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Almost-additive and almost-multiplicative functions with regularity properties

By Jean-Marie De Koninck, Imre Kátai and Bui Minh Phong

Dedicated to Professor M. V. Subbarao on the occasion of the centenary of his birth

Abstract. We introduce the concepts of almost-additive and almost-multiplicative functions. We then prove some results concerning such functions which satisfy certain regularity conditions.

1. Introduction

Let, as usual, $\mathcal{P}$, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{C}$ stand for the set of prime numbers, positive integers, real numbers and complex numbers, respectively.

In 1985, M. V. Subbarao [10] introduced the concept of *weakly multiplicative* arithmetic function (later renamed *quasi-multiplicative*) as those functions f for which

$$f(np) = f(n)f(p)$$

for every $p \in \mathcal{P}$ and $n \in \mathbb{N}$ coprime to p.

Similarly, g is said to be *quasi-additive* if

$$g(np) = g(n) + g(p)$$

for every $p \in \mathcal{P}$ and $n \in \mathbb{N}$ coprime to p.

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Clearly, multiplicative (resp. additive) functions are quasi-multiplicative (resp. quasi-additive) functions.

Many interesting papers have been published on this topic, in particular those of J. Fabrykowski and M. V. Subbarao [2], J. Fehér and B. M. Phong [3], as well as B. M. Phong [9].

2. Some known results

The following is an old result proved independently by I. Kátai [4] and E. Wirsing [12].

Proposition 1. *Let f be an additive function satisfying*

$$\frac{1}{x}\sum_{n\le x}|f(n+1)-f(n)|\to 0 \quad \text{as} \quad x\to\infty,$$

Then there exists a constant c such that $f(n)=c\log n$ for all positive integers n.

In 2000, I. Kátai and M. V. Subbarao [5] proved the following four results regarding wider classes of arithmetical functions.

Theorem A. *If a quasi-additive function f is monotonic, then it is a constant multiple of* $\log n$.

Theorem B. *If f is a quasi-additive function and*

$$\frac{1}{x}\sum_{n\le x}|f(n+1)-f(n)|\to 0 \quad \text{as} \quad x\to\infty,$$

then there exists a constant C such that $f(n)=C\log n$.

Theorem C. *If g is a quasi-multiplicative function, $|g(n)|=1$ and*

$$\Delta g(n):=g(n+1)-g(n)\to 0 \quad \text{as} \quad n\to\infty,$$

then $g(n)=n^{i\tau}$ for some $\tau\in\mathbb{R}$.

Theorem D. *If g is a quasi-multiplicative function, $|g(n)|=1$ and*

$$\frac{1}{x}\sum_{n\le x}|g(n+1)-g(n)|\to 0 \quad \text{as} \quad x\to\infty,$$

then g is a completely multiplicative function.

Observe that Theorem C and Theorem D also hold for multiplicative functions (see [13], [14] and [8]).

3. Almost-additive and almost-multiplicative functions

Let $\mathcal{B}$ be a subset of primes for which

$$\sum_{p\in\mathcal{B}} \frac{1}{p} < \infty$$

and let $\mathcal{B}^*$ be the multiplicative semigroup generated by $\mathcal{B}$. Moreover, let $\mathcal{M}$ be the set of squarefree numbers coprime to $\mathcal{B}^*$. It is clear that every integer n can be uniquely written in the form $n = Km$, $(K, m) = 1$, where m is the largest divisor of n that belongs to $\mathcal{M}$ and for which $(n/m, m) = 1$.

Definition. *Let $f : \mathbb{N} \to \mathbb{R}$7. We say that f is almost-additive if for every $p \in \mathcal{P}$ and $n \in \mathbb{N}$ with $(p, n) = 1$ and $(p, \mathcal{B}) = 1$, we have*

$$f(np) = f(n) + f(p).$$

Definition. *Let $g : \mathbb{N} \to \mathbb{C}$. We say that g is almost-multiplicative if for every $p \in \mathcal{P}$ and $n \in \mathbb{N}$ with $(p, n) = 1$ and $(p, \mathcal{B}) = 1$, we have*

$$g(np) = g(n)g(p).$$

Our purpose in this paper is to generalize Theorems A–D of I. Kátai and M. V. Subbarao [5] by proving the following results.

Theorem 1. *If some given almost-additive function f is monotonic, then $f = C \log n$ for some $C \in \mathbb{R}$.*

Theorem 2. *If f is an almost-additive function and*

$$\frac{1}{x}\sum_{n\le x} |f(n+1) - f(n)| \to 0 \quad \text{as} \quad x \to \infty, \tag{1}$$

then $f = C \log n$ for some $C \in \mathbb{R}$.

Theorem 3. *If g is an almost-multiplicative function, $|g(n)| = 1$ and*

$$\frac{1}{x}\sum_{n\le x} |g(n+1) - g(n)| \to 0 \quad \text{as} \quad x \to \infty, \tag{2}$$

then $g(n) = n^{i\tau}$ for some $\tau \in \mathbb{R}$.

Corollary 1. *If g is an almost-multiplicative function, $|g(n)| = 1$ and*

$$\Delta g(n) := g(n+1) - g(n) \to 0 \quad \text{as} \quad n \to \infty,$$

then $g(n) = n^{i\tau}$ for some $\tau \in \mathbb{R}$.

4. Lemmas

Lemma 1. *(P. Erdős) Assume that f is additive and that there are positive constants c_1, c_2 and a sequence $x_\nu \to \infty$ $(\nu \to \infty)$, such that for all ν one can find suitable integers $1 \le a_1 \le \dots \le a_t \le x_\nu$ such that $t > c_1 x_\nu$ and*

$$|f(a_j) - f(a_k)| \le c_2$$

for every $j, k \le t$. Then f is finitely distributed, i.e. $f(n) = c \log n + t(n)$, where

$$\sum_p \frac{\min(1, t^2(p))}{p} < \infty.$$

Proof. Thsi result is Lemma V in the 1946 paper of P. Erdős [1]. □

Lemma 2. *Let h be an additive function defined on $\mathcal{M}$ and let $\mathcal{R}_0$ be a subset of primes for which*

$$\sum_{p \in \mathcal{R}_0} \frac{1}{p} < \infty.$$

Moreover, let $\mathcal{M}_{\mathcal{R}_0} := \{m \in \mathcal{M} : (m, \mathcal{R}_0) = 1\}$ and assume that $1 \le Y_0$, $x \ge e^{Y_0}$. Finally, letting $p(m)$ stand for the smallest prime factor of m, set

$$S(x|Y_0) := \sum_{\substack{m \le x \\ m \in \mathcal{M}_{\mathcal{R}_0}, p(m) > Y_0}} 1.$$

Then

$$S(x|Y_0) = \frac{c_0}{\log Y_0}(1 + o(1))x \qquad (x \to \infty),$$

where c_0 is a positive constant which may depend on $\mathcal{B}$ and $\mathcal{R}_0$.

Proof. The proof uses standard techniques from analytic number theory, and we therefore omit it. □

Lemma 3. *(Turán-Kubilius) Let h and $\mathcal{R}_0$ be as in Lemma 2. Further let*

$$a(x) = \sum_{\substack{Y_0 < p \le x \\ p \in \mathcal{M}_{\mathcal{R}_0}}} \frac{h(p)}{p}, \qquad b_1^2(x) = \sum_{\substack{Y_0 < p \le x \\ p \in \mathcal{M}_{\mathcal{R}_0}}} \frac{h^2(p)}{p}$$

and

$$b_2^2(x) = \sum_{x^{\frac{1}{4}} \le p < x} \frac{h^2(p)}{p},$$

and assume that $|h(p)| \leq 1$ *if* $p \in \mathcal{M}_{\mathcal{R}_0}$*. Then*

$$\sum_{\substack{m \leq x \\ m \in \mathcal{M}_{\mathcal{R}_0}, p(m) > Y_0}} \Big(h(m) - a(x)\Big)^2 \leq c_1 S(x|Y_0) b_1^2(x) + c_2 x b_2^2(x),$$

where c_1 *and* c_2 *are absolute constants.*

Proof. This a generalisation of the well-known Turán-Kubilius inequality. The proof is on the same lines as the standard proof of this famous inequality, which can be found for instance in the book of G. Tenenbaum [11]. □

5. Proof of Theorem 1

Let $F(m) = f(m)$ if $m \in \mathcal{M}$, and $F(p^\alpha) = 0$ if $p \in \mathcal{B}$ and $\alpha \geq 1$, as well as for every p if $\alpha \geq 2$. Then F is an additive function which satisfies Lemma 1. Indeed, let a be a positive integer coprime with $\mathcal{B}$ and let m be a positive integer coprime with $a\mathcal{B}$. Further let I_m be the set of positive integers coprime to $\mathcal{B}$ and belonging to the interval $[m, am]$. Then $\#I_m > cm$ for some positive constant c. Now it follows from the monotonicity of the function F on I_m that $F(n) \in [F(m), F(am)]$ for each $n \in I_m$, thereby establishing that the conditions of Lemma 1 are satisfied.

Hence, we have

$$F(n) = c \log n + t(n)$$

where

$$\sum_p \frac{\min(1, t^2(p))}{p} < \infty. \tag{3}$$

Let $\mathcal{R} = \{p \in \mathcal{P} : |t(p)| \geq 1\}$. Then, from (3) we have

$$\sum_{p \in \mathcal{R}} \frac{1}{p} < \infty.$$

Let us now consider $S(x|Y_0)$ with $\mathcal{R}$ instead of $\mathcal{R}_0$.

Let N and M be arbitrary positive integers and let $\epsilon_1, \epsilon_2, \delta_1, \delta_2$ be arbitrary positive numbers. Let us choose $Y_0 > \max(N, M)$ and set

$$\mathcal{L}_1 := \Big(\frac{x(1-\delta)}{N}, \frac{x}{N}\Big) \quad \text{and} \quad \mathcal{L}_2 := \Big(\frac{x(1+\delta)}{M}, \frac{x}{M}\Big).$$

Also, let J be the set of those integers $m \in \mathcal{M}_{\mathcal{R}}$ with $p(m) > Y_0$ and $m \le x$, and set

$$\tilde{a}(x) := \sum_{\substack{Y_0 < p \le x \\ p \notin \mathcal{R}}} \frac{t(p)}{p}.$$

We then let ν run over $\mathcal{L}_1 \cap J$ and μ run over $\mathcal{L}_2 \cap J$. Assuming that x is large, it follows from Lemmas 2 and 3 that with the possible exception of at most $\epsilon S(\frac{x}{N}, Y_0)$ integers $\nu \in \mathcal{L}_1 \cap J$, and at most $\epsilon S(\frac{x}{M}, Y_0)$ integers $\mu \in \mathcal{L}_2 \cap J$, we have

$$t(\nu) - \tilde{a}\Big(\frac{x}{N}\Big) \in [-\epsilon, \epsilon] \quad \text{and} \quad t(\mu) - \tilde{a}\Big(\frac{x}{M}\Big) \in [-\epsilon, \epsilon],$$

since $\displaystyle\sum_{\substack{Y_0 < p \le x \\ p \notin \mathcal{R}}} \frac{t^2(p)}{p} \to 0$ as $x \to \infty$.

Here,

$$\tilde{a}\Big(\frac{x}{N}\Big) - \tilde{a}\Big(\frac{x}{M}\Big) \to 0 \quad \text{as} \quad x \to \infty. \tag{4}$$

Indeed, using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\Big|\tilde{a}\Big(\frac{x}{N}\Big) - \tilde{a}\Big(\frac{x}{M}\Big)\Big| &= \left| \sum_{\substack{\frac{x}{N} < p \le \frac{x}{M} \\ p \notin \mathcal{R}}} \frac{t(p)}{p} \right| = \left| \sum_{\substack{\frac{x}{N} < p \le \frac{x}{M} \\ p \notin \mathcal{R}}} \frac{1}{\sqrt{p}} \frac{t(p)}{\sqrt{p}} \right| \\
&\le \left(\sum_{\substack{\frac{x}{N} < p \le \frac{x}{M} \\ p \notin \mathcal{R}}} \frac{1}{p} \right)^{1/2} \cdot \left(\sum_{\substack{\frac{x}{N} < p \le \frac{x}{M} \\ p \notin \mathcal{R}}} \frac{t^2(p)}{p} \right)^{1/2} \to 0 \quad \text{as } x \to \infty,
\end{aligned}$$

thereby justifying why (4) is true.

We can therefore find a pair $(\nu^*, \mu^*) \in \mathcal{L}_1 \times \mathcal{L}_2$ for which

$$t(\nu^*) - t(\mu^*) \in [-2\epsilon, 2\epsilon].$$

Since $\nu^* N < \mu^* M$, we have $f(\nu^* N) < f(\mu^* M)$, and so $f(N) + f(\nu^*) < f(M) + f(\mu^*)$, that is,

$$\begin{aligned}
& f(N) - f(M) < f(\mu^*) - f(\nu^*) = F(\mu^*) - F(\nu^*) = \\
& = c \log\Big(\frac{\mu^*}{\nu^*}\Big) + t(\mu^*) - t(\nu^*) \le c \log\Big(\frac{x(1+\delta)N}{Mx(1-\delta)}\Big) + 2\epsilon = \\
& = c \log\Big(\frac{N}{M}\Big) + c \log\Big(\frac{1+\delta}{1-\delta}\Big) + 2\epsilon.
\end{aligned}$$

Since ϵ and δ can be chosen arbitrarily small, it follows that $f(N) - f(M) < c\log\left(\frac{N}{M}\right)$. Interchanging the values N and M, the inequality $f(M) - f(N) < c\log\left(\frac{M}{N}\right)$ holds as well, implying that

$$f(N) - f(M) = c\log\left(\frac{M}{N}\right),$$

and therefore that $f(N) = c\log N$, thus completing the proof of Theorem 1.

6. Proof of Theorem 2

Since Theorem 2 is true for additive functions, it is enough to prove that (1) implies that f is additive. The proof is very similar to the proof of Theorem 2 in [5].

Let $K = K_1K_2$, $2|K_2$, $(K_1, K_2) = 1$. Let $\mathcal{H}$ be the set of those $m \in \mathcal{M}$ for which

(1) $(m, K) = 1$,

(2) $(mK_2 + 1, K_1) = 1$, $mK_2 + 1$ is squarefree and belongs to $\mathcal{M}$.

Let us first prove that there exists a positive constant C_0 such that

$$\lim_{x\to\infty} \frac{1}{x} \sum_{\substack{m\le x\\ m\in\mathcal{H}}} 1 = C_0. \tag{5}$$

To do so, first consider the arithmetic function

$$u(n) := \begin{cases} 1 & \text{if } n \in \mathcal{M} \text{ with } (n, K) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Since $u(n)$ is a multiplicative function, we have that

$$\begin{aligned} h(s) &:= \sum_{n=1}^{\infty} \frac{u(n)}{n^s} = \prod_{p\nmid K\mathcal{B}} \left(1 + \frac{1}{p^s}\right) \\ &= \zeta(s) \cdot \prod_p \left(1 - \frac{1}{p^s}\right) \cdot \prod_{p\nmid K\mathcal{B}} \left(1 + \frac{1}{p^s}\right) \\ &= \zeta(s) \cdot \prod_p \left(1 - \frac{1}{p^s}\right) \cdot \frac{\prod_p \left(1 + \frac{1}{p^s}\right)}{\prod_{p|K\mathcal{B}} \left(1 + \frac{1}{p^s}\right)} \end{aligned}$$

$$\begin{aligned}
&= \zeta(s)\cdot\prod_{p}\left(1-\frac{1}{p^{2s}}\right)\frac{1}{\prod_{p\mid K\mathcal{B}}\left(1+\frac{1}{p^s}\right)}\\
&= \zeta(s)\prod_{p\mid K\mathcal{B}}\left(1-\frac{1}{p^{s}}\right)\prod_{p\nmid K\mathcal{B}}\left(1-\frac{1}{p^{2s}}\right)\\
&= \zeta(s)H(s),
\end{aligned}$$

say. Let $U(n)$ be defined implicitly by the relation

$$H(s)=\sum_{n=1}^{\infty}\frac{U(n)}{n^s}.$$

Observe that $U(n)$ is a multiplicative function defined at prime powers p^α as follows:

- If $p \mid K\mathcal{B}$, then $U(p)=-1$ and $U(p^\alpha)=0$ for each $\alpha\ge 2$.
- If $p\nmid K\mathcal{B}$, then $U(p^2)=-1$ and $U(p^\alpha)=0$ if $\alpha\ne 2$.

On the other hand, it easily follows from the definition of $\mathcal{B}$ that

$$\sum_{d=1}^{\infty}\frac{|U(d)|}{d}<\infty. \tag{6}$$

Moreover,

$$\begin{aligned}
S(x) &:= \sum_{\substack{n\le x\\ n\in\mathcal{H}}} 1 = \sum_{n\le x} u(n)u(K_2n+1)\\
&= \sum_{(d,\delta)=1} U(d)U(\delta) \sum_{\substack{n\le x\\ d\mid n,\ \delta\mid K_2n+1}} 1.
\end{aligned} \tag{7}$$

For fixed d,δ, assuming that $(\delta,K_2)=1$, we have that

$$\sum_{\substack{n\le x\\ d\mid n,\ \delta\mid K_2n+1}} 1 = \frac{x}{d\delta}+o(x) \qquad (x\to\infty).$$

Using this in (7) and taking into account (6), we may conclude that

$$\lim_{x\to\infty}\frac{S(x)}{x}=C_0,$$

where

$$C_0=\prod_{p\mid K_2}\left(1-\frac{1}{p}\right)\prod_{\substack{p\mid K\mathcal{B}\\ p\nmid K_2}}\left(1-\frac{2}{p}\right)\prod_{p\nmid K\mathcal{B}}\left(1-\frac{1}{p^2}\right),$$

thus completing the proof of (5).

Now, given $m \in \mathcal{H}$, we have that

$$
\begin{aligned}
& f(Km+K_1) - f(Km) = f(K_1) + f(K_2m+1) - f(K) - f(m) \\
& = [f(K_1) + f(K_2) - f(K)] + f(K_2m+1) - f(K_2m),
\end{aligned}
$$

so that

$$|f(K_1) + f(K_2) - f(K)| \le |f(Km+K_1) - f(Km)| + |f(K_2m+1) - f(K_2m)|.$$

Letting

$$\delta_K(n) = \max_{j=1,\ldots,K} |f(n+j) - f(n)|,$$

from the assumption (1) of Theorem 2, we find that

$$\frac{1}{x} \sum_{n \le x} \delta_K(n) \to 0 \quad \text{as} \quad x \to \infty.$$

Therefore

$$|f(K_1) + f(K_2) - f(K)| \cdot \sum_{\substack{m \le x \\ m \in \mathcal{H}}} 1 = o(x) \qquad (x \to \infty).$$

Now, in light of (5), we obtain that

$$f(K_1) + f(K_2) = f(K),$$

from which it follows that

$$f(2^\alpha m) + f(n) = f(2^\alpha mn)$$

if $(n, m) = 1$. Consequently,

$$f(nm) = f(n) + f(m)$$

if $(n, m) = 1$, thus establishing that f is an additive function. Theorem 2 then follows from Proposition 1.

7. Proofs of Theorem 3 and Corollary 1

Repeating the argument used in the proof of Theorem 2, we can deduce that if (2) holds, then g is multiplicative, and by a result of J.-L. Mauclaire and L. Murata [8], we can conclude that g is completely multiplicative. According to the famous theorem of O. Klurman [6] and of O. Klurman and A. Mangerel [7], we have that $g(n) = n^{i\tau}$ for some $\tau \in \mathbb{R}$.

Thus, Theorem 3 is true. On the other hand, it is clear that Corollary 1 is an immediate consequence of Theorem 3.

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DÉPARTEMENT DE MATHÉMATIQUES ET DE STATISTIQUE
UNIVERSITÉ LAVAL
QUÉBEC, QUÉBEC G1V 0A6
CANADA

E-mail: jmdk@mat.ulaval.ca
URL: http://www.jeanmariedekoninck.mat.ulaval.ca

COMPUTER ALGEBRA DEPARTMENT, FACULTY OF INFORMATICS
EÖTVÖS LORÁND UNIVERSITY
H-1117 BUDAPEST, PÁZMÁNY PÉTER SÉTÁNY 1/C
HUNGARY

E-mail: katai@inf.elte.hu

COMPUTER ALGEBRA DEPARTMENT, FACULTY OF INFORMATICS
EÖTVÖS LORÁND UNIVERSITY
H-1117 BUDAPEST, PÁZMÁNY PÉTER SÉTÁNY 1/C
HUNGARY

E-mail: bui@inf.elte.hu