Summing the largest prime factor over integer sequences

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Abstract

Given an integer $n \geq 2$, let $P(n)$ stand for its largest prime factor. We examine the behaviour of $\displaystyle\sum_{\substack{n\leq x\\ n\in A}} P(n)$ in the case of two sets A, namely the set of r-free numbers and the set of h-full numbers.

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1 Introduction

Given an integer $n \geq 2$, let $P(n)$ stand for its largest prime factor, with $P(1) = 1$. Even though this function is very chaotic as the values of $P(n)$ alternate between small and large values as n varies, its average value over large intervals is more smooth and can be estimated.

The first published estimate regarding the sum $\sum_{n\leq x} P(n)$ is due to Alladi and Erdős [1] as they proved that

$$\sum_{n\leq x} P(n) = \frac{\pi^2}{12}\frac{x^2}{\log x} + O\left(\frac{x^2}{\log^2 x}\right). \tag{1.1}$$

This result was later improved by De Koninck and Ivić [2] when they showed that, given any positive integer k, there exist computable constants $c_2, \ldots, c_k$ such that

$$\sum_{n\leq x} P(n) = \frac{\pi^2}{12}\frac{x^2}{\log x} + c_2\frac{x^2}{\log^2 x} + \cdots + c_k\frac{x^2}{\log^k x} + O\left(\frac{x^2}{\log^{k+1} x}\right). \tag{1.2}$$

A natural question to ask is how the above formula changes if instead of summing $P(n)$ over all natural numbers $n \leq x$, we restrict these numbers n to a particular subset A of $\mathbb{N}$. For this purpose, we will consider here two large families of integers, namely the set of r-free numbers and the set of h-full numbers.

Given an integer $n \geq 2$, write its prime factorisation as $n = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_s^{\alpha_s}$, where $q_1 < q_2 < \cdots < q_s$ are primes and $\alpha_1, \alpha_2, \ldots, \alpha_s \in \mathbb{N}$. Given fixed integers $r \geq 2$ and $h \geq 2$, we say that n is a *r-free number* if $\max(\alpha_1, \alpha_2, \ldots, \alpha_s) \leq r - 1$, whereas we say that n is a *h-full number* if $\min(\alpha_1, \alpha_2, \ldots, \alpha_s) \geq h$. We will denote by $\mathbb{F}_r$ the set of r-free numbers; amongst these sets, the sets $\mathbb{F}_2$ of *square-free numbers* and the set $\mathbb{F}_3$ of *cube-free numbers* are often mentioned in the literature. On the other hand, we will denote by $\mathbb{P}_h$ the set of h-full numbers. Particular cases are the set $\mathbb{P}_2$, known as the set of *powerful numbers* or *square-full numbers*, and the set $\mathbb{P}_3$, the set of *cube-full numbers*.

In what follows we will make frequent use of the Riemann zeta function $\zeta(s)$ defined by

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \qquad (s > 1).$$

Let $\mu_r(n)$ be the characteristic function of the r-free numbers, that is,

$$\mu_r(n) = \begin{cases} 1 & \text{if } n \text{ is } r\text{-free}, \\ 0 & \text{otherwise}, \end{cases}$$

implying in particular that its generating function is

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\mu_r(n)}{n^s} &= \prod_p \left(1 + \frac{1}{p^s} + \cdots + \frac{1}{p^{(r-1)s}}\right) \\ &= \frac{\prod_p \left(1 - \frac{1}{p^{rs}}\right)}{\prod_p \left(1 - \frac{1}{p^s}\right)} = \frac{\zeta(s)}{\zeta(rs)} \qquad (s > 1). \end{aligned} \tag{1.3}$$

Let $\chi_h(n)$ be the characteristic function of the h-full numbers, that is,

$$\chi_h(n) = \begin{cases} 1 & \text{if } n \text{ is } h\text{-full}, \\ 0 & \text{otherwise}, \end{cases}$$

implying in particular that its generating function is

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{\chi_h(n)}{n^s} &= \prod_p \left(1 + \frac{1}{p^{hs}} + \frac{1}{p^{(h+1)s}} + \cdots\right) \\ &= \zeta(hs) \prod_p \left(1 - \frac{1}{p^{hs}}\right) \prod_p \left(1 + \frac{1}{p^{hs}} + \frac{1}{p^{(h+1)s}} + \cdots\right) \\ &= \zeta(hs) \prod_p \left(1 + \frac{1}{p^{(h+1)s}} + \frac{1}{p^{(h+2)s}} + \cdots + \frac{1}{p^{(2h-1)s}}\right) \qquad (s > 1). \end{aligned} \tag{1.4}$$

Finally, let us mention that the counting functions $\mathbb{F}_r(x)$ and $\mathbb{P}_h(x)$ of these two families of numbers are well-known. These are, for fixed integers $r \geq 2$ and $h \geq 2$,

$$\mathbb{F}_r(x) = \frac{1}{\zeta(r)} x + O\left(x^{1/r}\right), \tag{1.5}$$

$$\mathbb{P}_r(x) = \gamma_h x^{1/h} + O\left(x^{1/(h+1)}\right) \tag{1.6}$$

for some positive constant γ_h. For a proof of (1.5) in the simplest case, that is for $r = 2$, see Theorem 8.25 in the book of Niven, Zuckerman and Montgomery [4]; for a proof of the general case, that is for any $r \geq 2$, see the survey paper of Pappalardi [5]. For a proof of (1.6), see the paper of Ivíc and Shiu [3], where in fact a much more accurate formula is proved.

2 Main results

For our first set A, we choose the set of r-free numbers $\mathbb{F}_r$. In this case we can prove the following.

Theorem 1. *Let $r \geq 2$ be a fixed integer. Then, given any positive integer k, there exist computable constants $d_1, d_2, \ldots, d_k$ such that*

$$\sum_{\substack{n \leq x \\ n \in \mathbb{F}_r}} P(n) = \sum_{n \leq x} \mu_r(n) P(n) = d_1 \frac{x^2}{\log x} + d_2 \frac{x^2}{\log^2 x} + \cdots + d_k \frac{x^2}{\log^k x} + O\left(\frac{x^2}{\log^{k+1} x}\right), \tag{2.1}$$

where in particular, in light of (1.5),

$$d_1 = d_1^{(r)} = \frac{1}{2} \sum_{n=1}^{\infty} \frac{\mu_r(n)}{n^2} = \frac{\zeta(2)}{2\zeta(2r)}$$

Remark 2.1. *In the case $r = 2$, that is, the case of square-free numbers, we have*

$$d_1^{(2)} = \frac{\zeta(2)}{2\zeta(4)} = \frac{15}{2\pi^2} = 0.759909\ldots.$$

In the case $r = 3$, that is, the case of cube-free numbers, we have

$$d_1^{(3)} = \frac{\zeta(2)}{2\zeta(6)} = \frac{315}{4\pi^4} = 0.808446\ldots.$$

When choosing $A = \mathbb{P}_h$, we can prove the following general result.

Theorem 2. *Let $h \geq 2$ be a fixed integer. Then, given any positive integer k, there exist computable constants $e_1, e_2, \ldots, e_k$ such that*

$$\sum_{\substack{n \leq x \\ n \in \mathbb{P}_h}} P(n) = e_1 \frac{x^{2/h}}{\log x} + e_2 \frac{x^{2/h}}{\log^2 x} + \cdots + e_k \frac{x^{2/h}}{\log^k x} + O\left(\frac{x^{2/h}}{\log^{k+1} x}\right), \tag{2.2}$$

where

$$e_1 = \frac{h}{2} \sum_{n \in \mathbb{P}_h} \frac{1}{n^{2/h}} = \frac{h}{2} \prod_p \left(1 + \frac{1}{(p^h)^{2/h}} + \frac{1}{(p^{h+1})^{2/h}} + \cdots\right).$$

Remark 2.2. *In the particular case of square-full numbers, we have, in light of (1.4) with $h = 2$ and $s = 1$,*

$$e_1 = \sum_{n \in \mathbb{P}_2} \frac{1}{n} = \prod_p \left(1 + \frac{1}{p^2} + \frac{1}{p^3} + \cdots\right) = \frac{\zeta(2)\zeta(3)}{\zeta(6)} = 1.9436\ldots$$

In the case of cube-full numbers, we find

$$e_1 = \frac{3}{2} \sum_{n \in \mathbb{P}_3} \frac{1}{n^{2/3}} = \frac{3}{2} \prod_p \left(1 + \frac{1}{p^2} + \frac{1}{p^{8/3}} + \frac{1}{p^{10/3}} + \cdots\right) = 3.44967\ldots.$$

3 Preliminary results

Let $\pi(x)$ stand for the number of primes not exceeding x. Using the prime number theorem in the form

$$\pi(x) = \frac{x}{\log x} + \frac{1!x}{\log^2 x} + \frac{2!x}{\log^3 x} + \cdots + \frac{(k-1)!x}{\log^k x} + O\left(\frac{x}{\log^{k+1} x}\right),$$

one can easily prove the following.

Lemma 1. *Given any positive integer k, there exist computable constants $a_2, \ldots, a_k$ such that*

$$\sum_{p \le X} p = \frac{1}{2}\frac{X^2}{\log X} + a_2 \frac{X^2}{\log^2 X} + \cdots + a_k \frac{X^2}{\log^k X} + O\left(\frac{X^2}{\log^{k+1} X}\right). \tag{3.1}$$

We also have the following.

Lemma 2. *Given fixed positive integers s and k, there exist computable constants $c_{0,s}, c_{1,s}, \ldots, c_{k,s}$ such that*

$$\sum_{n \le \exp\{\sqrt{\log x}\}} \frac{1}{n^2 \log^s(x/n)} = \frac{c_{0,s}}{\log^s x} + \frac{c_{1,s}}{\log^{s+1} x} + \cdots + \frac{c_{k,s}}{\log^{s+k} x} + O\left(\frac{1}{\log^{s+k+1} x}\right). \tag{3.2}$$

On the other hand, given a fixed integer $r \ge 2$, as well as fixed integers $s, k \in \mathbb{N}$, there exist computable constants $d_{0,s}, d_{1,s}, \ldots, d_{k,s}$ such that

$$\sum_{n \le \sqrt{x}} \frac{\mu_r(n)}{n^2 \log^s(x/n)} = \frac{d_{0,s}}{\log^s x} + \frac{d_{1,s}}{\log^{s+1} x} + \cdots + \frac{d_{k,s}}{\log^{s+k} x} + O\left(\frac{1}{\log^{s+k+1} x}\right). \tag{3.3}$$

Proof. We only provide the proof of (3.3) since the proof of (3.2) is similar. Since we assumed that $n \le \sqrt{x}$, we have that $\dfrac{\log n}{\log x} \le \dfrac{1}{2}$. Therefore, for a fixed integer $k \ge 1$ and all $y \le \frac{1}{2}$, we may use the expansion

$$\frac{1}{1-y} = 1 + y + y^2 + \cdots + y^k + O(y^{k+1})$$

to write that, in the case $s = 1$,

$$\begin{aligned}
\frac{\mu_r(n)}{n^2 \log(x/n)} &= \frac{\mu_r(n)}{n^2 \log x \left(1 - \frac{\log n}{\log x}\right)} = \frac{\mu_r(n)}{n^2 \log x}\left(1 + \frac{\log n}{\log x} + \cdots + \frac{\log^{k-1} n}{\log^{k-1} x} + O\left(\frac{\log^k n}{\log^k x}\right)\right) \\
&= \frac{\mu_r(n)}{n^2}\frac{1}{\log x} + \frac{\mu_r(n)\log n}{n^2}\frac{1}{\log^2 x} + \cdots + \frac{\mu_r(n)\log^{k-1} n}{n^2}\frac{1}{\log^k x} + O\left(\frac{1}{\log^{k+1} x}\right).
\end{aligned}$$

Then, observing that for each integer $j \ge 1$, the corresponding series $\displaystyle\sum_{n=1}^{\infty} \frac{\mu_r(n)\log^j n}{n^2}$ converges, estimate (3.3) in the case $s = 1$ easily follows. The case of an arbitrary $s \in \mathbb{N}$ can be handled similarly. □

Lemma 3. *For each integer $h \geq 2$,*

$$\sum_{\substack{m\leq x\\ m\in\mathbb{P}_h}} \frac{1}{m^{2/h}} = \sum_{m\in\mathbb{P}_h} \frac{1}{m^{2/h}} + O\left(\frac{1}{x^{1/h}}\right) \tag{3.4}$$

and, for each fixed $j \in \mathbb{N}$,

$$\sum_{\substack{m\leq x\\ m\in\mathbb{P}_h}} \frac{\log^j m}{m^{2/h}} = O(1). \tag{3.5}$$

Proof. First observe that, replacing the sum $\sum_{\substack{m>x\\ m\in\mathbb{P}_h}} \frac{1}{m^{2/h}}$ by a Stieltjes integral, using integration by parts and thereafter using the bound $\mathbb{P}_h(t) = O(t^{1/h})$ provided by (1.6), we obtain that

$$\begin{aligned}
\sum_{\substack{m>x\\ m\in\mathbb{P}_h}} \frac{1}{m^{2/h}} &= \int_x^\infty \frac{1}{t^{2/h}}\, d\,\mathbb{P}_h(t) = \left.\frac{\mathbb{P}_h(t)}{t^{2/h}}\right|_x^\infty + \frac{2}{h}\int_x^\infty t^{-\frac{2}{h}-1}\mathbb{P}_h(t)\, dt \\
&\ll \frac{1}{x^{2/h-1/h}} + \frac{2}{h}\int_x^\infty t^{1/h-1}\, dt \ll \frac{1}{x^{1/h}}.
\end{aligned} \tag{3.6}$$

Using (3.6), it follows that

$$\sum_{\substack{m\leq x\\ m\in\mathbb{P}_h}} \frac{1}{m^{2/h}} = \sum_{m\in\mathbb{P}_h} \frac{1}{m^{2/h}} - \sum_{\substack{m>x\\ m\in\mathbb{P}_h}} \frac{1}{m^{2/h}} = \sum_{m\in\mathbb{P}_h} \frac{1}{m^{2/h}} + O\left(\frac{1}{x^{1/h}}\right),$$

thus completing the proof of (3.4).

The proof of (3.5) can easily be established using the same technique as that employed to prove (3.4). □

4 Proof of Theorem 1

First observe that

$$\begin{aligned}
M(x) &:= \sum_{n\leq x} \mu_r(n)P(n) = \sum_{p\leq x} p \sum_{\substack{mp\leq x\\ P(m)<p}} \mu_r(m) \\
&= \sum_{p\leq\sqrt{x}} p \sum_{\substack{m\leq x/p\\ P(m)<p}} \mu_r(m) + \sum_{\sqrt{x}<p\leq x} p \sum_{\substack{m\leq x/p\\ P(m)<p}} \mu_r(m) \\
&= M_1(x) + M_2(x).
\end{aligned} \tag{4.1}$$

It is trivial that

$$M_1(x) \leq \sum_{p\leq\sqrt{x}} p\cdot\frac{x}{p} = x\,\pi(\sqrt{x}) \ll \frac{x^{3/2}}{\log x}. \tag{4.2}$$

To estimate $M_2(x)$, first observe that if $p > \sqrt{x}$, we have that $x/p < p$, in which case the condition $P(m) < p$ appearing in the definition of $M_2(x)$ can be dropped. Hence, inverting the sum over p with the sum over m, and then using Lemma 1 with $X = \sqrt{x}$, we obtain that

$$\begin{aligned} M_2(x) &= \sum_{\sqrt{x}<p\le x} p \sum_{m\le x/p} \mu_r(m) = \sum_{m\le\sqrt{x}} \mu_r(m) \sum_{\sqrt{x}<p\le x/m} p \\ &= \sum_{m\le\sqrt{x}} \mu_r(m) \sum_{p\le x/m} p - \sum_{m\le\sqrt{x}} \mu_r(m) \sum_{p\le\sqrt{x}} p \\ &= \sum_{m\le\sqrt{x}} \mu_r(m) \sum_{p\le x/m} p + O\left(\frac{x^{3/2}}{\log x}\right) \\ &= M_3(x) + O\left(\frac{x^{3/2}}{\log x}\right). \qquad (4.3) \end{aligned}$$

Gain using Lemma 1 but this time with $X = x/m$ and thereafter formula (3.3) of Lemma 2, we obtain

$$\begin{aligned} M_3(x) &= \sum_{m\le\sqrt{x}} \mu_r(m) \left\{ \frac{1}{2}\frac{(x/m)^2}{\log(x/m)} + a_2\frac{(x/m)^2}{\log^2(x/m)} \right. \\ &\qquad \left. + \cdots + a_k\frac{(x/m)^2}{\log^k(x/m)} + O\left(\frac{(x/m)^2}{\log^{k+1}(x/m)}\right) \right\} \\ &= \frac{1}{2}\sum_{m=1}^{\infty}\frac{\mu_r(m)}{m^2}\frac{x^2}{\log x} + d_2\frac{x^2}{\log^2 x} + \cdots + d_k\frac{x^2}{\log^k x} + O\left(\frac{x^2}{\log^{k+1} x}\right), \qquad (4.4) \end{aligned}$$

where we took the liberty to replace $\sum_{m\le\sqrt{x}} \frac{\mu_r(m)}{m^2}$ by $\sum_{m=1}^{\infty} \frac{\mu_r(m)}{m^2}$, a justified move since

$$\begin{aligned} \sum_{m\le\sqrt{x}} \frac{\mu_r(m)}{m^2} &= \sum_{m=1}^{\infty} \frac{\mu_r(m)}{m^2} - \sum_{m>\sqrt{x}} \frac{\mu_r(m)}{m^2} \\ &= \sum_{m=1}^{\infty} \frac{\mu_r(m)}{m^2} + O\left(\int_{\sqrt{x}}^{\infty} \frac{dt}{t^2}\right) \\ &= \sum_{m=1}^{\infty} \frac{\mu_r(m)}{m^2} + O\left(\frac{1}{\sqrt{x}}\right). \end{aligned}$$

Finally, gathering (4.2), (4.3) and (4.4) in (4.1) completes the proof of Theorem 1.

5 Proof of Theorem 2

First, we write

$$U(x) := \sum_{\substack{n\le x \\ n\in\mathbb{P}_h}} P(n) = \sum_{p\le x^{1/h}} p \sum_{\substack{mp^h\le x \\ m\in\mathbb{P}_h \\ P(m)\le p}} 1 = \sum_{p\le x^{1/h}} p \sum_{\substack{m\le x/p^h \\ m\in\mathbb{P}_h \\ P(m)\le p}} 1$$

$$
\begin{aligned}
&= \sum_{p\le x^{1/(h+1)}} p \sum_{\substack{m\le x/p^h\\ m\in\mathbb{P}_h\\ P(m)\le p}} 1 + \sum_{x^{1/(h+1)}<p\le x^{1/h}} p \sum_{\substack{m\le x/p^h\\ m\in\mathbb{P}_h\\ P(m)\le p}} 1\\
&= U_1(x)+U_2(x). \qquad (5.1)
\end{aligned}
$$

It follows from estimate (1.6) that

$$
\begin{aligned}
U_1(x) &\le \sum_{p\le x^{1/(h+1)}} p \sum_{\substack{m\le x/p^h\\ m\in\mathbb{P}_h}} 1 = \sum_{p\le x^{1/(h+1)}} p\, \mathbb{P}_h\left(\frac{x}{p^h}\right)\\
&\ll \sum_{p\le x^{1/(h+1)}} p\frac{x^{1/h}}{p} = x^{1/h}\,\pi(x^{1/(h+1)}) \ll \frac{x^{\frac{2h+1}{h(h+1)}}}{\log x}. \qquad (5.2)
\end{aligned}
$$

To evaluate $U_2(x)$, first observe that for $p > x^{1/(h+1)}$, we have that $\dfrac{x}{p^h} < p$, implying that in this case the condition $P(m) \le p$ appearing in the second sum defining $U_2(x)$ can be dropped and therefore that

$$
\begin{aligned}
U_2(x) &= \sum_{x^{1/(h+1)}<p\le x^{1/h}} p \sum_{\substack{mp^h\le x\\ m\in\mathbb{P}_h}} 1 = \sum_{p\le x^{1/h}} p \sum_{\substack{mp^h\le x\\ m\in\mathbb{P}_h}} 1 - \sum_{p\le x^{1/(h+1)}} p \sum_{\substack{mp^h\le x\\ m\in\mathbb{P}_h}} 1\\
&= \sum_{p\le x^{1/h}} p \sum_{\substack{mp^h\le x\\ m\in\mathbb{P}_h}} 1 + O\left(\frac{x^{\frac{2h+1}{h(h+1)}}}{\log x}\right)\\
&= T(x) + O\left(\frac{x^{\frac{2h+1}{h(h+1)}}}{\log x}\right) = T(x) + O\left(\frac{x^{2/h}}{\log^{k+1} x}\right), \qquad (5.3)
\end{aligned}
$$

where we made use of (5.2) and the fact that $\dfrac{2h+1}{h(h+1)} < \dfrac{2}{h}$.

Inverting the two sums appearing in the definition of $T(x)$, we can rewrite $T(x)$ as follows.

$$
\begin{aligned}
T(x) &= \sum_{\substack{m\le x\\ m\in\mathbb{P}_h}} \sum_{p^h\le x/m} p = \sum_{\substack{m\le x\\ m\in\mathbb{P}_h}} \sum_{p\le (x/m)^{1/h}} p\\
&= \sum_{\substack{m\le \exp\{\sqrt{\log x}\}\\ m\in\mathbb{P}_h}} \sum_{p\le (x/m)^{1/h}} p + \sum_{\substack{\exp\{\sqrt{\log x}\}<m\le p^{1/h}\\ m\in\mathbb{P}_h}} \sum_{p\le (x/m)^{1/h}} p\\
&= T_1(x)+T_2(x). \qquad (5.4)
\end{aligned}
$$

Again, using the bound $\mathbb{P}_h(t) \ll t^{1/h}$ ensured by estimate (1.6), we have, arguing as we did in Lemma 3,

$$
\begin{aligned}
T_2(x) &\le \sum_{\substack{\exp\{\sqrt{\log x}\}<m\le p^{1/h}\\ m\in\mathbb{P}_h}} \left(\frac{x}{m}\right)^{2/h} = x^{2/h} \sum_{\substack{\exp\{\sqrt{\log x}\}<m\le p^{1/h}\\ m\in\mathbb{P}_h}} \frac{1}{m^{2/h}}\\
&= x^{2/h} \int_{\exp\{\sqrt{\log x}\}}^{x} t^{-2/h}\, d\,\mathbb{P}_h(t)
\end{aligned}
$$

$$\ll x^{2/h}\left(t^{-2/h}t^{1/h}\Big|_{\exp\{\sqrt{\log x}\}}^{x}+\int_{\exp\{\sqrt{\log x}\}}^{x} t^{-\frac{2}{h}-1}t^{1/h}\,dt\right)$$

$$\ll x^{2/h}\cdot\frac{1}{t^{1/h}}\Big|_{\exp\{\sqrt{\log x}\}}^{x}\ll\frac{x^{2/h}}{\exp\{\frac{1}{h}\sqrt{\log x}\}}\ll\frac{x^{2/h}}{\log^{k+1}x}. \tag{5.5}$$

Making use of Lemma 1 with $X=(x/m)^{1/h}$ and thereafter of formula (3.2) of Lemma 2, we obtain

$$T_1(x)=\sum_{\substack{m\le\exp\{\sqrt{\log x}\}\\ m\in\mathbb{P}_h}}\left\{\frac{1}{2}\frac{(x/m)^{2/h}}{\frac{1}{h}\log(x/m)}+a_2\frac{(x/m)^{2/h}}{\frac{1}{h^2}\log^2(x/m)}+\cdots+a_k\frac{(x/m)^{2/h}}{\frac{1}{h^k}\log^k(x/m)}+O\left(\frac{(x/m)^{2/h}}{\log^{k+1}(x/m)}\right)\right\}$$

$$=e_1\frac{x^{2/h}}{\log x}+e_2\frac{x^{2/h}}{\log^2 x}+\cdots+e_k\frac{x^{2/h}}{\log^k x}+O\left(\frac{x^{2/h}}{\log^{k+1}x}\right), \tag{5.6}$$

where we used the fact that, in light of estimate (3.4) of Lemma 3,

$$\frac{h}{2}\frac{x^{2/h}}{\log x}\sum_{\substack{m\le\exp\{\sqrt{\log x}\}\\ m\in\mathbb{P}_h}}\frac{1}{m^{2/h}}=\frac{h}{2}\frac{x^{2/h}}{\log x}\left(\sum_{\substack{m=1\\ m\in\mathbb{P}_h}}^{\infty}\frac{1}{m^{2/h}}-\sum_{\substack{m>\exp\{\sqrt{\log x}\}\\ m\in\mathbb{P}_h}}\frac{1}{m^{2/h}}\right)$$

$$=\frac{x^{2/h}}{\log x}\frac{h}{2}\sum_{\substack{m=1\\ m\in\mathbb{P}_h}}^{\infty}\frac{1}{m^{2/h}}\left(1+O\left(\frac{1}{x^{1/h}}\right)\right)$$

and where we used estimate (3.5) of Lemma 3 to estimate the other coefficients e_i appearing in (5.6).

Finally, gathering estimates (5.2), (5.3), (5.4), (5.5) and (5.6) in (5.1) completes the proof of Theorem 2.

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