Expanding on results of Wirsing and Klurman

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Abstract

Building on two conjectures formulated by the second author in the 1980's regarding characterisations of multiplicative functions and their respective proofs by Wirsing in the late 1990's and by Klurman in 2017, we expand on the achievements of these two authors by proving new results.

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1 Introduction

As is common, we let $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$ stand the sets of positive integers, real and complex numbers, respectively. Let also $\mathcal{M}$ (resp. $\mathcal{M}^*$) stand for the set of complex-valued multiplicative (resp. completely multiplicative) functions. We say that $f \in \mathcal{M}_1$ (resp. $\mathcal{M}_1^*$), if $f \in \mathcal{M}$ (resp. $\mathcal{M}^*$) and $|f(n)| = 1$ for every $n \in \mathbb{N}$.

Let us first recall two classical conjectures due to the second author.

Conjecture 1. (Kátai [3]) *If $g \in \mathcal{M}$ and*

$$g(n+1) - g(n) \to 0 \quad \textit{as} \quad n \to \infty,$$

then either $g(n) \to 0$ as $n \to \infty$, or $g(n) = n^s$ for some complex number s with $\Re s < 1$.

Conjecture 2. (Kátai [3]) *If $g \in \mathcal{M}_1$ and*

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n \le x} |g(n+1) - g(n)| = 0 \qquad \textit{or} \qquad \lim_{x\to\infty} \frac{1}{\log x} \sum_{n \le x} \frac{|g(n+1) - g(n)|}{n} = 0,$$

then there exists a real number τ for which

$$g(n) = n^{i\tau} \quad \textit{for all } n \in \mathbb{N}.$$

Conjecture 1 was proved by E. Wirsing in 1996. The proof is given in Wirsing, Yuansheng and Pintsung [8] as well as in Wirsing and Zagier [9].

A particular case of Conjecture 2 was proved by O. Klurman [4] in 2017 and we state it as follows.

Theorem A. (Klurman) *Conjecture 2 is true for $g \in \mathcal{M}_1^*$.*

Here, we expand on the above results of Wirsing and Klurman.

2 A generalisation of Wirsing's result

In a recent paper [1], we proved that if $f \in \mathcal{M}_1^*$ and if c_0, c_1, c_2 are three complex numbers satisfying $(c_0, c_1, c_2) \neq (0,0,0)$ for which

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n \le x} |c_0 f(n) + c_1 f(n+1) + c_2 f(n+2)| = 0,$$

then there exists a real number τ for which $f(n) = n^{i\tau}$ for all positive integers n.

In the same paper [1], we formulated the following conjecture.

Conjecture 3. *Given $f \in \mathcal{M}_1^*$ and complex numbers $c_0, c_1, \ldots, c_k$ such that $(c_0, c_1, \ldots, c_k) \neq (0,0,\ldots,0)$. Assuming that*

$$c_0 f(n) + c_1 f(n+1) + \cdots + c_k f(n+k) \to 0 \qquad \text{as } n \to \infty,$$

then there exists a real number τ for which $f(n) = n^{i\tau}$ for all positive integers n.

In [1], we also formulated the following weaker conjecture.

Conjecture 4. *Given $f \in \mathcal{M}_1^*$ and complex numbers $c_0, c_1, \ldots, c_k$ such that $(c_0, c_1, \ldots, c_k) \neq (0,0,\ldots,0)$. Assuming that*

$$\frac{1}{x} \sum_{n \le x} |c_0 f(n) + c_1 f(n+1) + \cdots + c_k f(n+k)| \to 0 \qquad \text{as } x \to \infty,$$

then $c_0 + c_1 + \cdots + c_k = 0$ and there exists a real number τ for which $f(n) = n^{i\tau}$ for all positive integers n.

Observe that in [1] we were able to prove Conjecture 4 in the particular case $k = 1$. However, we were unable to prove the more general case $k \ge 2$.

Now, consider the three operators E, I and Δ defined on the sequence $(x_n)_{n\in\mathbb{N}}$ by

$$E\, x_n = x_{n+1}, \qquad I\, x_n = x_n, \qquad \Delta\, x_n = E\, x_n - I\, x_n = x_{n+1} - x_n.$$

We also consider iterations of $\Delta f(n)$. For this, we let

$$\Delta^2 f(n) := \Delta\Delta f(n) = \Delta(f(n+1) - f(n)) = f(n+2) - 2f(n+1) + f(n),$$

and for an arbitrary integer $k \ge 3$, we let $\Delta^k f(n) := \Delta\Delta^{k-1} f(n)$.

Our purpose in this section and the next one is the following generalisation of the result of Wirsing mentioned in the Introduction.

Theorem 1. *Let $f \in \mathcal{M}_1^*$, set $s_f(n) := \Delta^7 f(n)$ and assume that $\lim_{n\to\infty} s_f(n) = 0$. Then, there exists a real number τ such that $f(n) = n^{i\tau}$ for all $n \in \mathbb{N}$.*

3 Proof of Theorem 1

In our recent paper, we proved (see Theorem 10 in [2]) that, assuming that

$$|\Delta^7 f(n)| \le K := 12 - \delta,$$

then there exists some real number t such that $f(n) = n^{it}F(n)$ for all $n \in \mathbb{N}$, where $F^k(n) = 1$ for all $n \in \mathbb{N}$, and $|\Delta^7 F(n)| \le K + \varepsilon$ provided $n \ge n_0(\varepsilon)$.

As we will see, Theorem 1 is an easy consequence of our Theorem 10 in [2]. Indeed, since we assumed that $s_f(n) \to 0$ as $n \to \infty$ and since $n^{i\tau} - (n+1)^{i\tau} \to 0$ as $n \to \infty$, it follows that $s_F(n) \to 0$ as $n \to \infty$. But since $s_F(n)$ can assume only finitely many distinct values, it follows that

$$s_F(n) = \Delta^7 F(n) = 0 \quad \text{for every large } n.$$

On the other hand, since the roots of the polynomial $(x-1)^7$ are $\omega_\ell := e^{2\pi i \ell/7}$ $(i = 0, 1, \ldots, 6)$, we have that

$$F(m + n_0) = \sum_{\ell=0}^{6} c_\ell \omega_\ell^m \qquad (m = 0, 1, 2, \ldots). \tag{3.1}$$

Observe that the right-hand side of (3.1) is periodic mod 7, that is, $F(n_1) = F(n_2)$ if $n_1 \equiv n_2 \pmod 7$ with $n_1, n_2 \ge n_0$. From this, it follows that $F(7n) = F(7(n+1))$, and since $F(7) \ne 0$, this implies that $F(n) = 1$ for all positive integers n, thereby completing the proof of Theorem 1.

In concluding this topic, we formulate yet another conjecture.

Conjecture 5. *Let $f \in \mathcal{M}_1^*$ and assume that*

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n \le x} \left|\Delta^7 f(n)\right| = 0.$$

Then, there exists a real number τ for which $f(n) = n^{i\tau}$ for every $n \in \mathbb{N}$.

4 Consequences of Klurman's theorem

We now move to some interesting consequences of Klurman's theorem, namely by establishing four new results.

Theorem 2. *Assume that $a, b, c \in \mathbb{N}$ and $d \in \mathbb{C}\backslash\{0\}$. Then functions $g_1 \in \mathcal{M}_1$ and $g_2 \in \mathcal{M}_1$ satisfy the condition*

$$\lim_{x\to\infty} \frac{1}{\log x} \sum_{n \le x} \frac{|g_1(an+b) - dg_2(cn)|}{n} = 0$$

if and only if there are functions $G_1, G_2 \in \mathcal{M}_1$ and a number $\tau \in \mathbb{R}$ such that

$$g_1(n) = n^{i\tau} G_1(n), \quad g_2(n) = n^{i\tau} G_2(n)$$

and

$$G_1(an+b) - d\Big(\frac{c}{a}\Big)^{i\tau} G_2(cn) = 0$$

are satisfied for every $n \in \mathbb{N}$.

Theorem 3. *Assume that $a, b, c \in \mathbb{N}$ and $d \in \mathbb{C}\setminus\{0\}$. Then functions $g_1 \in \mathcal{M}_1$ and $g_2 \in \mathcal{M}_1$ satisfy the condition*

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} |g_1(an+b) - dg_2(cn)| = 0$$

if and only if there are functions $G_1, G_2 \in \mathcal{M}_1$ and a number $\tau \in \mathbb{R}$ such that

$$g_1(n) = n^{i\tau} G_1(n), \quad g_2(n) = n^{i\tau} G_2(n)$$

and

$$G_1(an+b) - d\Big(\frac{c}{a}\Big)^{i\tau} G_2(cn) = 0$$

are satisfied for every $n \in \mathbb{N}$.

Theorem 4. *Assume that $a > 0$, b, $A > 0$, $B \in \mathbb{N}$ are integers with $(a, A) = 1$, $\Delta = Ab - aB \neq 0$ and $C \in \mathbb{C} \setminus \{0\}$. If $g \in \mathcal{M}_1^*$ satisfies the condition*

$$\lim_{x\to\infty} \frac{1}{\log x} \sum_{n\le x} \frac{|g(an+b) - Cg(An+B)|}{n} = 0,$$

then there is a real number τ such that $g(n) = n^{i\tau}$ for every $n \in \mathbb{N}$.

Theorem 5. *Assume that $a > 0$, b, $A > 0$, $B \in \mathbb{N}$ are integers with $(a, A) = 1$, $\Delta = Ab - aB \neq 0$ and $C \in \mathbb{C} \setminus \{0\}$. If $g \in \mathcal{M}_1^*$ satisfies the condition*

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} |g(an+b) - Cg(An+B)| = 0,$$

then there is a real number τ such that $g(n) = n^{i\tau}$ for every $n \in \mathbb{N}$.

5 Proofs of Theorems 2 through 5

For the proofs of Theorems 2 and 3, we will be using the following result.

Theorem B. (Phong [6], Theorems 1 and 2) *Assume that $a, b, c \in \mathbb{N}$ and $d \in \mathbb{C} \setminus \{0\}$. Then functions $g_1 \in \mathcal{M}_1$ and $g_2 \in \mathcal{M}_1$ satisfy the condition either*

$$\lim_{x\to\infty} \frac{1}{\log x} \sum_{n\le x} \frac{|g_1(an+b) - dg_2(cn)|}{n} = 0$$

or

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} |g_1(an+b) - dg_2(cn)| = 0$$

if and only if there are functions $g^* \in \mathcal{M}_1^*$ and $G_1, G_2 \in \mathcal{M}_1$ such that

$$g_1(n) = g^*(n)G_1(n), \quad g_2(n) = g^*(n)G_2(n),$$

$$G_1(an+b) - d\Big(\frac{g^*(c)}{g^*(a)}\Big)G_2(cn) = 0$$

and either

$$\lim_{x\to\infty} \frac{1}{\log x} \sum_{n\le x} \frac{|g^*(n+1) - g^*(n)|}{n} = 0 \tag{5.1}$$

or

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} |g^*(n+1) - g^*(n)| = 0 \tag{5.2}$$

are satisfied for every $n \in \mathbb{N}$.

Now, we infer from Theorem A that the conditions (5.1)-(5.2) imply that there is a number $\tau \in \mathbb{R}$ such that $g^*(n) = n^{i\tau}$ for every $n \in \mathbb{N}$.

Hence, Theorems 2 and 3 are proved.

In similar way, we prove Theorem 4. This time, we will be using the following result.

Theorem C. (Phong [7], Theorem 1) *Assume that $a > 0,\ b,\ A > 0,\ B \in \mathbb{N}$ be integers with $(a, A) = 1,\ \Delta = Ab - aB \neq 0$ and $C \in \mathbb{C} \setminus \{0\}$. If $g \in \mathcal{M}_1^*$ satisfies the condition*

$$\lim_{x\to\infty} \frac{1}{\log x} \sum_{n\le x} \frac{|g(an+b) - Cg(An+B)|}{n} = 0$$

then

$$\lim_{x\to\infty} \frac{1}{\log x} \sum_{n\le x} \frac{|g(n+1) - g(n)|}{n} = 0 \tag{5.3}$$

We also infer from Theorem A that the condition (5.3) implies that there is a number $\tau \in \mathbb{R}$ such that $g(n) = n^{i\tau}$ for every $n \in \mathbb{N}$, thereby completing the proof of Theorem 4.

We now move to the proof of Theorem 5. By adopting the method used in the proof of Theorem 1 in [7], we have the following result.

Theorem D. *Assume that $a > 0,\ b,\ A > 0,\ B \in \mathbb{N}$ be integers with $(a, A) = 1,\ \Delta = Ab - aB \neq 0$ and $C \in \mathbb{C} \setminus \{0\}$. If $g \in \mathcal{M}_1^*$ satisfies the condition*

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} |g(an+b) - Cg(An+B)| = 0,$$

then

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} |g(n+1) - g(n)| = 0. \tag{5.4}$$

The condition (5.4) with Theorem A implies the proof of Theorem 5.

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