25}$$

Combining (23) and (25) in (22) proves (21).

We will now prove that

$$\left(\int_1^2 \frac{1 - \log u}{u(u+1)} \, du + \int_3^\infty \frac{u-1}{u+1} \frac{\rho(u-1)}{u} \, du\right) x + O\left(\frac{x}{\log x} \log\log x\right)$$
$$< T_2(x) < (1 - \log 2)x + O\left(\frac{x}{\log x}\right). \tag{26}$$

First, the lower bound. For this, we split the sum $T_2(x)$ as follows.

$$T_2(x) = \sum_{\substack{2 \le n \le x \\ n^{1/3} \le P(n) < \sqrt{x}}} \frac{\log \rho_1(n)}{\log \rho_2(n)} + \sum_{\substack{2 \le n \le x \\ P(n) < n^{1/3}}} \frac{\log \rho_1(n)}{\log \rho_2(n)} = U_1(x) + U_2(x), \tag{27}$$

say.

Since in $U_1(x)$, we count those integers $n \in [2, x]$ for which $P(n) < \sqrt{n}$, it follows from the inequalities $P(n) \le \rho_1(n) < \sqrt{n}$ that

$$\begin{aligned} U_1(x) &= \sum_{\substack{2 \le n \le x \\ n^{1/3} \le P(n) < \sqrt{x}}} \frac{\log P(n)}{\log n - \log P(n)} \\ &= \sum_{x^{1/3} \le p < \sqrt{x}} \sum_{\substack{mp \le x \\ P(m) \le p}} \frac{\log p}{\log(mp) - \log p} - E_2(x), \end{aligned} \tag{28}$$

where

$$E_2(x) = \sum_{n^{1/3} \le p < x^{1/3}} \sum_{\substack{mp \le x \\ P(m) \le p}} \frac{\log p}{\log(mp) - \log p}.$$

To show that $E_2(x)$ is "small" compared to the main term in (28), we proceed essentially along the same lines as with the evaluation of $E(x)$ in (11), that is by first putting aside those $n \le x$ which are larger that $x/\log x$, thus introducing an error no larger than $x/\log x$, so that we may write that

$$\begin{aligned} E_2(x) &\ll \frac{x}{\log x} + \sum_{\left(\frac{x}{\log x}\right)^{1/3} \le p < x^{1/3}} \sum_{\substack{mp \le x \\ P(m) \le p}} 1 \\ &\le \frac{x}{\log x} + \sum_{\left(\frac{x}{\log x}\right)^{1/3} \le p < x^{1/3}} \frac{x}{p} \\ &= \frac{x}{\log x} + x \left(\log\log x^{1/3} - \log\log \left(\left(\frac{x}{\log x} \right)^{1/3} \right) + O\left(\frac{1}{\log x} \right) \right) \\ &\ll \frac{x \log\log x}{\log x}. \end{aligned} \tag{29}$$

Getting back to the double sum in (28), we have

$$\begin{aligned} \sum_{x^{1/3} \le p < \sqrt{x}} \sum_{\substack{mp \le x \\ P(m) \le p}} \frac{\log p}{\log(mp) - \log p} &= \sum_{x^{1/3} \le p < \sqrt{x}} \log p \sum_{\substack{2 \le m \le x/p \\ P(m) \le p}} \frac{1}{\log m} \\ &= \sum_{x^{1/3} \le p < \sqrt{x}} \log p \int_2^{x/p} \frac{1}{\log t} d\, \Psi(t,p) \\ &= \sum_{x^{1/3} \le p < \sqrt{x}} \log p \left\{ \frac{\Psi(t,p)}{\log t} \Big|_2^{x/p} + \int_2^{x/p} \frac{\Psi(t,p)}{t \log^2 t}\, dt \right\} \\ &> \sum_{x^{1/3} \le p < \sqrt{x}} \frac{\log p}{\log(x/p)} \Psi(x/p, p) \\ &= U_1'(x), \end{aligned} \tag{30}$$

say.

Using Proposition 4, it follows that

$$\begin{aligned} U_1'(x) &= \sum_{x^{1/3} \le p < \sqrt{x}} \frac{\log p}{\log(x/p)} \left(\frac{x}{p} \rho \left(\frac{\log x}{\log p} - 1 \right) + O\left(\frac{x}{p \log p} \right) \right) \\ &= \sum_{x^{1/3} \le p < \sqrt{x}} \frac{\log p}{\log(x/p)} \frac{x}{p} \rho \left(\frac{\log x}{\log p} - 1 \right) + O\left(\sum_{x^{1/3} \le p < \sqrt{x}} \frac{x/p}{\log(x/p)} \right) \\ &= \int_{x^{1/3}}^{\sqrt{x}} \frac{\log t}{\log(x/t)} \frac{x}{t} \rho \left(\frac{\log x}{\log t} - 1 \right) d\,\pi(t) + O\left(\frac{x}{\log x} \sum_{x^{1/3} \le p < \sqrt{x}} \frac{1}{p} \right) \end{aligned}$$

$$= \int_{x^{1/3}}^{\sqrt{x}} \frac{1}{\log(x/t)} \frac{x}{t} \rho\left(\frac{\log x}{\log t} - 1\right)\left(1 + O\left(\frac{1}{\log t}\right)\right) dt + O\left(\frac{x}{\log x}\right)$$
$$= x\left(1 + O\left(\frac{1}{\log x}\right)\right) \int_2^3 \frac{\rho(u-1)}{(u-1)u}\, du + O\left(\frac{x}{\log x}\right), \tag{31}$$

where we used the change of variable $u = \log x/\log t$. Using (31) in (30) and taking into account (29), it follows from estimate (28) that

$$U_1(x) > x \int_1^2 \frac{\rho(u)}{u(u+1)}\, du + O\left(\frac{x}{\log x}\right) = x \int_1^2 \frac{1 - \log u}{u(u+1)}\, du + O\left(\frac{x}{\log x}\right). \tag{32}$$

As for the second sum in (27), we have, in light of Lemma 6,

$$U_2(x) = \sum_{\substack{2 \le n \le x \\ P(n) < n^{1/3}}} \frac{\log \rho_1(n)}{\log n - \log \rho_1(n)} \ge \sum_{\substack{2 \le n \le x \\ P(n) < n^{1/3}}} \frac{\frac{1}{2}\log n - \frac{1}{2}\log P(n)}{\frac{1}{2}\log n + \frac{1}{2}\log P(n)}$$
$$= \sum_{p < x^{1/3}} \sum_{\substack{mp \le x \\ P(m) \le p}} \frac{\log m}{\log m + 2\log p} - E_3(x), \tag{33}$$

where $E_3(x)$ accounts for the error term created by counting those integers $n \in [2, x]$ such that $n^{1/3} \le P(n) < x^{1/3}$. Using the same technique as that employed earlier to evaluate the size of $E_2(x)$, we find that

$$E_3(x) \ll \frac{x \log\log x}{\log x}. \tag{34}$$

We now evaluate the inner sum on the right hand side of (33), writing it as a Stieltjes integral and thereafter using integration by parts,

$$\sum_{\substack{m \le x/p \\ P(m) \le p}} \frac{\log m}{\log m + 2\log p} = \int_1^{x/p} \frac{\log t}{\log t + 2\log p} d\,\Psi(t,p)$$
$$= \frac{\log t}{\log t + 2\log p} \Psi(x,p)\Big|_1^{x/p}$$
$$- \int_1^{x/p} \frac{\Psi(t,p)}{t}\left(\frac{1}{\log t + 2\log p} + \frac{\log t}{(\log t + 2\log p)^2}\right) dt$$
$$= A(x,p) - B(x,p), \tag{35}$$

say.

On the one hand, observing that for each $p < x^{1/3}$, we have $x^{2/3} < \frac{x}{p} < x$ and using Proposition 4, we may write that

$$\begin{aligned}\sum_{p<x^{1/3}} A(x,p) &= \left(1+O\left(\frac{1}{\log x}\right)\right)\int_2^{x^{1/3}} \frac{\log x - \log t}{\log x + \log t}\frac{x}{t}\rho\left(\frac{\log x}{\log t}-1\right)d\pi(t)\\ &= x\left(1+O\left(\frac{1}{\log x}\right)\right)\\ &\qquad \times\int_2^{x^{1/3}} \frac{\log x - \log t}{\log x + \log t}\frac{1}{t\log t}\rho\left(\frac{\log x}{\log t}-1\right)dt\\ &= x\int_3^\infty \frac{1-1/u}{1+1/u}\frac{\rho(u-1)}{u}\,du + O\left(\frac{x}{\log x}\right)\\ &= x\int_3^\infty \frac{u-1}{u+1}\frac{\rho(u-1)}{u}\,du + O\left(\frac{x}{\log x}\right), \qquad (36)\end{aligned}$$

where we used the change of variable $u = \log x/\log t$.

To complete our evaluation, we will show that

$$\sum_{p<x^{1/3}} B(x,p) \ll \frac{x}{\log x}. \qquad (37)$$

Indeed, first we have

$$\begin{aligned}\sum_{p<x^{1/3}} B(x,p) &\ll \sum_{p<x^{1/3}} \int_2^{x/p} \frac{\Psi(x,p)}{t\log t}\,dt\\ &= \sum_{p<x^{1/3}} \int_2^{x^{2/3}} \frac{\Psi(x,p)}{t\log t}\,dt + \sum_{p<x^{1/3}} \int_{x^{2/3}}^{x/p} \frac{\Psi(x,p)}{t\log t}\,dt\\ &= I_1(x)+I_2(x), \qquad (38)\end{aligned}$$

say. On the one hand, using Proposition 3, we obtain that

$$\begin{aligned}I_1(x) &\ll \sum_{p<x^{1/3}} \int_2^{x^{2/3}} \exp\left\{-\frac{1}{2}\frac{\log t}{\log p}\right\}\frac{dt}{\log t}\\ &\le \sum_{p<x^{1/3}} \exp\left\{-\frac{1}{3}\frac{\log x}{\log p}\right\}\int_2^{x^{2/3}} \frac{dt}{\log t}\\ &\le e^{-1}\sum_{p<x^{1/3}} \int_2^{x^{2/3}} \frac{dt}{\log t} \ll \pi(x^{1/3})\frac{x^{2/3}}{\log x} \ll \frac{x}{\log^2 x}, \qquad (39)\end{aligned}$$

where we used the fact that $\displaystyle\int_2^y \frac{dt}{\log t} \ll \frac{y}{\log y}$ and the Chebyshev inequality $\pi(y) \ll y/\log y$.

On the other hand, again using Proposition 3, we get that

$$\begin{aligned} I_2(x) &\ll \sum_{p<x^{1/3}} \frac{1}{\log x} \int_{x^{2/3}}^{x/p} \exp\left\{-\frac{1}{3}\frac{\log x}{\log p}\right\} dt \\ &< \frac{1}{\log x} \sum_{p<x^{1/3}} \exp\left\{-\frac{1}{3}\frac{\log x}{\log p}\right\} \frac{x}{p} \\ &\ll \frac{x}{\log x} \int_2^{x^{1/3}} \exp\left\{-\frac{1}{3}\frac{\log x}{\log t}\right\} \frac{dt}{t\log t}. \end{aligned} \tag{40}$$

Since

$$\begin{aligned} \int_2^{x^{1/3}} \exp\left\{-\frac{1}{3}\frac{\log x}{\log t}\right\} \frac{dt}{t\log t} &\le \int_2^{\exp(\sqrt{\log x})} \exp\left\{-\frac{1}{3}\frac{\log x}{\log t}\right\} \frac{dt}{t\log t} \\ &\qquad + \int_{\exp(\sqrt{\log x})}^{x^{1/3}} \exp\left\{-\frac{1}{3}\frac{\log x}{\log t}\right\} \frac{dt}{t\log t} \\ &\le \exp\left\{-\frac{1}{3}\sqrt{\log x}\right\} \int_2^{\exp(\sqrt{\log x})} \frac{dt}{t\log t} \\ &\qquad + \int_{\sqrt{\log x}}^{\frac{1}{3}\log x} \exp\left\{-\frac{1}{3}\frac{\log x}{u}\right\} \frac{du}{u} \\ &\ll \exp\left\{-\frac{1}{3}\sqrt{\log x}\right\} \log\log e^{\sqrt{\log x}} \\ &\qquad + \int_3^{\sqrt{\log x}} \frac{1}{ve^{\frac{1}{3}v}}\, dv \\ &\ll \exp\left\{-\frac{1}{3}\sqrt{\log x}\right\} \log\log x + O(1) = O(1), \end{aligned}$$

estimate (40) can be replaced by

$$I_2(x) \ll \frac{x}{\log x}. \tag{41}$$

Using (39) and (41) in (38), we obtain (37). Then, gathering estimates (36) and (37) in (35) and taking into account (34), we get from (33) that

$$U_2(x) \ll \frac{x\log\log x}{\log x}.$$

Using this last bound along with inequality (32) in (27) completes the proof of the first inequality in (26).

Finally, it follows from Proposition 4 that

$$T_2(x) < \sum_{\substack{n\le x\\ P(n)\le\sqrt{x}}} 1 = \Psi(x,\sqrt{x}) = x\rho(2)+O\left(\frac{x}{\log x}\right) = x(1-\log 2)+O\left(\frac{x}{\log x}\right),$$

which establishes the second inequality in (26).

To complete the proof of Theorem 2, we only need to observe that the first inequality in (1) follows from relations (20), (21) and the first inequality in (26), whereas the second inequality in (1) follows from relations (20), (21) and the second inequality in (26).

6 Final remark

Regarding possible improvements to Theorem 2, one might wonder if there exists a positive constant c such that $T(x) = (c + o(1))x$ as $x \to \infty$. If there is such a constant c, using a computer to calculate $T(x)$ up to $x = 4 \times 10^9$ seems to indicate that c is approximately 0.566.

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Declarations

The authors have no conflicts of interest.

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