On an open question regarding generalized number systems in Euclidean spaces

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Abstract

We raise an open question regarding generalized number systems in Euclidean spaces and formulate a partial answer.

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1 Introduction

Given an integer M, let $t = |M|$ and assume that $t \neq 0, 1$. Let

$$\mathcal{A} := \{a_0 = 0, a_1, \dots, a_{t-1}\}$$

be a complete residue system modulo M. Further set

$$\mathcal{B} := \mathcal{A} - \mathcal{A} = \{a_i - a_j : a_i, a_j \in \mathcal{A}\}.$$

Michalek [4], [5] proved the following.

Theorem A. *Let $\mathcal{A}$ and $\mathcal{B}$ be as above. Any integer n can be written in the form*

$$n = c_0 + c_1 M + \cdots + c_h M^h \text{ with each } c_i \in \mathcal{B} \tag{1.1}$$

for some positive integer h if and only if $GCD(a_1, a_2, \dots, a_{t-1}) = 1$.

Why is Theorem A interesting ? Consider the sets

$$\begin{aligned} H &:= \left\{ \sum_{\nu=1}^{\infty} \frac{c_\nu}{M^\nu} : c_\nu \in \mathcal{A} \right\}, \\ \Gamma_\ell &:= \left\{ \sum_{\nu=0}^{\ell} c_\nu M^\nu : c_\nu \in \mathcal{A} \right\}, \\ \Gamma &:= \bigcup_{\ell=0}^{\infty} \Gamma_\ell. \end{aligned}$$

The following statements were proved in [1], [2] and [3]:

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(1) H is a compact set and $\lambda(H) > 0$ (here, λ is the Lebesgue measure in $\mathbb{R}$).

(2) $\lambda(H + \gamma_1 \cap H + \gamma_2) = 0$ if $\gamma_1, \gamma_2 \in \Gamma$, $\gamma_1 \neq \gamma_2$.

In light of the above setup, we say that $(\mathcal{A}, M)$ is *generalized number system* if $\Gamma = \mathbb{Z}$, or in other words, if every $n \in \mathbb{Z}$ can be written as $n = \sum_{\nu=0}^{h} c_\nu M^\nu$ for some positive integer h. In this case,

$$\lambda(H + n_1 \cap H + n_2) = 0 \text{ for all } n_1, n_2 \in \mathbb{Z},\ n_1 \neq n_2. \tag{1.2}$$

It may occur that (1.2) holds despite the fact that $\Gamma \neq \mathbb{Z}$. Observe that it was proved earlier that (1.2) holds if and only if

$$\Gamma - \Gamma = \mathbb{Z}, \tag{1.3}$$

that is if every $n \in \mathbb{Z}$ can be written in the form (1.1). In this case, we say that $(\mathcal{A}, M)$ is a *just touching covering system*.

2 Generalized number systems in Euclidean spaces

Given a positive integer k, let $\mathbb{R}_k$ and $\mathbb{Z}_k$ stand respectively for the k-dimensional real Euclidean space and the ring of k-dimensional vectors with integer entries. Fix k and let M be a $k \times k$ matrix with integer elements. Assume that M has k distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ such that $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_k| > 1$. Let $\mathcal{L} := M\,\mathbb{Z}_k$. Then, $\mathcal{L}$ is a subgroup of $\mathbb{Z}_k$. Let t stand for the order of $\mathbb{Z}_k/\mathcal{L}$, so that $t = |\det M|$. Further let $A_0, A_1, \ldots, A_{t-1}$ stand for the residue classes mod $\mathcal{L}$ and let $A_0 = \mathcal{L}$. For each $j \in \{0, 1, \ldots, t-1\}$, choose an arbitrary element $\underline{a}_j \in A_j$ such that the vector $\underline{a}_0$ is the zero vector $\underline{0} = (0, 0, \ldots, 0)$, and then write

$$\mathcal{A} := \{\underline{a}_0, \underline{a}_1, \ldots, \underline{a}_{t-1}\},$$

so that $\mathcal{A}$ is a k-dimensional complete residue system modulo M.

We now introduce the sets

$$\begin{aligned} H &:= \left\{ \sum_{\nu=1}^{\infty} M^{-\nu} \underline{c}_\nu : \underline{c}_\nu \in \mathcal{A} \right\}, \\ \Gamma_\ell &:= \left\{ \sum_{\nu=0}^{\ell} M^{\nu} \underline{d}_\nu : \underline{d}_\nu \in \mathcal{A} \right\}, \\ \Gamma &:= \bigcup_{\ell=0}^{\infty} \Gamma_\ell. \end{aligned}$$

We will say that $(\mathcal{A}, M)$ is *just touching covering system* (JTCS) if

$$\lambda(H + \underline{n}_1 \cap H + \underline{n}_2) = 0 \text{ for all } \underline{n}_1, \underline{n}_2 \in \mathbb{Z}_k,\ \underline{n}_1 \neq \underline{n}_2.$$

In [3], it was proved that $(\mathcal{A}, M)$ is a JTCS if and only if $\Gamma - \Gamma = \mathbb{Z}_k$, that is if every $\underline{n} \in \mathbb{Z}_k$ can be written as

$$\underline{n} = \sum_{j=0}^{h} M^j \underline{d}_j, \tag{2.1}$$

where the $\underline{d}_j$'s belong to the set $\mathcal{B} = \mathcal{A} - \mathcal{A}$.

We now state the following.

Open question. *Under what condition is it true that $(\mathcal{A}, M)$ is a JTCS?*

Let us consider the special case where $M = \operatorname{diag}(m_1, \ldots, m_k)$, that is the "diagonal matrix", whose elements are all 0 except the elements on the diagonal whose values are $m_1, \ldots, m_k$. Let $t = |m_1 \cdots m_k|$ and assume that $|m_j| > 1$ for $j = 1, \ldots, k$. Let $\mathcal{A} = \{\underline{a}_0 = 0, \underline{a}_1, \ldots, \underline{a}_{t-1}\}$ be the complete residue system modulo t. If $(\mathcal{A}, M)$ is a JTCS, let $\mathcal{A}^{(j)}$ be the set of the j-th coordinates of $\mathcal{A}$, then $(\mathcal{A}^{(j)}, m_j)$ should be a JTCS in $\mathbb{R}$, in which case every $n \in \mathbb{Z}$ can be written as

$$n = \sum_{\nu=0}^{h} b_\nu m_j^\nu \quad \text{with} \quad b_\nu \in \mathcal{A}^{(j)} - \mathcal{A}^{(j)}.$$

This means that if $(\mathcal{A}, M)$ is a JTCS, then

$$\operatorname{GCD}(a_1^{(j)}, \ldots, a_{t-1}^{(j)}) = 1 \quad \text{for each } j = 1, \ldots, k.$$

The question is: Is this condition sufficient?

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