Integers with a sum of co-divisors yielding a square

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Abstract

Finding elliptic curves with high ranks has been the focus of much research. Recently, with the goal of generating elliptic curves with a large rank, some authors used large integers n which have many divisors, amongst which one can find divisors d such that $d + n/d$ is a perfect square. This strategy is in itself a motivation for studying the function $\tau_\square(n)$ which counts the number of divisors d of an integer n for which $d + n/d$ is a perfect square. We show that $\sum_{n\le x} \tau_\square(n) = c_\square x^{3/4} + O(\sqrt{x})$ for some explicit constant $c_\square$. Moreover, letting $\rho_1(n) := \max\{d \mid n : d \le \sqrt{n}\}$ and $\rho_2(n) := \min\{d \mid n : d \ge \sqrt{n}\}$ stand for the middle divisors of n, we show that the order of magnitude of the number of positive integers $n \le x$ for which $\rho_1(n) + \rho_2(n)$ is a perfect square is $x^{3/4}/\log x$.

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1 The connection with elliptic curves

Let E be an elliptic curve over $\mathbb{Q}$. According to the Mordell-Weil theorem, the set $E(\mathbb{Q})$ of rational points $(x, y) \in E$ is a finitely generated abelian group, whose structure is given by $E(\mathbb{Q}) = T \bigoplus \mathbb{Z}^r$, where T is a finite torsion group and r is the *rank* of the elliptic curve. It is widely believed (but not yet proved) that there is no maximal rank for an elliptic curve. Nevertheless, the highest rank ever found is 28 (N.D. Elkies, 2006). Since there are no known algorithm for establishing the rank of an elliptic curve, finding elliptic curves with a large rank can end up being quite a challenge. It is in this context that in 1974 and 1975, Penney and Pomerance ([7], [8]) had the idea of considering the group of rational points of the elliptic curve

$$y^2 = x^3 + ax^2 + bx,$$

where $a, b \in \mathbb{Z}$ and $a^2 - 4b$ is not a perfect square and then examine those divisors d of b with the property that $d + b/d + a$ is an integral perfect square. They found that with an appropriate choice of the integers a and b, they could establish that the corresponding elliptic curve is of rank 7.

Later, Aguirre, Castañeda, and Peral [1], using a slightly different approach, came up with an elliptic curve of rank 8. More specifically, they considered the elliptic curve

$$y^2 = x^3 + Bx, \tag{1}$$

where B is a large negative integer with many divisors, amongst which at least two distinct ones, d_1 and d_2, are such that the numbers $d_1 + B/d_1$ and $d_2 + B/d_2$ are both perfect squares. Then, by choosing

$$B = -14\,752\,493\,461\,692 = -2^2 \cdot 3^2 \cdot 7 \cdot 23 \cdot 71 \cdot 113 \cdot 281 \cdot 1129$$

(which has 576 positive divisors), they identified eight positive divisors $d_1 < \cdots < d_8$ of B which have the property that $d_i + B/d_i$ is a perfect square for $i = 1, \ldots, 8$, and then, using a clever argument, were able to show that the elliptic curve (1) is of rank 8.

Further exploiting their method, Aguirre, Castañeda, and Peral [2] later found elliptic curves of rank 13.

The fact that such achievements rely essentially on numbers n with many pairs of co-divisors d and n/d whose sum is a perfect square is certainly a motivation for investigating the function $n \mapsto \#\{d \mid n : d + n/d \text{ is a square}\}$. On the other hand, since the numbers

$$\rho_1(n) := \max\{d \mid n : d \leq \sqrt{n}\} \quad \text{and} \quad \rho_2(n) := \min\{d \mid n : d \geq \sqrt{n}\},$$

called the *middle divisors* of n, are of special interest for number theorists (see for instance the papers of Tenenbaum [10] and Ford [4]), we also investigate the particular case of those integers n for which $\rho_1(n) + \rho_2(n)$ is a perfect square.

2 Main results

Given a divisor d of n, we say that d and n/d are *co-divisors* of n. We first introduce the function

$$\tau_\square(n) := \#\left\{d \mid n : d + \frac{n}{d} = c^2 \text{ for some } c \in \mathbb{N}\right\}$$

and the sum $\mathcal{T}(x) := \sum_{n \le x} \tau_\square(n)$. We also introduce the set

$$\mathcal{R} := \{n \in \mathbb{N} : \rho_1(n) + \rho_2(n) = c^2 \text{ for some } c \in \mathbb{N}\}$$

and its counting function $\mathcal{R}(x) := \#\{n \le x : n \in \mathcal{R}\}$.

Our main results are the following.

Theorem 1. *With* $c_\square := \dfrac{4\sqrt{2\pi}\Gamma(5/4)}{3\Gamma(3/4)} \approx 2.4721$, *where* Γ *stands for the Gamma function, we have*

$$\mathcal{T}(x) = c_\square x^{3/4} + O\left(\sqrt{x}\right),$$

so that in particular

$$\tau_\square(n) = 0 \qquad a.e.$$

Theorem 2. *Letting* $c_\square$ *be defined as in Theorem 1, then for* x *sufficiently large,*

$$c_\square \frac{x^{3/4}}{\log x} < \mathcal{R}(x) < 2c_\square \frac{x^{3/4}}{\log x}. \tag{2}$$

3 Preliminary results

Given an integer $a \ge 2$, consider the number $n = a^2 - 1$. Since 1 and $a^2 - 1$ are co-divisors of n and since $1 + a^2 - 1 = a^2$, it follows that $\tau_\square(n) \ge 1$ and therefore that $\mathcal{T}(x) \gg \sqrt{x}$. On the other hand, if $a \ge 3$ is an arbitrary odd integer, then there exists $m \in \mathbb{N}$ such that $a^2 = 2m + 1$. Setting $n = m(m+1)$, so that $\rho_1(n) = m$ and $\rho_2(n) = m + 1$, we find that $\rho_1(n) + \rho_2(n) = a^2$. It then easily follows from this observation that $\mathcal{R}(x) \gg \sqrt{x}$.

With little effort, we have thus established somewhat weak lower bounds for $\mathcal{T}(x)$ and $\mathcal{R}(x)$. Clearly we can do better.

First, some notation and preliminary results. Given an integer $n \ge 2$, we let $P(n)$ stand for its largest prime factor. We let $\pi(x)$ stand for the number of primes not exceeding x. In what follows, the letter p will always represent a prime number, so that in particular we may write $\pi(x) = \sum_{p \le x} 1$. We will often be using the prime number theorem in the form

$$\pi(x) = \frac{x}{\log x} + R(x), \qquad \text{where } R(x) = O\left(\frac{x}{\log^2 x}\right). \tag{3}$$

Frequently, given a large number x, we shall encounter sums running over primes $p \le x$, such as the ones given in the following result.

Lemma 1. *For large X,*

(a) $\displaystyle\sum_{p\le X}\sqrt{p}=\frac{2}{3}\frac{X^{3/2}}{\log X}\left(1+O\left(\frac{1}{\log X}\right)\right).$

(b) $\displaystyle\sum_{p\le X}\frac{1}{\sqrt{p}}=2\frac{\sqrt{X}}{\log X}\left(1+O\left(\frac{1}{\log X}\right)\right).$

Proof. We start with part (a). We separate the sum in two parts as follows.

$$\sum_{p\le X}\sqrt{p}=\sum_{p<\sqrt{X}}\sqrt{p}+\sum_{\sqrt{X}\le p\le X}\sqrt{p}. \tag{4}$$

On the one hand,

$$\sum_{p<\sqrt{X}}\sqrt{p}<\sum_{n\le\sqrt{X}}\sqrt{n}\ll\int_1^{\sqrt{X}}\sqrt{t}\,dt\ll X^{3/4}. \tag{5}$$

On the other hand, using the representation of a sum as a Stieltjes integral and then using integration by parts, we obtain

$$\begin{aligned}\sum_{\sqrt{X}\le p\le X}\sqrt{p} &= \int_{\sqrt{X}}^{X}\sqrt{t}\,d\pi(t) = \sqrt{t}\,\pi(t)\Big|_{t=\sqrt{X}}^{t=X}-\frac{1}{2}\int_{\sqrt{X}}^{X}\frac{\pi(t)}{t^{1/2}}\,dt\\ &= \sqrt{t}\frac{t}{\log t}\left(1+O\left(\frac{1}{\log t}\right)\right)\Bigg|_{t=\sqrt{X}}^{t=X}\\ &\quad -\frac{1}{2}\int_{\sqrt{X}}^{X}\frac{t}{\log t}\left(1+O\left(\frac{1}{\log t}\right)\right)\frac{dt}{t^{1/2}},\end{aligned} \tag{6}$$

where we used the prime number theorem in the form (3). Observe that in the above, the range of t is $\sqrt{X}\le t\le X$, which means that twice in (6) we may replace $\left(1+O\left(\frac{1}{\log t}\right)\right)$ by $\left(1+O\left(\frac{1}{\log X}\right)\right)$, thereby implying that (6) becomes

$$\begin{aligned}\sum_{\sqrt{X}\le p\le X}\sqrt{p} &= \frac{X^{3/2}}{\log X}\left(1+O\left(\frac{1}{\log X}\right)\right)-\frac{1}{2}\frac{X^{3/2}}{\log X^{3/2}}\left(1+O\left(\frac{1}{\log X}\right)\right)\\ &= \frac{2}{3}\frac{X^{3/2}}{\log X}\left(1+O\left(\frac{1}{\log X}\right)\right).\end{aligned} \tag{7}$$

Substituting the estimates (5) and (7) in (4) completes the proof of part (a).

The proof of part (b) uses the same technique and we will therefore skip it. □

The following result already appeared as Lemma 5 in [3]. For the sake of completeness, we also include its proof here.

Lemma 2. *Given any integer $n \geq 2$, $P(n) \geq \sqrt{n}$ if and only if $\rho_2(n) = P(n)$.*

Proof. If n is prime, then the result is obvious. If n is composite and $P(n) \geq \sqrt{n}$, then all other divisors of n smaller than n must not exceed $\sqrt{n}$, in which case it is clear that $\rho_2(n) = P(n)$. Conversely, if $\rho_2(n) = P(n)$, we have $\sqrt{n} \leq \rho_2(n) = P(n)$, which proves our claim. □

Lemma 3. *For all integers $n \geq 1$,*

$$\tau_\square(n) = 2\tau_0(n) + \chi(n),$$

where

$$\tau_0(n) := \#\left\{d \mid n : d < \sqrt{n} \text{ and } d + \frac{n}{d} = c^2\right\}$$

and

$$\chi(n) := \begin{cases} 1 & \text{if } n = 2^{4s+2}\ell^4 \text{ for some integers } s \geq 0 \text{ and } \ell \text{ odd,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. It is obvious that if n is not a perfect square, then $\tau_\square(n) = 2\tau_0(n)$. On the other hand, if n is a perfect square, say $n = m^2$, then in order for n to satisfy $m + n/m = c^2$ for a certain positive integer c, we must have $2\sqrt{n} = c^2$ and therefore $4n = c^4$. Hence, c must be even and similarly for n. Writing $c = 2^r\ell$ for certain positive integers r and ℓ with ℓ odd, we have $4n = 2^{4r}\ell^4$, which implies that

$$n = 2^{4r-2}\ell^4 = 2^{4s+2}\ell^4 \qquad (s \geq 0),$$

thus completing the proof of the lemma. □

Lemma 4. *Let $\chi(n)$ be the function defined in the statement of Lemma 3. Then,*

$$\sum_{n \leq x} \chi(n) = \frac{x^{1/4}}{\sqrt{2}} + O(\log x).$$

Proof. It follows from the definition of $\chi(n)$ that

$$\begin{aligned}
\sum_{n \leq x} \chi(n) &= \sum_{\substack{2^{4s+2}\ell^4 \leq x \\ s \geq 0,\ \ell \geq 1,\ \ell \text{ odd}}} 1 = \sum_{0 \leq s \leq \frac{1}{4}\frac{\log x}{\log 2} - \frac{1}{2}} \sum_{\substack{\ell^4 \leq x/2^{4s+2} \\ \ell \text{ odd}}} 1 \\
&= \sum_{0 \leq s \leq \frac{1}{4}\frac{\log x}{\log 2} - \frac{1}{2}} \left(\frac{x^{1/4}}{2 \cdot 2^{s+1/2}} + O(1)\right) \\
&= \frac{x^{1/4}}{2\sqrt{2}} \sum_{0 \leq s \leq \frac{1}{4}\frac{\log x}{\log 2} - \frac{1}{2}} \frac{1}{2^s} + O(\log x). \qquad (8)
\end{aligned}$$

Let us write

$$\sum_{0\le s\le \frac{1}{4}\frac{\log x}{\log 2}-\frac{1}{2}} \frac{1}{2^s} = \sum_{s=0}^{\infty}\frac{1}{2^s} - \sum_{s>\frac{1}{4}\frac{\log x}{\log 2}-\frac{1}{2}} \frac{1}{2^s} = 2 - \sum_{s>\frac{1}{4}\frac{\log x}{\log 2}-\frac{1}{2}} \frac{1}{2^s}. \tag{9}$$

It is clear that

$$\sum_{s>\frac{1}{4}\frac{\log x}{\log 2}-\frac{1}{2}} \frac{1}{2^s} \ll \int_{\frac{\log x}{4\log 2}-\frac{1}{2}}^{\infty} \frac{1}{2^t}\,dt = -\frac{1}{2^t\log 2}\Big|_{t=\frac{\log x}{4\log 2}-\frac{1}{2}}^{t=\infty} = \frac{\sqrt{2}}{x^{1/4}\log 2}. \tag{10}$$

Combining the relations (9) and (10), relation (8) becomes

$$\sum_{n\le x}\chi(n) = \sum_{\substack{2^{4s+2}\ell^4\le x\\ s\ge 0,\ \ell\ge 1,\ \ell \text{ odd}}} 1 = \frac{x^{1/4}}{\sqrt{2}} + O(1) + O(\log x) = \frac{x^{1/4}}{\sqrt{2}} + O(\log x),$$

which completes the proof of Lemma 4. □

Let

$$\mathcal{B}(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}dt \qquad (Re(x)>0, Re(y)>0)$$

and

$$\Gamma(z) := \int_0^{\infty} t^{z-1}e^{-t}dt \qquad (Re(z)>0)$$

stand respectively for the *Beta function* and the *Gamma function*.

Basic properties of these two functions can be found in Chapter 2 of the book of Rainville [9]. We will be needing the additional properties detailed in the following lemma.

Lemma 5. *We have*

$$\mathcal{B}(x,y) = \int_0^1 \frac{t^{x-1}+t^{y-1}}{(1+t)^{x+y}}\,dt \qquad (Re(x)>0, Re(y)>0)\,, \tag{11}$$

$$\Gamma(x+y)\cdot\mathcal{B}(x,y) = \Gamma(x)\Gamma(y) \qquad (Re(x)>0, Re(y)>0) \tag{12}$$

and

$$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\,\Gamma(2z) \qquad (Re(z)>0)\,. \tag{13}$$

Proof. We begin by establishing relation (11). In the definition of the Beta function, we make the change of variable $t = s/(1+s)$ and get

$$\mathcal{B}(x,y) = \int_0^{\infty}\left(\frac{s}{1+s}\right)^{x-1}\left(\frac{1}{1+s}\right)^{y-1}\frac{1}{(1+s)^2}\,ds = \int_0^{\infty}\frac{s^{x-1}}{(1+s)^{x+y}}\,ds$$

$$= \int_0^1 \frac{s^{x-1}}{(1+s)^{x+y}}\, ds + \int_1^\infty \frac{s^{x-1}}{(1+s)^{x+y}}\, ds. \tag{14}$$

Setting $s = 1/u$ in the last integral, we obtain

$$\int_1^\infty \frac{s^{x-1}}{(1+s)^{x+y}}\, ds = -\int_1^0 \frac{(1/u)^{x-1}(1/u)^2}{(1+1/u)^{x+y}}\, du = \int_0^1 \frac{u^{x+y-1-x}}{(1+u)^{x+y}}\, du = \int_0^1 \frac{u^{y-1}}{(1+u)^{x+y}}\, du.$$

Using this last relation in (14), we immediately obtain (11).

To prove (12), we first expand its right hand side as follows.

$$\Gamma(x)\Gamma(y) = \int_0^\infty t^{x-1}e^{-t}\, dt \cdot \int_0^\infty r^{y-1}e^{-r}\, dr = \int_0^\infty \int_0^\infty t^{x-1}r^{y-1}e^{-(t+r)}\, dr\, dt. \tag{15}$$

We introduce the variables u and v defined by $u = t + r$ and $v = t/(t + r)$, imposing new limits of integration for u and v, namely with u going from 0 to ∞, and v going from 0 to 1. Thus, taking into account that the Jacobian appearing in the integrand is equal to $|\Delta|$, where

$$\Delta := \begin{vmatrix} \frac{\partial t}{\partial u} & \frac{\partial t}{\partial v} \\ \frac{\partial r}{\partial u} & \frac{\partial r}{\partial v} \end{vmatrix} = -uv - u(1-v) = -u,$$

relation (15) becomes

$$\begin{aligned} \Gamma(x)\Gamma(y) &= \int_0^1 \int_0^\infty (uv)^{x-1}(u(1-v))^{y-1}e^{-(uv+u(1-v))}u\, du\, dv \\ &= \int_0^1 \int_0^\infty u^{x+y-1}v^{x-1}(1-v)^{y-1}e^{-u}\, du\, dv \\ &= \int_0^1 v^{x-1}(1-v)^{y-1}\, dv \cdot \int_0^\infty u^{x+y-1}e^{-u}\, du \\ &= \mathcal{B}(x,y) \cdot \Gamma(x+y), \end{aligned}$$

thus completing the proof of (12).

Relation (13) is known as the *Legendre duplication formula* and its original proof can be found in the 1809 paper of Legendre [5]. For the sake of completeness, we provide here a classical proof which in fact uses the first two identities of this lemma. First, we choose $x = z$ and $y = z$ in formula (12) and thereafter use identity (11) which gives

$$\Gamma(z)^2 = \Gamma(2z) \int_0^1 t^{z-1}(1-t)^{z-1}\, dt.$$

Replacing t by $(s + 1)/2$ in the above integral, we obtain

$$\Gamma(z)^2 = \Gamma(2z) \int_{-1}^1 \left(\frac{1+s}{2}\right)^{z-1} \left(\frac{1-s}{2}\right)^{z-1} \frac{ds}{2} = \frac{\Gamma(2z)}{2^{2z-1}} \int_{-1}^1 (1-s^2)^{z-1}\, ds.$$

Rearranging this last formula, we get

$$2^{2z-1}\Gamma(z)^2 = 2\Gamma(2z)\int_0^1 (1-s^2)^{z-1}\,ds. \tag{16}$$

If we make the change of variable $t = s^2$ in the integral appearing in (11), we obtain

$$\mathcal{B}(x,y) = \int_0^1 s^{2x-2}(1-s^2)^{y-1}\,2s\,ds = 2\int_0^1 s^{2x-1}(1-s^2)^{y-1}\,ds.$$

Replacing x by $1/2$ and y by z in the above, we get that

$$\mathcal{B}(1/2,z) = 2\int_0^1 (1-s^2)^{z-1}\,ds. \tag{17}$$

Combining (16) and (17), and thereafter using (12), we find

$$2^{2z-1}\Gamma(z)^2 = \Gamma(2z)\frac{\Gamma(1/2)\Gamma(z)}{\Gamma(1/2+z)},$$

which, taking into account the fact that $\Gamma(1/2) = \sqrt{\pi}$, completes the proof of (13).

□

The following are standard identities in real analysis.

Lemma 6. *In a neighborhood of $t = 0$, we have the two series expansion*

(a) $\sqrt{1+t} = 1 + \frac{t}{2} - \frac{1}{8}t^2 + O(t^3)$,

(b) $\frac{1}{\sqrt{1+t}} = 1 - \frac{t}{2} + \frac{3}{8}t^2 + O(t^3)$.

4 Proof of Theorem 1

In light of Lemmas 3 and 4, we have

$$\sum_{n\le x} \tau_\square(n) = 2\sum_{n\le x} \tau_0(n) + O(x^{1/4}). \tag{18}$$

Observe that $\tau_0(n)$ counts the number of ways that one can write n as $n = k(a^2-k)$ for some positive integer a with $k \in \mathbb{N}$ satisfying $k < a^2 - k$, that is,

$$a^2 > 2k. \tag{19}$$

We are interested in counting those $n \le x$ for which $n = k(a^2 - k) \le x$, that is, $a^2 - k \le x/k$. This amounts to counting those positive integers a for which

$$a^2 \le \frac{x}{k} + k. \tag{20}$$

Thus it is clear that in light of (19) and (20), we may write that

$$\begin{aligned}
\sum_{n \le x} \tau_0(n) &= \sum_{1 \le k < \sqrt{x}} \sum_{\sqrt{2k} < a \le \sqrt{\frac{x}{k}+k}} 1 \\
&= \sum_{1 \le k < \sqrt{x}} \left(\left\lfloor \sqrt{\frac{x}{k} + k} \right\rfloor - \left\lfloor \sqrt{2k} \right\rfloor \right) \\
&= \sum_{1 \le k < \sqrt{x}} \sqrt{\frac{x}{k} + k} - \sum_{1 \le k < \sqrt{x}} \sqrt{2k} + O\left(\sqrt{x}\right) \\
&= \int_1^{\sqrt{x}} \sqrt{\frac{x}{u} + u}\, du - \int_1^{\sqrt{x}} \sqrt{2u}\, du + O(\sqrt{x}) \\
&= S_1(x) - S_2(x) + O(\sqrt{x}),
\end{aligned} \tag{21}$$

say, where the error term $O(\sqrt{x})$ accounts for replacing $\left(\left\lfloor \sqrt{\frac{x}{k} + k} \right\rfloor - \left\lfloor \sqrt{2k} \right\rfloor \right)$ by $\sqrt{\frac{x}{k} + k} - \sqrt{2k}$. The estimation of $S_2(x)$ is very simple since

$$S_2(x) = \sqrt{2}\, \frac{u^{3/2}}{3/2} \Big|_{u=1}^{u=\sqrt{x}} = \frac{2\sqrt{2}}{3} x^{3/4} + O(1). \tag{22}$$

On the other hand, using integration by parts, we find that

$$\begin{aligned}
S_1(x) &= \int_1^{\sqrt{x}} \sqrt{u} \cdot \sqrt{1 + \frac{x}{u^2}}\, du \\
&= \frac{2}{3} u^{3/2} \sqrt{1 + \frac{x}{u^2}} \Big|_{u=1}^{u=\sqrt{x}} + \frac{2}{3} x \int_1^{\sqrt{x}} \frac{du}{u^{3/2} \sqrt{1 + \frac{x}{u^2}}} \\
&= \frac{2\sqrt{2}}{3} x^{3/4} + \frac{2}{3} x \int_1^{\sqrt{x}} \frac{du}{u^{3/2} \sqrt{1 + \frac{x}{u^2}}} + O(\sqrt{x}).
\end{aligned} \tag{23}$$

It remains to estimate

$$I(x) := \frac{2}{3} x \int_1^{\sqrt{x}} \frac{du}{u^{3/2} \sqrt{1 + \frac{x}{u^2}}}.$$

We have

$$I(x) = \frac{2}{3}\sqrt{x}\int_1^{\sqrt{x}} \frac{du}{\sqrt{u}\sqrt{1+\frac{u^2}{x}}},$$

and by the change of variable $t = u^2/x$, we obtain

$$\begin{aligned} I(x) &= \frac{1}{3}x^{3/4}\int_{1/x}^1 \frac{dt}{t^{3/4}\sqrt{1+t}} = \frac{1}{3}x^{3/4}\left(\int_0^1 \frac{dt}{t^{3/4}\sqrt{1+t}} - \int_0^{1/x} \frac{dt}{t^{3/4}\sqrt{1+t}}\right) \\ &= \frac{1}{3}x^{3/4}\left(J - U(x)\right), \end{aligned} \tag{24}$$

say.

To estimate J, we use relation (11) with $x = y = 1/4$, which gives

$$J = \int_0^1 \frac{dt}{t^{3/4}\sqrt{1+t}} = \frac{1}{2}\mathcal{B}(1/4, 1/4). \tag{25}$$

Then, using (12) and then (13) first with $z = 1/4$ and then with $z = 3/4$, we find that

$$J = \frac{1}{2}\frac{\Gamma(1/4)^2}{\Gamma(1/2)} = \frac{\pi\sqrt{\pi}}{\Gamma(3/4)^2} = \frac{2\sqrt{2\pi}\Gamma(5/4)}{\Gamma(3/4)}, \tag{26}$$

where we used the fact that $\Gamma(1/2) = \sqrt{\pi}$ and that $\Gamma(3/2) = \frac{1}{2}\Gamma(1/2) = \frac{1}{2}\sqrt{\pi}$. Let us now evaluate $U(x)$. Using Lemma 6(b), we obtain

$$U(x) = \int_0^{1/x} \frac{1}{t^{3/4}}\left(1 - \frac{t}{2} + O(t^2)\right)dt = \frac{4}{x^{1/4}} + O\left(\frac{1}{x^{5/4}}\right) = O\left(\frac{1}{x^{1/4}}\right). \tag{27}$$

Combining (27) and (26) in (24), we obtain

$$I(x) = \frac{2\sqrt{2\pi}\Gamma(5/4)}{3\Gamma(3/4)}x^{3/4} + O(\sqrt{x}).$$

Bringing this last estimate in (23) gives

$$S_1(x) = \left(\frac{2\sqrt{2}}{3} + \frac{2\sqrt{2\pi}\Gamma(5/4)}{3\Gamma(3/4)}\right)x^{3/4} + O(\sqrt{x}). \tag{28}$$

Finally, combining relations (22) and (28) in (21) and thereafter in (18), the proof of Theorem 1 is complete.

5 Proof of Theorem 2

Let x be a large number and let $n \in \mathcal{R}$. If $n = P(n)^2$, then $n = p^2$ for some prime p, in which case we easily obtain that the only number n with this property is $n = 4$. Hence, we will examine separately the two sets

$$\mathcal{R}_1 = \left\{n \in \mathcal{R} : P(n) > \sqrt{n}\right\} \qquad \text{and} \qquad \mathcal{R}_2 = \left\{n \in \mathcal{R} : P(n) < \sqrt{n}\right\}$$

and estimate the size of their respective counting functions $\mathcal{R}_1(x)$ and $\mathcal{R}_2(x)$, so that

$$\mathcal{R}(x) = \mathcal{R}_1(x) + \mathcal{R}_2(x) + 1.$$

5.1 The estimation of $\mathcal{R}_1(x)$

Let $n \in \mathcal{R}_1$, $n \leq x$. For such numbers n, in light of Lemma 2, we have $\rho_2(n) = P(n)$. Therefore, setting $k = \rho_1(n)$ and $p = \rho_2(n) = P(n)$, we may write that $n = k \cdot p$, where $k \in \{1, 2, \ldots, p-1\}$ and $p + k = c^2$ for some positive integer c. Since $n \leq x$, we have $n = k \cdot p \leq x$, so that $k \leq x/p$. This is why we have

$$n = k \cdot p \leq x \text{ and } p + k = \text{a square, where } 1 \leq k \leq \alpha(p, x) := \min\left(p - 1, \frac{x}{p}\right).$$

It is clear that the number of perfect squares amongst the numbers

$$p + 1, p + 2, \ldots, p + \alpha(p, x)$$

is equal to

$$\left\lfloor \sqrt{p + \alpha(p, x)} \right\rfloor - \left\lfloor \sqrt{p + 1} \right\rfloor = \left\lfloor \sqrt{p + \alpha(p, x)} \right\rfloor - \lfloor \sqrt{p} \rfloor.$$

This is why

$$\begin{aligned}
\mathcal{R}_1(x) &= \sum_{p \leq x} \left(\left\lfloor \sqrt{p + \alpha(p, x)} \right\rfloor - \lfloor \sqrt{p} \rfloor \right) \\
&= \sum_{p \leq \sqrt{x}} \left(\left\lfloor \sqrt{2p - 1} \right\rfloor - \lfloor \sqrt{p} \rfloor \right) + \sum_{\sqrt{x} < p \leq x} \left(\left\lfloor \sqrt{p + \frac{x}{p}} \right\rfloor - \lfloor \sqrt{p} \rfloor \right) \\
&= U_1(x) + U_2(x), \qquad (29)
\end{aligned}$$

say. On the one hand, since

$$\left\lfloor \sqrt{2p - 1} \right\rfloor - \lfloor \sqrt{p} \rfloor = \sqrt{2p - 1} - \sqrt{p} + O(1) = (\sqrt{2} - 1)\sqrt{p} + O(1),$$

it follows, using Lemma 1(a), that

$$\begin{aligned} U_1(x) &= \sum_{p\le\sqrt{x}}(\sqrt{2}-1)\sqrt{p}+O(\pi(\sqrt{x})) = (\sqrt{2}-1)\sum_{p\le\sqrt{x}}\sqrt{p}+O\left(\frac{\sqrt{x}}{\log x}\right) \\ &= \frac{4}{3}(\sqrt{2}-1)\,\frac{x^{3/4}}{\log x}+O\left(\frac{x^{3/4}}{\log^2 x}\right). \end{aligned} \tag{30}$$

To estimate $U_2(x)$, we first write

$$\begin{aligned} U_2(x) &= \sum_{\sqrt{x}<p\le x^{2/3}}\left(\left\lfloor\sqrt{p+\frac{x}{p}}\right\rfloor-\lfloor\sqrt{p}\rfloor\right)+\sum_{x^{2/3}<p\le x}\left(\left\lfloor\sqrt{p+\frac{x}{p}}\right\rfloor-\lfloor\sqrt{p}\rfloor\right) \\ &= T_1(x)+T_2(x), \end{aligned} \tag{31}$$

say.

We begin by showing that

$$T_2(x)\ll\frac{x^{2/3}}{\log x}. \tag{32}$$

To do so, we introduce the function

$$g(x,p):=\left\lfloor\sqrt{p+\frac{x}{p}}\right\rfloor-\lfloor\sqrt{p}\rfloor$$

and verify that

$$g(x,p)\in\{0,1\}\qquad\text{for each prime } p\in I(x):=(x^{2/3},x]. \tag{33}$$

Indeed, for each prime $p\in I(x)$, setting $d:=x/p$, we have

$$\begin{aligned} g(x,p) &= \left\lfloor\sqrt{p+d}\right\rfloor-\lfloor\sqrt{p}\rfloor\le\sqrt{p+d}-\sqrt{p}+1 \\ &= \frac{(\sqrt{p+d}-\sqrt{p})(\sqrt{p+d}+\sqrt{p})}{\sqrt{p+d}+\sqrt{p}}+1 \\ &\le \frac{d}{2\sqrt{p}}+1. \end{aligned} \tag{34}$$

It is obvious that, for each $p\in I(x)$, we have $d\le x^{1/3}$. Therefore, for each $p>x^{2/3}$, we have $4p>x^{2/3}$, and consequently $2\sqrt{p}>x^{1/3}\ge d$. This last inequality implies that

$$\frac{d}{2\sqrt{p}}<1. \tag{35}$$

Combining (34) and (35), we obtained that $g(x,p)<2$, and since $g(x,p)$ is an integer, it proves (33).

In light of (33), we have

$$g(x,p) = \begin{cases} 1 & \text{if there exists an integer } r \text{ such that } p < r^2 \le p + \dfrac{x}{p}, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, since $T_2(x) = \sum_{p \in I(x)} g(x,p)$, the function $T_2(x)$ counts the number of primes $p \le x$ for which there exists an integer r such that $p < r^2 \le p + x/p$. This is equivalent to counting the number of integers r for which there exists a prime p such that $p < r^2 \le p + x/p$. To this effect, we introduce the function

$$h(x,r) = \begin{cases} 1 & \text{if there exists a prime } p \text{ such that } p < r^2 \le p + \dfrac{x}{p}, \\ 0 & \text{otherwise,} \end{cases}$$

and subdivide the set of primes $p \in I(x)$ into disjoint subsets

$$E_k := \left\{ p : \frac{x}{k+1} < p \le \frac{x}{k} \right\} \qquad (k = 1, 2, \ldots, \lfloor x^{1/3} \rfloor).$$

Now, let $p \in E_k$, k fixed, and assume that there exists an integer r such that $p < r^2 \le p + x/p$. Then,

$$\begin{aligned} p < r^2 \le p + x/p &\Longrightarrow \sqrt{\frac{x}{k+1}} < r < \sqrt{\frac{x}{k} + 2k} \\ &\Longrightarrow \sqrt{\frac{x}{k+1}} < r < \sqrt{\frac{x}{k}} \sqrt{1 + \frac{2k^2}{x}} \\ &\Longrightarrow \sqrt{\frac{x}{k+1}} < r < \sqrt{\frac{x}{k}} \left(1 + \frac{k^2}{x}\right) \qquad \text{because } 1 + \frac{2k^2}{x} < \left(1 + \frac{k^2}{x}\right)^2 \\ &\Longrightarrow \sqrt{\frac{x}{k+1}} < r < \sqrt{\frac{x}{k}} + \frac{k^{3/2}}{\sqrt{x}} \\ &\Longrightarrow \sqrt{\frac{x}{k+1}} < r < \sqrt{\frac{x}{k}} + 1 \qquad \text{because } k \le x^{1/3} \end{aligned}$$

and that

$$p < r^2 \le p + x/p \Longrightarrow r^2 - \frac{x}{p} \le p < r^2 \Longrightarrow r^2 - k - 1 < p < r^2 \Longrightarrow r^2 - k \le p < r^2.$$

Gathering these relations, we may write that

$$T_2(x) = \sum_{p \in I(x)} g(x,p) = \sum_{1 \le k \le x^{1/3}} \sum_{p \in E_k} g(x,p)$$

$$\le \sum_{1\le k\le x^{1/3}} \sum_{\sqrt{\frac{x}{k+1}}<r<\sqrt{\frac{x}{k}}+1} h(x,r)$$
$$\le \sum_{1\le k\le x^{1/3}} \sum_{\sqrt{\frac{x}{k+1}}<r<\sqrt{\frac{x}{k}}+1} \sum_{r^2-k\le p<r^2} 1$$
$$= \sum_{1\le k\le x^{1/3}} \sum_{\sqrt{\frac{x}{k+1}}<r<\sqrt{\frac{x}{k}}+1} \left(\pi(r^2)-\pi(r^2-k)\right).$$

Recalling the inequality

$$\pi(y+d)-\pi(y) \le \frac{2d}{\log d} \qquad (y,d\ge 2)$$

(see formula (4.11) in the book of Montgomery [6]), we have

$$\begin{aligned}
T_2(x) &\le 2\sum_{1\le k\le x^{1/3}} \frac{k}{\log k} \sum_{\sqrt{\frac{x}{k+1}}<r<\sqrt{\frac{x}{k}}+1} 1 \le 2\sum_{1\le k\le x^{1/3}} \frac{k}{\log k}\left(\sqrt{\frac{x}{k}}-\sqrt{\frac{x}{k+1}}+2\right)\\
&= 2\sum_{1\le k\le x^{1/3}} \frac{k}{\log k}\left(\sqrt{\frac{x}{k}}-\sqrt{\frac{x}{k+1}}\right)+4\sum_{1\le k\le x^{1/3}} \frac{k}{\log k}\\
&= 2\sqrt{x}\sum_{1\le k\le x^{1/3}} \frac{k}{\log k}\left(\frac{1}{\sqrt{k}}-\frac{1}{\sqrt{k+1}}\right)+O\left(\frac{x^{2/3}}{\log x}\right)\\
&= 2\sqrt{x}\sum_{1\le k\le x^{1/3}} \frac{k}{\log k}\left(\frac{\sqrt{k+1}-\sqrt{k}}{\sqrt{k}\sqrt{k+1}}\right)+O\left(\frac{x^{2/3}}{\log x}\right)\\
&= 2\sqrt{x}\sum_{1\le k\le x^{1/3}} \frac{k}{\log k}\left(\frac{1}{(\sqrt{k}\sqrt{k+1})(\sqrt{k}+\sqrt{k+1})}\right)+O\left(\frac{x^{2/3}}{\log x}\right)\\
&\le \sqrt{x}\sum_{1\le k\le x^{1/3}} \frac{1}{\sqrt{k}\log k}+O\left(\frac{x^{2/3}}{\log x}\right)\\
&\ll \frac{x^{2/3}}{\log x},
\end{aligned}$$

thus proving (32).

Now, for $T_1(x)$, we have

$$\begin{aligned}
T_1(x) &= \sum_{\sqrt{x}<p\le x^{2/3}} \left(\left\lfloor\sqrt{p+\frac{x}{p}}\right\rfloor-\lfloor\sqrt{p}\rfloor\right) = \sum_{\sqrt{x}<p\le x^{2/3}} \left(\sqrt{p+\frac{x}{p}}-\sqrt{p}+O(1)\right)\\
&= \sum_{\sqrt{x}<p\le x^{2/3}} \left(\sqrt{p+\frac{x}{p}}-\sqrt{p}\right)+O\left(\pi\left(x^{2/3}\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\sqrt{x}<p\le x^{2/3}} \sqrt{p}\left(\sqrt{1+\frac{x}{p^2}}-1\right)+O\left(\frac{x^{2/3}}{\log x}\right)\\
&= Z(x)+O\left(\frac{x^{2/3}}{\log x}\right), \qquad (36)
\end{aligned}
$$

say. Writing $Z(x)$ as a Stieltjes integral, integrating by parts and using the fact that, as a consequence of Lemma 6(b), we have

$$
\sqrt{1+\frac{1}{x^{1/3}}}-1=O\left(\frac{1}{x^{1/3}}\right),
$$

and therefore,

$$
\pi(x^{2/3})x^{1/3}\left(\sqrt{1+\frac{1}{x^{1/3}}}-1\right)=O\left(\frac{x^{2/3}}{\log x}\right),
$$

we obtain

$$
\begin{aligned}
Z(x) &= \int_{\sqrt{x}}^{x^{2/3}} \sqrt{t}\left(\sqrt{1+\frac{x}{t^2}}-1\right)\,d\pi(t)\\
&= \left(\pi(t)\sqrt{t}\left(\sqrt{1+\frac{x}{t^2}}-1\right)\right)\Bigg|_{t=\sqrt{x}}^{t=x^{2/3}} - \int_{\sqrt{x}}^{x^{2/3}} \pi(t)\,\frac{d}{dt}\left(\sqrt{t}\left(\sqrt{1+\frac{x}{t^2}}-1\right)\right)\\
&= \pi(x^{2/3})x^{1/3}\left(\sqrt{1+\frac{1}{x^{1/3}}}-1\right)-\pi\left(\sqrt{x}\right)x^{1/4}(\sqrt{2}-1)\\
&\quad -\int_{\sqrt{x}}^{x^{2/3}} \pi(t)\left(\frac{1}{2\sqrt{t}}\left(\sqrt{1+\frac{x}{t^2}}-1\right)-\frac{x\sqrt{t}}{t^3\sqrt{1+\frac{x}{t^2}}}\right)dt\\
&= O\left(\frac{x^{2/3}}{\log x}\right)-\pi\left(\sqrt{x}\right)x^{1/4}(\sqrt{2}-1)\\
&\quad -\frac{1}{2}\left(1+O\left(\frac{1}{\log x}\right)\right)\int_{\sqrt{x}}^{x^{2/3}} \sqrt{t}\left(\sqrt{1+\frac{x}{t^2}}-1\right)\frac{dt}{\log t}\\
&\quad +\left(1+O\left(\frac{1}{\log x}\right)\right)\int_{\sqrt{x}}^{x^{2/3}} \frac{x}{t^{3/2}\sqrt{1+\frac{x}{t^2}}}\,\frac{dt}{\log t}\\
&=O\left(\frac{x^{2/3}}{\log x}\right)-2(\sqrt{2}-1)\frac{x^{3/4}}{\log x}+O\left(\frac{x^{3/4}}{\log^2 x}\right)\\
&\quad -\frac{1}{2}\left(1+O\left(\frac{1}{\log x}\right)\right)K_1(x)+\left(1+O\left(\frac{1}{\log x}\right)\right)K_2(x), \qquad (37)
\end{aligned}
$$

where

$$
K_1(x)=\int_{\sqrt{x}}^{x^{2/3}} \sqrt{t}\left(\sqrt{1+\frac{x}{t^2}}-1\right)\frac{dt}{\log t}
$$

and

$$K_2(x) = \int_{\sqrt{x}}^{x^{2/3}} \frac{x}{t^{3/2}\sqrt{1+\frac{x}{t^2}}} \frac{dt}{\log t}.$$

For the evaluation of $K_2(x)$, we first set

$$f_2(t) := \frac{x}{t^{3/2}\sqrt{1+\frac{x}{t^2}}} \quad \text{and} \quad F_2(u) := \int_1^u f_2(t)\,dt. \tag{38}$$

Using integration by parts, we get

$$\begin{aligned} K_2(x) &= \frac{F_2(u)}{\log u}\Big|_{u=\sqrt{x}}^{u=x^{2/3}} + \int_{\sqrt{x}}^{x^{2/3}} \frac{F_2(u)}{u\log^2 u}\,du \\ &= \frac{3}{2}\frac{F_2(x^{2/3})}{\log x} - 2\frac{F_2(\sqrt{x})}{\log x} + \int_{\sqrt{x}}^{x^{2/3}} \frac{F_2(u)}{u\log^2 u}\,du \\ &= \frac{3}{2}\frac{F_2(x^{2/3})}{\log x} - 2\frac{F_2(\sqrt{x})}{\log x} + L(x), \end{aligned} \tag{39}$$

say. Since $\sqrt{x} \le u \le x^{2/3}$,

$$\begin{aligned} L(x) &= \int_{\sqrt{x}}^{x^{2/3}} \left(\int_1^{x^{2/3}} f_2(t)\,dt - \int_u^{x^{2/3}} f_2(t)\,dt\right) \frac{du}{u\log^2 u} \\ &= F_2(x^{2/3}) \int_{\sqrt{x}}^{x^{2/3}} \frac{du}{u\log^2 u} - \int_{\sqrt{x}}^{x^{2/3}} \left(\int_u^{x^{2/3}} f_2(t)\,dt\right) \frac{du}{u\log^2 u} \\ &= F_2(x^{2/3}) \left(-\frac{1}{\log u}\Big|_{u=\sqrt{x}}^{u=x^{2/3}}\right) - \int_{\sqrt{x}}^{x^{2/3}} \left(\int_u^{x^{2/3}} f_2(t)\,dt\right) \frac{du}{u\log^2 u} \\ &= \frac{1}{2}\frac{F_2(x^{2/3})}{\log x} - \int_{\sqrt{x}}^{x^{2/3}} \left(\int_u^{x^{2/3}} f_2(t)\,dt\right) \frac{du}{u\log^2 u}. \end{aligned} \tag{40}$$

Observe that

$$\begin{aligned} \int_u^{x^{2/3}} f_2(t)\,dt = x\int_u^{x^{2/3}} \frac{1}{t^{3/2}\sqrt{1+\frac{x}{t^2}}}\,dt \le x\int_u^{x^{2/3}} \frac{1}{t^{3/2}}\,dt \\ = x\left(-\frac{2}{\sqrt{x}}\Big|_{t=u}^{t=x^{2/3}}\right) \le 2\frac{x}{\sqrt{u}}, \end{aligned}$$

from which it follows that

$$\int_{\sqrt{x}}^{x^{2/3}} \left(\int_u^{x^{2/3}} f_2(t)\,dt\right) \frac{du}{u\log^2 u} \le 2\int_{\sqrt{x}}^{x^{2/3}} \frac{x}{u^{3/2}\log^2 u}\,du \ll \frac{x}{\log^2 x}\int_{\sqrt{x}}^{x^{2/3}} \frac{1}{u^{3/2}}\,du$$

$$\ll \frac{x^{3/4}}{\log^2 x},$$

and therefore that (40) can be replaced by

$$L(x) = \frac{1}{2}\frac{F_2(x^{2/3})}{\log x} + O\left(\frac{x^{3/4}}{\log^2 x}\right).$$

Using this last estimate in (39), we obtain

$$K_2(x) = 2\frac{F_2(x^{2/3})}{\log x} - 2\frac{F_2(\sqrt{x})}{\log x} + O\left(\frac{x^{3/4}}{\log^2 x}\right). \tag{41}$$

We already know from (25) that

$$F_2(\sqrt{x}) = \frac{1}{4}\mathcal{B}(1/4, 1/4)x^{3/4} + O(\sqrt{x}). \tag{42}$$

By a calculation similar to the one done in (24), we have

$$\begin{aligned}
F_2(x^{2/3}) &= \frac{1}{2}x^{3/4}\int_{1/x}^{x^{1/3}} \frac{1}{t^{3/4}\sqrt{1+t}}\,dt \\
&= \frac{1}{2}x^{3/4}\left(\int_0^\infty \frac{1}{t^{3/4}\sqrt{1+t}}\,dt - \int_0^{1/x} \frac{1}{t^{3/4}\sqrt{1+t}}\,dt - \int_{x^{1/3}}^\infty \frac{1}{t^{3/4}\sqrt{1+t}}\,dt\right) \\
&= \frac{1}{2}x^{3/4}\left(\int_0^\infty \frac{1}{t^{3/4}\sqrt{1+t}}\,dt - \int_{x^{1/3}}^\infty \frac{1}{t^{3/4}\sqrt{1+t}}\,dt\right) + O(\sqrt{x}).
\end{aligned} \tag{43}$$

We can easily see that

$$\int_{x^{1/3}}^\infty \frac{1}{t^{3/4}\sqrt{1+t}}\,dt \le \int_{x^{1/3}}^\infty \frac{1}{t^{5/4}}\,dt = \frac{4}{x^{1/12}}, \tag{44}$$

and it follows from Lemma 5 that

$$\int_0^\infty \frac{1}{t^{3/4}\sqrt{1+t}}\,dt = \mathcal{B}(1/4, 1/4). \tag{45}$$

Hence, using (44) and (45) in (43), we obtain

$$F_2(x^{2/3}) = \frac{1}{2}\mathcal{B}(1/4, 1/4)x^{3/4} + O(\sqrt{x}). \tag{46}$$

Using (42) and (46) in (41) gives

$$K_2(x) = \frac{1}{2}\mathcal{B}(1/4, 1/4)\frac{x^{3/4}}{\log x} + O\left(\frac{x^{3/4}}{\log^2 x}\right). \tag{47}$$

Similarly, for the evaluation of $K_1(x)$, we set

$$f_1(t) := \sqrt{t}\left(\sqrt{1+\frac{x}{t^2}}-1\right) \quad \text{and} \quad F_1(u) := \int_1^u f_1(t)\,dt. \tag{48}$$

Integrating by parts as we did before, we get

$$\begin{aligned} K_1(x) &= 2\frac{F_1(x^{2/3})}{\log x} - 2\frac{F_1(\sqrt{x})}{\log x} - \int_{\sqrt{x}}^{x^{2/3}} \left(\int_u^{x^{2/3}} f_1(t)\,dt\right) \frac{du}{u\log^2 u} \\ &= 2\frac{F_1(x^{2/3})}{\log x} - 2\frac{F_1(\sqrt{x})}{\log x} - N(x), \end{aligned} \tag{49}$$

say.

Observe that in the representation of $N(x)$, we have $\sqrt{x} \le u \le t \le x^{2/3}$, so that $x/t^2 \le 1$. Using Lemma 6(a), we have

$$\sqrt{1+\frac{x}{t^2}} - 1 = O\left(\frac{x}{t^2}\right),$$

implying that for some positive constant C,

$$\int_u^{x^{2/3}} \sqrt{t}\left(\sqrt{1+\frac{x}{t^2}}-1\right) dt \le Cx \int_u^{x^{2/3}} \frac{1}{t^{3/2}}\,dt = Cx\left(-\frac{2}{\sqrt{x}}\Big|_{t=u}^{t=x^{2/3}}\right) \le 2C\frac{x}{\sqrt{u}}.$$

It follows that

$$N(x) \le 2C\int_{\sqrt{x}}^{x^{2/3}} \frac{x}{u^{3/2}\log^2 u}\,du \ll \frac{x}{\log^2 x}\int_{\sqrt{x}}^{x^{2/3}} \frac{1}{u^{3/2}}\,du \ll \frac{x^{3/4}}{\log^2 x}. \tag{50}$$

Recalling the definition of $F_1(u)$ given in (48) and using integration by parts, we find that

$$\begin{aligned} F_1(u) &= \int_1^u \sqrt{t}\left(\sqrt{1+\frac{x}{t^2}}-1\right) dt \\ &= \frac{2}{3}t^{3/2}\left(\sqrt{1+\frac{x}{t^2}}-1\right)\Bigg|_{t=1}^{t=u} + \frac{2}{3}\int_1^u \frac{x}{t^{3/2}\sqrt{1+\frac{x}{t^2}}}\,dt \\ &= \frac{2}{3}\left(\sqrt{1+\frac{x}{u^2}}-1\right)u^{3/2} + \frac{2}{3}F_2(u) + O(\sqrt{x}). \end{aligned} \tag{51}$$

Using (42) in (51), we obtain

$$F_1(\sqrt{x}) = \left(\frac{2}{3}(\sqrt{2}-1) + \frac{1}{6}\mathcal{B}(1/4,1/4)\right)x^{3/4} + O(\sqrt{x}) \tag{52}$$

and using Lemma 6(a) and (46) in (51), it follows that

$$F_1(x^{2/3}) = \frac{2}{3}x\left(\sqrt{1+\frac{1}{x^{1/3}}}-1\right)+\frac{1}{3}\mathcal{B}(1/4,1/4)x^{3/4}+O(\sqrt{x})$$
$$= \frac{1}{3}\mathcal{B}(1/4,1/4)x^{3/4}+O(x^{2/3}). \tag{53}$$

Gathering (52), (53) and (50) in (49) gives us

$$K_1(x) = \left(\frac{1}{3}\mathcal{B}(1/4,1/4)-\frac{4}{3}(\sqrt{2}-1)\right)\frac{x^{3/4}}{\log x}+O\left(\frac{x^{3/4}}{\log^2 x}\right). \tag{54}$$

Combining (47) and (54) in (37), and then substituting the result thus obtained in (36), we obtain

$$Z(x) = \left(\frac{1}{3}\mathcal{B}(1/4,1/4)-\frac{4}{3}(\sqrt{2}-1)\right)\frac{x^{3/4}}{\log x}+O\left(\frac{x^{3/4}}{\log^2 x}\right)$$
$$= T_1(x)+O\left(\frac{x^{2/3}}{\log x}\right). \tag{55}$$

Using (32) and (55) in (31), we get

$$U_2(x) = \left(\frac{1}{3}\mathcal{B}(1/4,1/4)-\frac{4}{3}(\sqrt{2}-1)\right)\frac{x^{3/4}}{\log x}+O\left(\frac{x^{3/4}}{\log^2 x}\right). \tag{56}$$

Finally, using (30) and (56) in (29), and recalling the values of $\mathcal{B}(1/4,1/4)$ obtained through (25) and (26), we conclude that

$$\mathcal{R}_1(x) = \frac{1}{3}\mathcal{B}(1/4,1/4)\frac{x^{3/4}}{\log x}+O\left(\frac{x^{3/4}}{\log^2 x}\right)$$
$$= \frac{4\sqrt{2\pi}\Gamma(5/4)}{3\Gamma(3/4)}\frac{x^{3/4}}{\log x}+O\left(\frac{x^{3/4}}{\log^2 x}\right). \tag{57}$$

5.2 Evaluation of $\mathcal{R}_2(x)$

As we will see, the estimation of $\mathcal{R}_2(x)$ represents a much bigger challenge and we therefore only obtain an upper bound for it. We begin with a result of general interest.

Lemma 7. *For all integers $n \geq 2$,*

$$\rho_1(n)+\rho_2(n) \leq P(n)+\frac{n}{P(n)}. \tag{58}$$

Proof. If $P(n) \geq \sqrt{n}$, then according to Lemma 2, $P(n) = \rho_2(n)$, implying that the right hand side of (58) is $\rho_2(n) + \dfrac{n}{\rho_2(n)} = \rho_2(n) + \rho_1(n)$, so that in this case, (58) is in fact an equality.

If $P(n) < \sqrt{n}$, then $P(n) = \sqrt{n}/a$ for some number $a > 1$. Let also b be defined by $\rho_1(n) = \sqrt{n}/b$, so that

$$\frac{\sqrt{n}}{a} = P(n) \leq \rho_1(n) = \frac{\sqrt{n}}{b},$$

which implies that $a \geq b$. Therefore (58) is equivalent to

$$\frac{\sqrt{n}}{b} + \frac{n}{\sqrt{n}/b} \leq \frac{\sqrt{n}}{a} + \frac{n}{\sqrt{n}/a},$$

that is,

$$\frac{1}{b} + b \leq \frac{1}{a} + a, \tag{59}$$

which is indeed true for any $a \geq b > 1$, because the function $x \mapsto \dfrac{1}{x} + x$ is increasing on the interval $[1, \infty)$. $\square$

Let $n \in \mathcal{R}_2$, $n \leq x$. Since $p := P(n) < \sqrt{n}$, we may write

$$n = m \cdot p, \quad \text{with } p < m \leq x/p. \tag{60}$$

Moreover, it follows from the definition of $\rho_1(n)$ that $P(n) \leq \rho_1(n)$. Combining this with the fact that $\rho_1(n) \leq \rho_2(n)$, we have that

$$2P(n) < \rho_1(n) + \rho_2(n).$$

Also, from Lemma 7,

$$\rho_1(n) + \rho_2(n) \leq P(n) + \frac{n}{P(n)}.$$

Using these last two inequalities and assuming that $\rho_1(n)+\rho_2(n) = c^2$ for some integer c, we obtain that

$$2p = 2P(n) < c^2 = \rho_1(n) + \rho_2(n) \leq P(n) + \frac{n}{P(n)} = p + \frac{mp}{p}. \tag{61}$$

Combining (60) and (61), we get that

$$\begin{aligned} \mathcal{R}_2(x) = \sum_{\substack{n \leq x \\ \rho_1(n)+\rho_2(n)=c^2 \\ P(n)<\sqrt{n}}} 1 &\leq \sum_{n \leq x} \sum_{\substack{p \mid n,\ p<\sqrt{n} \\ 2p<c^2\leq p+\frac{n}{p}}} 1 \leq \sum_{p<\sqrt{x}} \sum_{\substack{2p<c^2\leq p+\frac{mp}{p} \\ p<m\leq x/p}} 1 \\ &\leq \sum_{p<\sqrt{x}} \sum_{2p<c^2\leq p+\frac{x}{p}} 1 = \sum_{p<\sqrt{x}} \left(\left\lfloor \sqrt{p+\frac{x}{p}} \right\rfloor - \left\lfloor \sqrt{2p} \right\rfloor \right) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{p<\sqrt{x}} \left(\sqrt{p+\frac{x}{p}} - \sqrt{2p} + O(1) \right) \\
&= \sum_{p<\sqrt{x}} \sqrt{p+\frac{x}{p}} - \sqrt{2} \sum_{p<\sqrt{x}} \sqrt{p} + O\left(\frac{\sqrt{x}}{\log x} \right) \\
&= \sum_{p<\sqrt{x}} \sqrt{p} \sqrt{1+\frac{x}{p^2}} - \frac{4\sqrt{2}}{3} \frac{x^{3/4}}{\log x} + O\left(\frac{x^{3/4}}{\log^2 x} \right) \qquad \bigl(\text{from Lemma 1(a)}\bigr) \\
&= W(x) - \frac{4\sqrt{2}}{3} \frac{x^{3/4}}{\log x} + O\left(\frac{x^{3/4}}{\log^2 x} \right), \qquad (62)
\end{aligned}
$$

say. Writing $W(x)$ as a Stieltjes integral, integrating by parts and using the function $R(t)$ defined in (3), we get

$$
\begin{aligned}
W(x) &= \int_2^{\sqrt{x}} \sqrt{t} \sqrt{1+\frac{x}{t^2}}\, d\pi(t) \\
&= \left(\pi(t)\sqrt{t} \sqrt{1+\frac{x}{t^2}} \right) \Bigg|_{t=2}^{t=\sqrt{x}} - \int_2^{\sqrt{x}} \pi(t) \frac{d}{dt} \left(\sqrt{t} \left(\sqrt{1+\frac{x}{t^2}} \right) \right) \\
&= 2\sqrt{2}\, \frac{x^{3/4}}{\log x} - \int_2^{\sqrt{x}} \pi(t) \left(\frac{1}{2\sqrt{t}} \sqrt{1+\frac{x}{t^2}} - \frac{x\sqrt{t}}{t^3 \sqrt{1+\frac{x}{t^2}}} \right) dt + O(\sqrt{x}) \\
&= 2\sqrt{2}\, \frac{x^{3/4}}{\log x} - \frac{1}{2} \left(\int_2^{\sqrt{x}} \sqrt{t} \sqrt{1+\frac{x}{t^2}} \frac{dt}{\log t} + \int_2^{\sqrt{x}} \frac{R(t)}{\sqrt{t}} \sqrt{1+\frac{x}{t^2}}\, dt \right) \\
&\quad + \int_2^{\sqrt{x}} \frac{x}{t^{3/2}\sqrt{1+\frac{x}{t^2}}} \frac{dt}{\log t} + \int_2^{\sqrt{x}} \frac{xR(t)}{t^{5/2}\sqrt{1+\frac{x}{t^2}}}\, dt \\
&= 2\sqrt{2}\, \frac{x^{3/4}}{\log x} - \frac{1}{2} \left(W_1(x) + R_1(x) \right) + W_2(x) + R_2(x), \qquad (63)
\end{aligned}
$$

say.

Using the prime number theorem in the form (3), there exists a constant $C > 0$ and real number $x_0 \geq 2$ such that for all $t \geq x_0$, we have $|R(t)| < C\dfrac{t}{\log^2 t}$, so that

$$
\begin{aligned}
|R_1(x)| &= \left| \int_2^{\sqrt{x}} \frac{R(t)}{\sqrt{t}} \sqrt{1+\frac{x}{t^2}}\, dt \right| \leq \int_2^{\sqrt{x}} \left| \frac{R(t)}{\sqrt{t}} \sqrt{1+\frac{x}{t^2}} \right| dt \\
&\leq \int_2^{x_0} \frac{|R(t)|}{\sqrt{t}} \sqrt{1+\frac{x}{t^2}}\, dt + \int_{x_0}^{\sqrt{x}} \frac{|R(t)|}{\sqrt{t}} \sqrt{1+\frac{x}{t^2}} dt \\
&< C \int_{x_0}^{\sqrt{x}} \sqrt{t} \sqrt{1+\frac{x}{t^2}} \frac{dt}{\log^2 t} + O(\sqrt{x}) \\
&\ll \frac{1}{\log^2 x} \int_{x_0}^{\sqrt{x}} \sqrt{t} \sqrt{1+\frac{x}{t^2}}\, dt \ll \frac{x^{3/4}}{\log^2 x}, \qquad (64)
\end{aligned}
$$

where we used Lemma 6(a) to manage the last integral.

Analogously, using Lemma 6(b), we find that

$$|R_2(x)| \ll \frac{1}{\log^2 x}\int_{x_0}^{\sqrt{x}} \frac{x}{t^{3/2}\sqrt{1+\frac{x}{t^2}}}\,dt \ll \frac{x^{3/4}}{\log^2 x}. \tag{65}$$

Recalling the definitions of $f_2(t)$ and $F_2(u)$ given in (38), we obtain, using integration by parts,

$$\begin{aligned} W_2(x) &= \int_2^{\sqrt{x}} \frac{x}{t^{3/2}\sqrt{1+\frac{x}{t^2}}}\,\frac{dt}{\log t} = \int_2^{\sqrt{x}} f_2(t)\,\frac{dt}{\log t} \\ &= \frac{F_2(u)}{\log u}\bigg|_{u=2}^{u=\sqrt{x}} + \int_2^{\sqrt{x}} \frac{F_2(u)}{u\log^2 u}\,du \\ &== 2\frac{F_2(\sqrt{x})}{\log x} - \frac{F_2(2)}{\log 2} + \int_2^{\sqrt{x}} \frac{F_2(u)}{u\log^2 u}\,du. \end{aligned} \tag{66}$$

Observe that for all $2 \le u \le \sqrt{x}$,

$$\begin{aligned} F_2(u) = \int_1^u \frac{x}{t^{3/2}\sqrt{1+\frac{x}{t^2}}}\,dt &= \int_1^u \frac{\sqrt{x}}{\sqrt{t}\sqrt{1+\frac{t^2}{x}}}\,dt \\ &\le \int_1^u \frac{\sqrt{x}}{\sqrt{t}}\,dt = \sqrt{x}\left(2\sqrt{t}\bigg|_{t=1}^{t=u}\right) \le 2\sqrt{x}\sqrt{u}. \end{aligned}$$

It follows that

$$\begin{aligned} \int_2^{\sqrt{x}} \frac{F_2(u)}{u\log^2 u}\,du \le 2\sqrt{x}\int_2^{\sqrt{x}} \frac{1}{\sqrt{u}\log^2 u}du &= \sqrt{x}\int_{\sqrt{2}}^{x^{1/4}} \frac{1}{\log^2 t}\,dt \\ \ll \sqrt{x}\frac{x^{1/4}}{\log^2 x} &= \frac{x^{3/4}}{\log^2 x} \end{aligned} \tag{67}$$

and that

$$F(2) \ll \sqrt{x}. \tag{68}$$

Using (42), (67) and (68) in (66), we get

$$W_2(x) = \frac{1}{2}\mathcal{B}(1/4,1/4)\frac{x^{3/4}}{\log x} + O\left(\frac{x^{3/4}}{\log^2 x}\right). \tag{69}$$

Set $h(t) := \sqrt{t}\sqrt{1+\frac{x}{t^2}}$ and $H(u) := \int_1^u h(t)\,dt$, so that

$$W_1(x) = \int_2^{\sqrt{x}} \sqrt{t}\,\sqrt{1+\frac{x}{t^2}}\,\frac{dt}{\log t} = \int_2^{\sqrt{x}} h(t)\,\frac{dt}{\log t}.$$

Using integration by parts, we get

$$W_1(x) = \frac{H(u)}{\log u}\bigg|_{u=2}^{u=\sqrt{x}} + \int_2^{\sqrt{x}} \frac{H(u)}{u\log^2 u}\,du$$
$$= 2\frac{H(\sqrt{x})}{\log x} - \frac{H(2)}{\log 2} + \int_2^{\sqrt{x}} \frac{H(u)}{u\log^2 u}\,du. \tag{70}$$

For $1 \le t \le \sqrt{x}$,

$$h(t) = \sqrt{t}\sqrt{1+\frac{x}{t^2}} \le \sqrt{t}\left(1+\frac{\sqrt{x}}{t}\right) = \sqrt{t} + \frac{\sqrt{x}}{\sqrt{t}} \le 2\frac{\sqrt{x}}{\sqrt{t}}.$$

Therefore, for $2 \le u \le \sqrt{x}$,

$$H(u) \le 2\int_1^u \frac{\sqrt{x}}{\sqrt{t}}\,dt = 2\sqrt{x}\left(2\sqrt{t}\bigg|_{t=1}^{t=u}\right) \le 4\sqrt{x}\sqrt{u}.$$

And similarly as with (67), we easily establish that

$$\int_2^{\sqrt{x}} \frac{H(u)}{u\log^2 u}\,du \ll \frac{x^{3/4}}{\log^2 x} \tag{71}$$

and that

$$H(2) \ll \sqrt{x}. \tag{72}$$

Observe that by definition, $H(\sqrt{x}) = S_1(x)$ (the function defined in (21) and handled in (23)), so that because of (28), we may write that

$$H(\sqrt{x}) = S_1(x) = \left(\frac{2\sqrt{2}}{3} + \frac{2\sqrt{2\pi}\Gamma(5/4)}{3\Gamma(3/4)}\right)x^{3/4} + O(\sqrt{x}). \tag{73}$$

Using (71), (72) and (73) in (70), we get

$$W_1(x) = \left(\frac{4\sqrt{2}}{3} + \frac{4\sqrt{2\pi}\Gamma(5/4)}{3\Gamma(3/4)}\right)\frac{x^{3/4}}{\log x} + O\left(\frac{x^{3/4}}{\log^2 x}\right). \tag{74}$$

Combining (64), (65), (69) and (74) in (63), we obtain

$$W(x) = \left(\frac{4\sqrt{2}}{3} + \frac{4\sqrt{2\pi}\Gamma(5/4)}{3\Gamma(3/4)}\right)\frac{x^{3/4}}{\log x} + O\left(\frac{x^{3/4}}{\log^2 x}\right).$$

Substituting this last equation in (62) yields

$$\mathcal{R}_2(x) \le \frac{4\sqrt{2\pi}\Gamma(5/4)}{3\Gamma(3/4)}\frac{x^{3/4}}{\log x} + O\left(\frac{x^{3/4}}{\log^2 x}\right). \tag{75}$$

5.3 Evaluation of $\mathcal{R}(x)$

Combining the estimation of $\mathcal{R}_1(x)$ and $\mathcal{R}_2(x)$ provided by (57) and (75), the inequalities in Theorem 2 follow immediately.

6 Final remarks, numerical data and open problems

The main reason we could obtain an asymptotic formula for $\mathcal{T}(x) := \sum_{n\le x}\tau_\square(n)$ (stated in Theorem 1) is that we could rely on an exact formula for $\sum_{n\le x}\tau_0(n)$, where $\tau_0(n) := \#\{d \mid n : d<\sqrt{n} \text{ and } d+n/d=c^2 \text{ for some } c\in\mathbb{N}\}$, since $\mathcal{T}(x)=2V(x)+1$, where

$$V(x) := \sum_{n\le x}\tau_0(n) = \sum_{1\le k<\sqrt{x}}\left(\left\lfloor\sqrt{\frac{x}{k}+k}\right\rfloor - \left\lfloor\sqrt{2k}\right\rfloor\right). \tag{76}$$

This allowed us to compute the values of $\mathcal{T}(x)$ for $x=10^k$, $k=1,2,\ldots,16$, using the exact formula (76). Unsurprisingly, as indicated in Table 1, the quotient $\mathcal{T}(x)/c_\square x^{3/4}$ tends to 1 rapidly (as predicted by Theorem 1).

Table 1: Values of $\mathcal{T}(x)$

x	$\mathcal{T}(x)$	$\mathcal{T}(x)/c_\square x^{3/4}$
10^1	5	0.359670
10^2	51	0.652385
10^3	349	0.793888
10^4	2 183	0.883055
10^5	12 997	0.934926
10^6	75 199	0.961936
10^7	430 251	0.978714
10^8	2 442 733	0.988121
10^9	13 808 741	0.993318
10^{10}	77 883 647	0.996277
10^{11}	438 686 005	0.997902
10^{12}	2 469 185 551	0.998821
10^{13}	13 892 386 569	0.999335
10^{14}	78 145 511 685	0.999627
10^{15}	439 515 879 593	0.999790
10^{16}	2 471 807 878 895	0.999882

Regarding the estimation of $\mathcal{R}(x) := \#\{n\le x : \rho_1(n)+\rho_2(n)=c^2 \text{ for some } c\in\mathbb{N}\}$, the outcome is very different. Indeed, recall that

$$\mathcal{R}(x)=\mathcal{R}_1(x)+\mathcal{R}_2(x)+1,$$

where $\mathcal{R}_1(x)=\#\{n\le x : P(n)>\sqrt{n} \text{ and } \rho_1(n)+\rho_2(n)=c^2 \text{ for some } c\in\mathbb{N}\}$ and $\mathcal{R}_2(x)=\#\{n\le x : P(n)<\sqrt{n} \text{ and } \rho_1(n)+\rho_2(n)=c^2 \text{ for some } c\in\mathbb{N}\}$.

Even though we could obtain an exact formula for $\mathcal{R}_1(x)$, namely

$$\mathcal{R}_1(x) = \sum_{p\le\sqrt{x}} \left(\left\lfloor\sqrt{2p-1}\right\rfloor - \lfloor\sqrt{p}\rfloor\right) + \sum_{\sqrt{x}<p\le x} \left(\left\lfloor\sqrt{p+\frac{x}{p}}\right\rfloor - \lfloor\sqrt{p}\rfloor\right),$$

from which we deduced the asymptotic formula $\mathcal{R}_1(x) \sim c_\Box x^{3/4}/\log x$ (as $x \to \infty$) – and actually the more accurate formula (57), we were unable to obtain an exact formula for $\mathcal{R}_2(x)$ and had to settle for an upper bound. Perhaps, eventually, one could prove that $\mathcal{R}(x) \sim c_\triangle x^{3/4}/\log x$ as $x \to \infty$, for some constant $c_\triangle \in (c_\Box, 2c_\Box)$. If such a constant exists, it could be near $2c_\Box$, as the data in Table 2 seems to indicate.

Table 2: Values of $\mathcal{R}(x)$

x	$\mathcal{R}(x)$	$\mathcal{R}(x)/(x^{3/4}/\log x)$
10	2	0.818928
10^2	12	1.74754
10^3	65	2.52494
10^4	325	2.99336
10^5	1 647	3.37194
10^6	8 517	3.72095
10^7	45 167	4.09388
10^8	241 394	4.44664
10^9	1 295 225	4.77313

Finally, as mentioned in Section 1, the underlying motivation for studying the function $\tau_\Box(n)$ originated in the search for elliptic curves with a high rank, and more precisely in a particular search that relies on integers n with a corresponding large value for $\tau_\Box(n)$. So, it seems natural to ask how large can $\tau_\Box(n)$ be. Most likely, it can be of any size, but we could not prove that. Nevertheless, through a computer search, we did find integers n with a large number of divisors $d < \sqrt{n}$ with the property that $d + n/d$ is a perfect square. For instance, the number

$$10\,631\,634\,411\,847\,680\,000 = 2^{26} \cdot 3^5 \cdot 5^4 \cdot 7 \cdot 11 \cdot 19 \cdot 23 \cdot 31$$

has 32 such divisors. More generally, letting $\mathcal{T}_k$ stand for the set of integers n which have k distinct divisors $d < \sqrt{n}$ such that $d + n/d$ is a perfect square, we conjecture that for each positive integer k, the corresponding set $\mathcal{T}_k$ is infinite.

Also, consider the numbers $n = 4k^4 - 1$, where $k = 2, 3, \ldots$. Since $\rho_1(n) + \rho_2(n) = (2k^2 - 1) + (2k^2 + 1) = 4k^2$, we have that $n \in \mathcal{R}$, and since $n + 1 = 4k^4$, we have that $\rho_1(n+1) + \rho_2(n+1) = 4k^2$, implying that $n + 1 \in \mathcal{R}$ as well. This observation establishes the fact that there exist infinitely many integers $n \in \mathcal{R}$ such that $n+1 \in \mathcal{R}$ as well. Setting $\mathcal{R}^{(2)}(x) := \#\{n \le x : n, n+1 \in \mathcal{R}\}$, we have thus proved that $\mathcal{R}^{(2)}(x) \gg x^{1/4}$. What about triplets? Are there infinitely many $n \in \mathcal{R}$ such that $n+1, n+2 \in \mathcal{R}$ as well? Setting $\mathcal{R}^{(3)} := \{n \in \mathbb{N} : n, n+1, n+2 \in \mathcal{R}\}$, one can check that 154, 282 674, 144 544 673 718 847 655 and 10 931 129 469 745 989 328 319 belong to $\mathcal{R}^{(3)}$. This set is most likely infinite, but we were unable to prove it.

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Data availability statement

Section 6 of the paper includes the data supporting the results proved in the paper. The data in Tables 1 and 2 was generated using the software system Mathematica.

Declarations

The authors have no conflict of interest.

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