ON THE VARIATIONS OF COMPLETELY MULTIPLICATIVE FUNCTIONS AT CONSECUTIVE ARGUMENTS

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Dedicated to the memory of Professor János Aczél

Abstract

We focus on the class $\mathcal{M}_1^*$ of completely multiplicative functions f whose set of values belong to the unit circle and their related function $\Delta f(n) := f(n+1) - f(n)$. For such functions f, we study the higher iterations $\Delta^m f(n)$ for fixed integers $m \in \{2, 3, \ldots, 7\}$, and for each of these we establish an absolute bound for $|\Delta^m f(n)|$. We also characterise those triplets of multiplicative functions f, g, h with unusually small gaps between their consecutive values. All our characterisations and bounds are obtained following new results of O. Klurman and A.P. Mangerel in the context of their proof of an old conjecture of Kátai characterising subclasses of $\mathcal{M}_1^*$.

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1 Introduction

Let $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$ stand for the set of points on the unit circle and let $\mathcal{M}_1^*$ stand for the set of completely multiplicative functions $f : \mathbb{N} \to \mathbb{T}$. Given $f \in \mathcal{M}_1^*$, we let $\Delta f(n) := f(n+1) - f(n)$.

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In 2017, Klurman [3] proved a 1983 conjecture of the second author [1], namely that given $f \in \mathcal{M}_1^*$ such that $\sum_{n \le x} |\Delta f(n)| = o(x)$ as $x \to \infty$ (or such that $\displaystyle\sum_{n \le x} \frac{|\Delta f(n)|}{n} = o(\log x)$ as $x \to \infty$), then there exists some real number t such that $f(n) = n^{it}$ for all $n \in \mathbb{N}$.

Given $f \in \mathcal{M}_1^*$, we shall denote by $S(f)$ the set of limit points of the set $\{f(n) : n \in \mathbb{N}\}$ and by $R(f)$ the set $\{p \in \wp : f(p) \ne 1\}$, where $\wp$ stands for the set of all primes. Also, given $k \in \mathbb{N}$, we set $W_k := \{e(a/k) : a = 0, 1, \ldots, k-1\} = \{\omega \in \mathbb{C} : \omega^k = 1\}$, where $e(y) := e^{2\pi i y}$. Finally, given a set of complex numbers $\{a_n : n \in \mathbb{N}\}$, we denote its closure by $\overline{\{a_n : n \in \mathbb{N}\}}$.

In 2018, Klurman and Mangerel [4] proved the following.

Theorem A. *Assume that $f, g \in \mathcal{M}_1^*$ are such that $S(f) = S(g) = \mathbb{T}$ and also that $\overline{\{(f(n), g(n+1)) : n \in \mathbb{N}\}} \ne \mathbb{T} \times \mathbb{T}$. Further assume that for infinitely many $j \in \mathbb{N}$, either $|R(f^j)| \cdot |R(g^j)| > 1$ or $R(f^j) \ne R(g^j)$. Then, for some real number t and positive integers k and ℓ, we have $f(n) = n^{it/k} F(n)$ and $g(n) = n^{it/\ell} G(n)$, where $F(\mathbb{N}) \in W_k$ and $G(\mathbb{N}) \in W_\ell$.*

This last theorem motivates the introduction of the set $\mathcal{H}$, namely the set made up of those pairs (f, g) of functions in $\mathcal{M}_1^*$ for which there exist infinitely many $j \in \mathbb{N}$ for which either $|R(f^j)| \cdot |R(g^j)| > 1$ or $R(f^j) \ne R(g^j)$.

Here, we apply the above results of Klurman and Mangerel to characterise those triplets of multiplicative functions f, g, h with unusually small gaps between their consecutive values. We also consider the higher iterations $\Delta^m f(n)$ for each of the integers $m = 2, 3, 4, 5, 6, 7$ and obtain bounds for $|\Delta^m f(n)|$.

2 Consequences of Klurman's result

Shortly after Klurman's 2017 result became known, Kátai and Phong [2] used his result to prove that if $f, g \in \mathcal{M}_1^*$ are such that $\displaystyle\sum_{n \le x} \frac{|g(2n+1) - Af(n)|}{n} = o(\log x)$ as $x \to \infty$ for some constant A, then there exists a real number t such that $f(n) = g(n) = n^{it}$ for all $n \in \mathbb{N}$ and moreover $A = f(2)$. They also proved that the same result holds if $\sum_{n \le x} |g(2n+1) - Af(n)| = o(x)$ as $x \to \infty$.

As a consequence of this result, we have the following.

Theorem B. *If $f, g \in \mathcal{M}_1^*$ are such that $\sum_{n \le x} |g(n+1) - f(n)| = o(x)$ as $x \to \infty$ or such that $\displaystyle\sum_{n \le x} \frac{|g(n+1) - f(n)|}{n} = o(\log x)$ as $x \to \infty$, then there exists a real number t such that $f(n) = g(n) = n^{it}$ for all $n \in \mathbb{N}$.*

Proof. By hypothesis, the sequence $g(n+1) - f(n)$ tends to 0 for almost all n. Hence, the same is true for the sequence $g(2n+1) - f(2)f(n)$, in which case Theorem B follows as a direct consequence of the above result of Kátai and Phong. $\square$

3 The case of three functions

We can prove the following.

Theorem 1. *Let $f, g, h \in \mathcal{M}_1^*$ be such that the function $s(n) := g(n+2) - 2h(n+1) + f(n)$ satisfies*

$$\sum_{n \le x} \frac{|s(n)|}{n} = o(\log x) \qquad (x \to \infty).$$

Then, there exists a real number t such that $f(n) = g(n) = h(n) = n^{it}$ for all $n \in \mathbb{N}$.

Proof. It follows from the definition of $s(n)$ that

$$-\frac{s(n)}{h(n+1)} = \left(1 - \frac{g(n+2)}{h(n+1)}\right) + \left(1 - \frac{f(n)}{h(n+1)}\right) = \gamma(n) + \delta(n),$$

say. Since $\Re(\gamma(n)) \ge 0$ with $2\Re(\gamma(n)) = |\gamma(n)|^2$ and also $\Re(\delta(n)) \ge 0$ with $2\Re(\delta(n)) = |\delta(n)|^2$ for all $n \in \mathbb{N}$, and since $|1-z|^2 = 2(1 - \Re(z))$ for all $z \in \mathbb{T}$, one easily obtains that

$$\sum_{n \le x} \frac{|\gamma(n)|^2}{n} = o(\log x) \quad \text{and} \quad \sum_{n \le x} \frac{|\delta(n)|^2}{n} = o(\log x) \qquad (x \to \infty). \tag{3.1}$$

Now, in general, given a bounded sequence of complex numbers $(a_n)_{n \ge 1}$, one can easily show that the statement $\sum_{n \le x} \frac{|a_n|^2}{n} = o(\log x)$ as $x \to \infty$ is equivalent to the statement $\sum_{n \le x} \frac{|a_n|}{n} = o(\log x)$ as $x \to \infty$. Similarly, one can show that the statement $\sum_{n \le x} |a_n|^2 = o(x)$ as $x \to \infty$ is equivalent to the statement $\sum_{n \le x} |a_n| = o(x)$ as $x \to \infty$.

In light of these observations, it follows from (3.1) that

$$\sum_{n \le x} \frac{|\gamma(n)|}{n} = o(\log x) \quad \text{and} \quad \sum_{n \le x} \frac{|\delta(n)|}{n} = o(\log x) \qquad (x \to \infty)$$

and therefore that

$$\sum_{n \le x} \frac{|g(n+1) - h(n)|}{n} = o(\log x) \;\; \text{and} \;\; \sum_{n \le x} \frac{|h(n+1) - f(n)|}{n} = o(\log x) \quad (x \to \infty).$$

Using Theorem B completes the proof of Theorem 1. $\square$

Theorem 2. *Let $f, g, h \in \mathcal{M}_1^*$ be such that $S(f) = S(g) = S(h) = \mathbb{T}$. Assume also that*

$$\overline{\{(g(n+1), h(n)) : n \in \mathbb{N}\}} \neq \mathbb{T} \times \mathbb{T} \quad \text{and} \quad \overline{\{(h(n+1), f(n)) : n \in \mathbb{N}\}} \neq \mathbb{T} \times \mathbb{T}$$

and that (f, h), $(h, g) \in \mathcal{H}$. Finally, let $\omega, \kappa \in \mathbb{T}$ be such that

$$s(n) := g(n+2)\omega - 2h(n+1) + f(n)\kappa \tag{3.2}$$

satisfies

$$\lim_{n\to\infty} s(n) = 0.$$

Then, there exists a real number t such that $f(n) = g(n) = h(n) = n^{it}$ for all $n \in \mathbb{N}$ and moreover $\omega = \kappa = 1$.

Proof. As in the proof of Theorem 1, we write

$$-\frac{s(n)}{h(n+1)} = \left(1 - \frac{g(n+2)\omega}{h(n+1)}\right) + \left(1 - \frac{f(n)\kappa}{h(n+1)}\right) = \gamma(n) + \delta(n),$$

say. Since $\Re(\gamma(n)) \geq 0$, $\Re(\delta(n)) \geq 0$, $|\gamma(n)|^2 = 2\Re(\gamma(n))$ and $|\delta(n)|^2 = 2\Re(\delta(n))$ for all $n \in \mathbb{N}$, it follows that $\lim_{n\to\infty} \gamma(n) = 0$ and $\lim_{n\to\infty} \delta(n) = 0$. From Theorem A, we then obtain that

$$f(n) = n^{it_1} F(n), \quad h(n) = n^{it_2} H(n), \quad g(n) = n^{it_3} G(n),$$

where $F \in W_{k_1}$, $H \in W_{k_2}$ and $G \in W_{k_3}$ for some positive integers k_1, k_2, k_3. Setting $\tau_1 = t_1 - t_2$ and $\tau_3 = t_3 - t_2$, we write

$$f_1(n) := n^{-it_2} f(n), \quad g_1(n) := n^{-it_2} g(n), \quad h_1(n) = H(n).$$

Using the estimate $(n+2)^{i\tau_3} = n^{i\tau_3} + o(1)$ as $n \to \infty$, we then have

$$2^{k_2} = (2H(n+1))^{k_2} = (n^{i\tau_1}\kappa F(n) + n^{i\tau_3}\omega G(n+2))^{k_2} + o(1) \qquad (n \to \infty). \tag{3.3}$$

Let us now introduce the function

$$\rho(n) := \frac{n^{i\tau_1}\kappa F(n) + n^{i\tau_3}\omega G(n+2)}{2}.$$

It clearly follows from (3.3) that $\rho(n)^{k_2} \to 1$ as $n \to \infty$. Therefore,

$$(n^{i\tau_1}\kappa F(n))^j \cdot (n^{i\tau_3}\omega G(n+2))^{k_2 - j} \to 1 \quad \text{as } n \to \infty \qquad (j = 0, 1, \ldots, k_2). \tag{3.4}$$

Then, in particular,

$$(n^{i\tau_1}\kappa F(n))^{k_2} = n^{ik_2\tau_1}\kappa^{k_2} \to 1 \quad \text{as } n \to \infty,$$

which is impossible if $\tau_1 \neq 0$. Hence, $\tau_1 = 0$. Proceeding in a similar manner, we obtain that $\tau_3 = 0$. In light of (3.4), we have thus established that

$$(\kappa F(n))^j \cdot (\omega G(n+2))^{k_2 - j} = 1 \qquad (j = 0, 1, \ldots, k_2).$$

We conclude from this that $(\omega G(n+2))^{k_2} = 1$ and therefore that

$$\frac{\kappa F(n)}{\omega G(n+2)} = 1.$$

This implies that if we set

$$\widetilde{\rho}(n) := \kappa F(n) - H(n+1),$$

we have $\widetilde{\rho}(n) = 0$ for all $n \geq n_0$ for some positive integer n_0. Then, we obtain $H(n^2) = \kappa F(n^2 - 1)$ and $H(n) = \kappa F(n-1)$. From this it follows that for some $n_0 \in \mathbb{N}$, we have $H(n) = F(n+1)$ for all $n \geq n_0$. Thus, $F(n+1)\overline{F}(n-1) = \overline{\kappa}$. This allows us to write

$$\begin{aligned} \overline{\kappa} &= F(m+1)\overline{F}(m) = F(2m+2)\overline{F}(2m) \\ &= F(2m+2)\overline{F}(2m+1)F(2m+1)\overline{F}(2m) \\ &= \overline{\kappa}^2 \quad \text{if } n \geq n_0, \end{aligned}$$

which clearly implies that $\kappa = 1$ and therefore that $F(n) = F(n+1)$ if $n \geq n_0$.

This obviously means that, for each $k \in \mathbb{N}$, $F(n) = F(n+k)$ if $n \geq n_0$. Therefore, $F(mn) = F(n)$ for $n \geq n_0$, from which we may conclude that $F(m) = 1$ for every positive integer m.

Now, since $\overline{\rho}(n) = 0$ for all $n \geq n_0$, we have that $H(n+1) = 1$ if $n \geq n_0$, and proceeding as above we may conclude that $H(n) = 1$ for every $n \in \mathbb{N}$.

Similarly, we can prove that $\omega = 1$ and that $G(n) = 1$ for every $n \in \mathbb{N}$, thereby completing the proof of Theorem 2. □

Interestingly, the situation is much simpler if at least one of the three sets $S(f)$, $S(g)$, $S(h)$ is not equal to $\mathbb{T}$, as can be seen in the following theorem.

Theorem 3. *Let $f, g, h \in \mathcal{M}_1^*$, where at least one of the three sets $S(f)$, $S(g)$, $S(h)$ is not equal to $\mathbb{T}$. Letting $s(n)$ be as in Theorem 2 and assuming that relation (3.2) of Theorem 2 holds, then*

$$\omega = \kappa = 1 \quad \text{and} \quad f(n) = g(n) = h(n) = 1 \ \text{ for all } n \in \mathbb{N}.$$

Essential for the proof of Theorem 3 are the following four lemmas.

Lemma 1. *Let $u \in \mathcal{M}_1^*$ and assume that this function is such that*

$$\#\overline{\{u(n+1)\overline{u}(n) : n \in \mathbb{N}\}} < \infty.$$

Then there exist $t \in \mathbb{R}$, $k \in \mathbb{N}$ and a function $U(n)$ with $U(\mathbb{N}) \in W_k$ such that $u(n) = n^{it}U(n)$ for all $n \in \mathbb{N}$.

Proof. Thus result is due to E. Wirsing [6]. □

As a consequence of Lemma 1, we have the following.

Lemma 2. *Let $u, v \in \mathcal{M}_1^*$ and let $\lambda_n := v(n+1)\overline{u}(n)$ for $n = 1, 2, \dots$ Assuming that $\#\overline{\{\lambda_n : n \in \mathbb{N}\}} < \infty$, then there exist $t \in \mathbb{R}$, $k_1, k_2 \in \mathbb{N}$ and two functions $U(n)$ and $V(n)$ with $U(\mathbb{N}) \in W_{k_1}$ and $V(\mathbb{N}) \in W_{k_2}$ such that $u(n) = n^{it}U(n)$ and $v(n) = n^{it}V(n)$ for all $n \in \mathbb{N}$.*

Proof. By hypothesis, we have

$$\lambda_{n^2-1}\overline{\lambda_{n-1}}^2 = u(n-1)\overline{u}(n+1)$$

and

$$\lambda_{(2k+1)^2-1}\overline{\lambda_{2m}}^2 = u(2m)\overline{u}(2m+2) = u(m)\overline{u}(m+1).$$

Hence, since the set of limit points of the sequence λ_n is finite, then the same is true for the sequence $u(m)\overline{u}(m+1)$. From Lemma 1, we then get that there exist $t_1 \in \mathbb{R}$, $k_1 \in \mathbb{N}$ and a function $U(n)$ with $U(\mathbb{N}) \in W_{k_1}$ such that $u(n) = n^{it_1}U(n)$ for all $n \in \mathbb{N}$. Similarly we can prove that the set of limit points of the sequence $v(n+1)\overline{v}(n)$ is finite and therefore that there exist $t_2 \in \mathbb{R}$, $k_2 \in \mathbb{N}$ and a function $V(n)$ with $V(\mathbb{N}) \in W_{k_2}$ such that $v(n) = n^{it_2}V(n)$ for all $n \in \mathbb{N}$. It follows from this that

$$\lambda_n = (n+1)^{it_2}n^{-it_1}V(n+1)\overline{U}(n) = n^{i(t_2-t_1)}V(n+1)\overline{U}(n) + o(1) \qquad (n \to \infty).$$

Since the set of limit points of the sequence λ_n is finite, we may conclude that $t_1 = t_2 (= t)$. □

Lemma 3. *Given $u, v \in \mathcal{M}_1^*$, set $t(n) := Av(n+1) - Bu(n)$, where $AB \neq 0$, and assume that $\#\overline{\{t(n) : n \in \mathbb{N}\}} < \infty$. Then, letting $\lambda_n := v(n+1)\overline{u}(n)$, we have $\#\overline{\{\lambda_n : n \in \mathbb{N}\}} < \infty$.*

Proof. Let the set of limit points of $t(n)$ be $\{c_1, \dots, c_r\}$. Let $f \in \mathcal{M}_1^*$ and assume that $S(f) \neq \mathbb{T}$. Then the set of limit points of the sequence

$$\left|\frac{t(n)}{A}\right| = \left|\frac{t(n)}{Af(n)}\right| = \left|\lambda_n - \frac{B}{A}\right|$$

is

$$\left\{d_j := \frac{|c_j|}{|A|} : j = 1, \dots, r\right\}.$$

Let $\alpha = \lim_{j\to\infty} \lambda_{n_j}$, where $n_1 < n_2 < \cdots$ Then $\left|\alpha - \dfrac{B}{A}\right| \in \{d_1, \dots, d_r\}$. No more than two numbers α may exist for which $|\alpha - B/A| = d_j$ and $|\alpha| = 1$ since $B/A \neq 0$, thereby completing the proof of Lemma 3. □

Lemma 4. *Given $f, g, h \in \mathcal{M}_1^*$, set $\sigma(n) := Ag(n+2) + Bh(n+1) + Cf(n)$, where $ABC \neq 0$. Assuming that $S(f) \neq \mathbb{T}$ and that $\lim_{n\to\infty} \sigma(n) = 0$. Then there exist $t \in \mathbb{R}$ and $k_1, k_2, k_3 \in \mathbb{N}$ and functions $F(n), G(n), H(n)$ with $F(\mathbb{N}) \in W_{k_1}$, $G(\mathbb{N}) \in W_{k_2}$, $H(\mathbb{N}) \in W_{k_3}$ such that*

$$f(n) = F(n), \quad g(n) = n^{it} G(n), \quad h(n) = n^{it} H(n)$$

and, for some $n_0 \in \mathbb{N}$,

$$AG(n+2) + BH(n+1) + CF(n) = 0 \quad \text{for all } n \geq n_0. \tag{3.5}$$

Proof. Since the set of limit points of the sequence $\sigma(n) - Cf(n)$ is finite, it follows from Lemma 3 that

$$\sigma(n) = n^{it} \left(AG(n+2) + BH(n+1) \right) + CF(n).$$

Since the number of possible values of $F(n)$, $G(n)$ and $H(n)$ is finite and since the sequence n^{it} is dense on $\mathbb{T}$, provided $t \neq 0$, we may conclude that $\sigma(n)$ does not tend to 0 as $n \to \infty$ provided $t \neq 0$. Thus, the set of possible values of $\sigma(n) = AG(n+2) + BH(n+1) + CF(n)$ is finite and since $\lim_{n\to\infty} \sigma(n) = 0$, we may conclude that $t = 0$ and that (3.5) holds. □

We are now ready to prove Theorem 3.

Proof of Theorem 3. Without any loss in generality, let us assume $S(f) \neq \mathbb{T}$. Let us use Lemma 4 with $B = -2$, $C = \kappa$ and $D = \omega$. Repeating the argument used in the proof of Theorem 2, we find that $\kappa = \omega = 1$, $A = C = 1$, $B = -2$ and $G(n+2) - 2H(n+1) + F(n) = 0$ provided $n \geq n_0$. We also obtain that $F(n) = G(n) = H(n) = 1$ if $n \geq n_0$. Then, similarly as above, we may conclude that $F(n) = G(n) = H(n) = 1$ for all $n \in \mathbb{N}$, thus completing the proof of Theorem 3.

□

4 Iterations of $\Delta f(n)$

From here on we will always assume that $f \in \mathcal{M}_1^*$, $S(f) = \mathbb{T}$ and also that $|R(f^m)| = \infty$ for infinitely many positive integers m. These conditions will allow us to freely use Theorem A. Moreover, we set

$$\xi(n) := f(n+1)\overline{f(n)}.$$

We then have the following result.

Theorem 4. *Assume that there exist $\delta > 0$, $\omega \in \mathbb{T}$ and some $n_0 \in \mathbb{N}$ such that*

$$|\xi(n)\omega - 1| < 2 - \delta \qquad (n \geq n_0). \tag{4.1}$$

Then, there exists a real number t such that $f(n) = n^{it} F(n)$ for all $n \in \mathbb{N}$, where $F(\mathbb{N}) \subseteq W_k$ and $\omega \in W_k$ for some positive integer k.

Proof. It follows from (4.1) that $|\xi(n)(-\omega) - 1| \geq \delta$ for all $n \geq n_0$. Therefore

$$\overline{\{(f(n+1), f(n)) : n \in \mathbb{N}\}} \neq \mathbb{T} \times \mathbb{T}.$$

Hence, applying Theorem A, the result follows. □

We now consider iterations of $\Delta f(n)$. For this, we let $\Delta^2 f(n) := \Delta\Delta f(n) = \Delta(f(n+1) - f(n)) = f(n+2) - 2f(n+1) + f(n)$, and for an arbitrary integer $k \geq 3$, we let $\Delta^k f(n) := \Delta\Delta^{k-1} f(n)$. Observe that we have the trivial bound $|\Delta^k f(n)| \leq 2^k$, with equality achieved in the case of the multiplicative function $f(n) = (-1)^{n+1}$.

In each of the following theorems, the real numbers $\varepsilon > 0$ and $\delta > 0$ are arbitrary but fixed.

Theorem 5. *Assume that $|\Delta^2 f(n)| \leq K := 2 - \delta$ for all $n \geq n_0$ for some positive integer n_0. Then, there exists a real number t and some positive integer k such that $f(n) = n^{it/k} F(n)$ for all $n \in \mathbb{N}$, where $F^k(n) = 1$ for all $n \geq 1$. Moreover, setting $E(n) := F(n+2) - 2F(n+1) + F(n)$, we have $|E(n)| \leq K + \varepsilon$ for all $n \geq n_0$.*

Proof. Observing that $\Delta^2 f(n) = f(n+2) - 2f(n+1) + f(n)$ and setting $\lambda_n := 1 + \xi(n+1)$ and $\mu_n := \overline{\xi(n)} - 3$, we have

$$\frac{\Delta^2 f(n)}{f(n+1)} = \lambda_n + \mu_n.$$

Since $\left|\dfrac{\Delta^2 f(n)}{f(n+1)}\right| \leq K$ for all $n \geq n_0$, it follows that $2 \leq |\mu_n| \leq K + |\lambda_n|$, implying that $|\lambda_n| > \delta$. From this it follows that $|\xi(n+1)(-1) - 1| \geq \delta > 0$ provided n is sufficiently large. Now, using Theorem A, we may conclude that there exists a real number t such that $f(n) = n^{it} F(n)$ for all $n \in \mathbb{N}$ and therefore that, as $n \to \infty$,

$$|E(n)| = \left|(\Delta^2 f(n)) n^{-it} + o(1)\right| < K + \varepsilon \qquad \text{provided } n \geq n_0,$$

thus completing the proof of Theorem 5. □

Theorem 6. *Assume that $|\Delta^3 f(n)| \leq K := 4 - \delta$ for all $n \geq n_0$ for some positive integer n_0. Then, there exists some real number t such that $f(n) = n^{it} F(n)$ for all $n \in \mathbb{N}$, where $F^\ell(n) = 1$ for all $n \in \mathbb{N}$ and $|\Delta^3 F(n)| \leq K + \varepsilon$ provided $n \geq n_1(\varepsilon)$.*

Proof. Set $s(n) := \Delta^3 f(n)$ and observe that $s(n) = f(n+3) - 3f(n+2) + 3f(n+1) - f(n)$. It follows that $3(\Delta f(n+1)) = f(n+3) - f(n) - s(n)$ and therefore that

$$3|\Delta f(n+1)| \leq K + 2 \text{ and therefore } |\Delta f(n+1)| \leq \frac{K+2}{3} = 2 - \frac{\delta}{3}.$$

Applying Theorem 4, the result follows. □

Theorem 7. *Assume that $|\Delta^4 f(n)| \leq K := 4 - \delta$ for all $n \geq n_0$ for some positive integer n_0. Then, there exists some real number t such that $f(n) = n^{it} F(n)$ for all $n \in \mathbb{N}$, where $F^k(n) = 1$ for all $n \in \mathbb{N}$, and $|\Delta^4 F(n)| \leq K + \varepsilon$ provided $n \geq n_0(\varepsilon)$.*

Proof. Set $s(n) := \Delta^4 f(n)$ and let $t(n) := f(n+4) - 2f(n+2) + f(n)$. Clearly, $s(n) = t(n) - 4\Delta^2 f(n+1)$. It follows that

$$4\left|\Delta^2 f(n+1)\right| \le |t(n)| + |s(n)| \le 8 - \delta,$$

so that

$$\left|\Delta^2 f(n+1)\right| \le 2 - \frac{\delta}{4}.$$

Applying Theorem 6, the result follows. □

Theorem 8. *Assume that $|\Delta^5 f(n)| \le K := 8 - \delta$ for all $n \ge n_0$ for some positive integer n_0. Then, there exists some real number t such that $f(n) = n^{it} F(n)$ for all $n \in \mathbb{N}$, where $F^k(n) = 1$ for all $n \in \mathbb{N}$, and $|\Delta^5 F(n)| \le K + \varepsilon$ provided $n \ge n_0(\varepsilon)$.*

Proof. Set $s(n) := \Delta^5 f(n)$. Observing that $s(n) = f(n+5) - f(n) - 5(f(n+4) - f(n+1)) + 10\Delta f(n+2)$, we have that

$$10|\Delta f(n+2)| \le |s(n)| + 10 + 2 \le K + 12.$$

Therefore,

$$|\Delta f(n+2)| \le \frac{K+12}{10} = 2 - \frac{\delta}{10}.$$

This implies that $|f(n+2) + f(n+1)| \ge \delta/10$ and therefore

$$|f(n+1)\overline{f(n)}(-1) - 1| \ge \frac{\delta}{10}.$$

Applying Theorem A, the result follows. □

Theorem 9. *Assume that $|\Delta^6 f(n)| \le K := 6 - \delta$ for all $n \ge n_0$ for some positive integer n_0. Then, there exists some real number t such that $f(n) = n^{it} F(n)$ for all $n \in \mathbb{N}$, where $F^k(n) = 1$ for all $n \in \mathbb{N}$, and $|\Delta^6 F(n)| \le K + \varepsilon$ provided $n \ge n_0(\varepsilon)$.*

Proof. Set $s(n) := \Delta^6 f(n)$. Observing that $s(n) = (f(n) + f(n+6)) - 6(f(n+1) + f(n+5)) + 15\Delta^2 f(n+2) - 10\Delta f(n+3)$. Therefore,

$$\left|15\Delta^2 f(n+2)\right| \le 24 + |s(n)| \le K + 24 = 30 - \delta,$$

implying that

$$|\Delta^2 f(n+2)| \le 2 - \frac{\delta}{15}.$$

Applying Theorem 5, the result follows. □

Theorem 10. *Assume that $|\Delta^7 f(n)| \le K := 12 - \delta$. Then, there exists some real number t such that $f(n) = n^{it} F(n)$ for all $n \in \mathbb{N}$, where $F^k(n) = 1$ for all $n \in \mathbb{N}$, and $|\Delta^7 F(n)| \le K + \varepsilon$ provided $n \ge n_0(\varepsilon)$.*

Proof. Set $s(n) := \Delta^7 f(n)$. Observing that $(x-1)^7 = (x^7-1) - 7(x^6-x) + 21(x^5-x^2) - 35(x^4-x^3)$, it follows that

$$s(n) = -35\Delta f(n+9) + 21(f(n+5)-f(n+2)) - 7(f(n+6)-f(n+1)) + (f(n+7)-f(n)),$$

implying that

$$|\Delta f(n+3)| \le \frac{K+58}{35} \le 2 - \frac{\delta}{35}.$$

Applying Theorem A, the result follows. □

Remark 1. *We are unable to obtain similar results for $\Delta^m f(n)$ for any of the integers $m \ge 8$.*

5 Final remarks

We believe that the following holds.

Conjecture 1. *Theorem A remains true if the condition $\overline{\{(f(n), g(n+1)) : n \in \mathbb{N}\}} \neq \mathbb{T} \times \mathbb{T}$ is weakened and replaced by the following: there exists a pair of points ξ, η located on the unit circle for which $\sum_{\substack{n \le x \\ |f(n)-\xi|<\varepsilon \\ |g(n+1)-\eta|<\varepsilon}} \frac{1}{n} = o(\log x)$ as $x \to \infty$, provided $\varepsilon > 0$ is sufficiently small.*

Observe that, perhaps a modification of the proof of Theorem A could lead to a proof of Conjecture 1. However, we could not find the right approach to reach that goal.

Finally, it is interesting to observe that if Conjecture 1 is true, then the theorems of the previous section remain true under a weaker condition, that is, instead of assuming that $|\Delta^m f(n)| \le K$ for all $n \ge n_0$, one can only assume that $|\Delta^m f(n)| \le K$ for all $n \in \mathbb{N}$ with the exception of some integers $n_1 < n_2 < \cdots$ for which $\sum_{n_j \le x} \frac{1}{n_j} = o(\log x)$ as $x \to \infty$.

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