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A FRONT ROW SEAT TO THE WORK OF ALEKSANDAR IVIĆ

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Dedicated to Professor Aleksandar Ivić (1949–2020)

A b s t r a c t. *We provide a survey of the vast contribution of Aleksandar Ivić to analytic number theory. Emphasis is put on Ivić's work on arithmetical functions. Highlights of his contribution to the study of the Riemann zeta-function are also presented.*

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1. *Introduction*

Professor Aleksandar Ivić will be remembered for his vast contribution to the study of the Riemann zeta-function and for providing a better understanding of the analytic behaviour of numerous arithmetical functions describing the multiplicative structure of integers.

I had the privilege of working with Ivić on various problems related to arithmetical functions. Our research collaboration was very productive, allowing us to write a book that gathered many of our results and to publish a dozen joint papers on the fascinating topic of arithmetical functions. In this paper, I will mainly focus on parts of Ivić's results on arithmetical functions, some of which were joint work with me.

I will conclude this paper by briefly mentioning the relevance of Ivić's tremendous contribution to the theory of the Riemann zeta-function.

2. *On the number of prime divisors of an integer*

An arithmetical function $f(n)$, that is, a function $f : \mathbb{N} \to \mathbb{C}$, is said to be *additive* if $f(mn) = f(m) + f(n)$ for all pairs of positive integers m, n such that $(m, n) = 1$ (here, (m, n) means GCD(m, n)). The functions

$$\omega(n) := \sum_{p|n} 1 \qquad \text{and} \qquad \Omega(n) := \sum_{p^\alpha \| n} \alpha$$

(here, $p^\alpha \| n$ means that p^α divides n, but $p^{\alpha+1}$ does not), which count, respectively, the number of distinct prime factors of n and the total number of prime divisors of n counting their multiplicity, are classical examples of additive functions. Finding the average value of additive functions is usually simple. For instance, one can easily establish that

$$\sum_{n \le x} \omega(n) = x \log\log x + C_1 x + O\left(\frac{x}{\log x}\right), \tag{2.1}$$

where

$$C_1 = \gamma + \sum_p \left(\log(1 - \frac{1}{p}) + \frac{1}{p}\right) \tag{2.2}$$

(here γ is Euler's constant). The proof of (2.1) is quite straightforward. Indeed, letting $\lfloor y \rfloor$ stand for the largest integer $\le y$, we have

$$\begin{aligned} \sum_{n \le x} \omega(n) &= \sum_{n \le x} \sum_{p|n} 1 = \sum_{p \le x} \left\lfloor \frac{x}{p} \right\rfloor \\ &= x \sum_{p \le x} \frac{1}{p} + \sum_{p \le x} \left(\left\lfloor \frac{x}{p} \right\rfloor - \frac{x}{p} \right) \\ &= x \sum_{p \le x} \frac{1}{p} + O(\pi(x)), \end{aligned} \tag{2.3}$$

where $\pi(x)$ stands for the number of primes $p \le x$ and where we used the fact that $|\lfloor y \rfloor - y| \le 1$ for any number y. Now, according to Mertens' theorem (see for instance Theorem 10.1 in the book of De Koninck and Doyon [6]),

$$\sum_{p \le x} \frac{1}{p} = \log\log x + C_1 + O\left(\frac{1}{\log x}\right),$$

where C_1 is defined above in (2.2) as well as the Chebyshev inequality $\pi(x) = O(x/\log x)$. Using these two estimates in (2.3), we obtain

$$\begin{aligned} \sum_{n\le x}\omega(n) &= x\left(\log\log x + C_1 + O\left(\frac{1}{\log x}\right)\right) + O\left(\frac{x}{\log x}\right) \\ &= x\log\log x + C_1 x + O\left(\frac{x}{\log x}\right), \end{aligned}$$

thereby proving our claim (2.1). Finally, combining (2.1) with the easily established asymptotic formula $\sum_{n\le x}\log\log n \sim x\log\log x$ (as $x\to\infty$), we may conclude that the average value of $\omega(n)$ is $\log\log n$.

Similarly, one can prove that

$$\sum_{n\le x}\Omega(n) = x\log\log x + C_2 x + O\left(\frac{x}{\log x}\right),$$

where $C_2 = C_1 + \sum_p \frac{1}{p(p-1)}$, confirming that the average value of $\Omega(n)$ is also $\log\log n$.

The above reasoning can be used to obtain a much more general result. Indeed, given any additive function $f(n)$ such that $f(p)=1$ for all primes p and such that $f(p^\alpha) - f(p^{\alpha-1}) = O(1)$ uniformly for primes p and integers $\alpha\ge 2$, one can prove that there exists a constant $D = D(f)$ such that

$$\sum_{n\le x} f(n) = x\log\log x + Dx + O\left(\frac{x}{\log x}\right).$$

On the other hand, finding an asymptotic estimate for $\sum_{n\le x}' \frac{1}{f(n)}$ for some given additive function $f(n)$ is a totally different challenge. Here the apostrophe on the sum indicates that the sum runs over those positive integers $n\le x$ for which $f(n)\ne 0$. For instance, in 1970, using the Turán-Kubilius inequality, Duncan [16] proved that

$$\sum_{n\le x}' \frac{1}{\omega(n)} \ll \frac{x}{\log\log x}, \tag{2.4}$$

falling short of providing an asymptotic estimate for the sum of the reciprocals of $\omega(n)$. In 1972, I was able to prove [4] that, given any positive integer k, there exist positive constants $a_1,\dots,a_k$ such that

$$\sum_{n\le x}' \frac{1}{\omega(n)} = a_1\frac{x}{\log\log x} + \cdots + a_k\frac{x}{(\log\log x)^k} + O\left(\frac{x}{(\log\log x)^{k+1}}\right). \tag{2.5}$$

A few years later, recognizing that a much more accurate and more general estimate could be obtained, Ivić reached out to me. This first contact marked the beginning of our collaborative work in number theory. As a starter, we considerably improved and generalized (2.5) by establishing [8] the following.

Theorem 2.1. *Let $k \in \mathbb{N}$ and let $f : \mathbb{N} \to \mathbb{N} \cup \{0\}$ be any additive function such that $f(p) = 1$ for all primes p and for which there exists a number B such that $|f(p^\alpha)| < B$ for all prime powers p^α. Then, there exist constants $b_1, b_2, \ldots, b_k$ such that*

$$\sum_{n \le x}{}' \frac{1}{f(n)} = b_1 x L_1(x) + b_2 x \frac{L_2(x)}{\log x} + \cdots + b_k x \frac{L_k(x)}{\log^{k-1} x} + O\left(\frac{x}{\log^k x}\right), \quad (2.6)$$

where each function $L_i(x)$ is a slowly varying function asymptotic to $1/\log\log x$ as $x \to \infty$.

Note that, in general, a function $L(x)$ is said to be *slowly varying* (or *slowly oscillating*) if it is positive, continuous, and for every number $c > 0$, satisfies

$$\lim_{x \to \infty} \frac{L(cx)}{L(x)} = 1.$$

Of course, the set of functions $f(n)$ satisfying (2.6) includes the very simple functions $\omega(n)$ and $\Omega(n)$.

Here, we will not prove Theorem 2.1 in its general form. We will only provide the proof of a particular case, namely the one where $f(n) = \omega(n)$, and with an error term weaker than the one appearing in (2.6). However, to get the general idea of the method, let us start with any additive function $f(n)$. Given any complex number $t \neq 0$, we easily see that the function $t^{f(n)}$ is a multiplicative function. We say that an arithmetic function $h(n)$ is *multiplicative* if $h(mn) = h(m)h(n)$ whenever $(m, n) = 1$. Now, consider the function

$$F(x,t) := \sum_{n \le x}{}' t^{f(n)}, \quad \text{so that} \quad \frac{F(x,t)}{t} = \sum_{n \le x}{}' t^{f(n)-1}.$$

This means that

$$\begin{aligned} \int_0^1 \frac{F(x,t)}{t}\,dt &= \int_0^1 \sum_{n \le x}{}' t^{f(n)-1}\,dt = \sum_{n \le x}{}' \int_0^1 t^{f(n)-1}\,dt \\ &= \sum_{n \le x}{}' \frac{t^{f(n)}}{f(n)}\bigg|_0^1 = \sum_{n \le x}{}' \frac{1}{f(n)}. \end{aligned} \quad (2.7)$$

Hence, providing an estimate for $\sum_{n\le x}' \frac{1}{f(n)}$ boils down to finding a way to estimate the integral $\int_0^1 \frac{F(x,t)}{t}\,dt$. However, for most functions $f(n)$, the corresponding expression $F(x,t)/t$ is unbounded at $t=0$. To overcome this obstacle, we choose to integrate $F(x,t)/t$ between $\varepsilon(x)$ and 1 for some function $\varepsilon(x)$ which tends to 0 as $x\to\infty$. So, instead of (2.7), we have

$$\int_{\varepsilon(x)}^1 \frac{F(x,t)}{t}\,dt = \sum_{n\le x}' \frac{1}{f(n)} - \sum_{n\le x}' \frac{\varepsilon(x)^{f(n)}}{f(n)}. \tag{2.8}$$

Choosing $\varepsilon(x)$ appropriately so that the size of the last sum on the right hand side of (2.8) is minimal when x is large will normally allow one to obtain a good estimate for the sum $\sum_{n\le x}' \frac{1}{f(n)}$.

To do the analytic work, we will be using the Riemann zeta-function $\zeta(s)$, which, let us recall, is defined by

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s} = \prod_p \left(1-\frac{1}{p^s}\right)^{-1} \qquad \text{for } \Re(s)>1$$

and otherwise by analytic continuation through the *functional equation*

$$\zeta(s) = \chi(s)\zeta(1-s), \quad \text{where } \chi(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s) \quad (s\in\mathbb{C}) \tag{2.9}$$

(here, Γ stands for the Gamma function).

To handle the function $F(x,t)$ appearing on the left hand side of (2.8), we will be using an important result of Atle Selberg [37], presented here in a simplified form that fits our purpose.

Theorem A (Selberg). *Let* $g(s,t) = \sum_{n=1}^\infty \frac{b_t(n)}{n^s}$ *for* $\Re(s)=\sigma>1$ *and assume that the series* $\sum_{n=1}^\infty \frac{|b_t(n)|}{n}\log^3 2n$ *is uniformly bounded for* $|t|\le 1$. *Furthermore, let* $a_t(n)$ *be the arithmetical function defined implicitly by*

$$\zeta(s)^t g(s,t) = \sum_{n=1}^\infty \frac{a_t(n)}{n^s} \qquad (\sigma>1).$$

Then, for $x \geq 2$,

$$\sum_{n \leq x} a_t(n) = \frac{g(1,t)}{\Gamma(t)} x \log^{t-1} x + O\left(x \log^{t-2} x\right) \quad \textit{uniformly for } |t| \leq 1.$$

Theorem A is the key ingredient in proving (2.6), and the details are contained in my book with Ivić [9]. From here on, the focus will be on the particular function $f(n) = \omega(n)$, aiming to obtain estimate (2.5).

First, observe that in the case $a_t(n) = t^{\omega(n)}$, we have

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{a_t(n)}{n^s} &= \sum_{n=1}^{\infty} \frac{t^{\omega(n)}}{n^s} = \prod_p \left(1 + \frac{t}{p^s} + \frac{t}{p^{2s}} + \cdots\right) \\ &= \zeta(s)^t \prod_p \left(1 - \frac{1}{p^s}\right)^t \prod_p \left(1 + \frac{t}{p^s} + \frac{t}{p^{2s}} + \cdots\right) \\ &= \zeta(s)^t \, g(s,t), \end{aligned}$$

say, so that, having checked that the conditions of Theorem A are satisfied, we obtain

$$\sum_{2 \leq n \leq x} t^{\omega(n)} = D(t) x \log^{t-1} x + R(x,t), \tag{2.10}$$

where $D(t) = \dfrac{g(1,t)}{\Gamma(t)}$ and $R(x,t) = O(x \log^{t-2} x)$.

For $t \in (0,1]$ and each $i = 1, 2, \ldots, \alpha + 2$, set $B_i(t) := \left(\dfrac{D(t)}{t}\right)^{(i-1)}$, that is the $i-1$-th derivative with respect to t of $D(t)/t$. Further set $A_i(t) := (-1)^{i-1} B_i(t)$ for each $i = 1, 2, \ldots, \alpha + 2$.

Dividing relation (2.10) by t, we get

$$\sum_{2 \leq n \leq x} t^{\omega(n)-1} = B_1(t) x \log^{t-1} x + \frac{R(x,t)}{t}. \tag{2.11}$$

Choose $\varepsilon(x) = (\log x)^{-1/2(\alpha+2)}$ and assume that $x \geq 3$ (so that $\varepsilon(x) < 1$).

Integrating both sides of (2.11), we obtain

$$\int_{\varepsilon(x)}^{1} \left(\sum_{2 \leq n \leq x} t^{\omega(n)-1}\right) dt = \int_{\varepsilon(x)}^{1} B_1(t) x \log^{t-1} x \, dt + \int_{\varepsilon(x)}^{1} \frac{R(x,t)}{t} \, dt$$

$$\sum_{2\le n\le x}\frac{1}{\omega(n)}-\sum_{2\le n\le x}\frac{\varepsilon(x)^{\omega(n)}}{\omega(n)} = \int_{\varepsilon(x)}^{1} B_1(t)x\log^{t-1}x\,dt+O\left(x\frac{\log\log x}{\log x}\right)$$

and therefore

$$\sum_{2\le n\le x}\frac{1}{\omega(n)}=\int_{\varepsilon(x)}^{1} B_1(t)x\log^{t-1}x\,dt+O\left(\frac{x}{(\log\log x)^{\alpha+1}}\right). \tag{2.12}$$

Integration by parts yields

$$\begin{aligned}\int_{\varepsilon(x)}^{1} B_1(t)x\log^{t-1}x\,dt &= x\Bigg\{\sum_{i=1}^{\alpha}\frac{A_i(t)\log^{t-1}x}{(\log\log x)^i}\Bigg|_{\varepsilon(x)}^{1}+\frac{A_{\alpha+1}(t)\log^{t-1}x}{(\log\log x)^{\alpha+1}}\Bigg|_{\varepsilon(x)}^{1}\\ &\quad+\frac{1}{(\log\log x)^{\alpha+1}}\int_{\varepsilon(x)}^{1}A_{\alpha+2}(t)\log^{t-1}x\,dt\Bigg\}.\end{aligned} \tag{2.13}$$

By the definition of $A_i(t)$ and $B_i(t)$, it is clear that there exists a positive number M such that

$$|A_i(t)|=|B_i(t)|\le\frac{M}{t^{\alpha+2}}\qquad\text{for each } i=1,2,\dots,\alpha+2. \tag{2.14}$$

Using (2.14), we have that for $1\le i\le\alpha+1$ and $\varepsilon(x)\le t\le 1$,

$$\left|\frac{A_i(\varepsilon(x))\log^{\varepsilon(x)-1}x}{(\log\log x)^i}\right|\le\frac{M\log^{\varepsilon(x)-1}x}{\varepsilon(x)^{\alpha+2}(\log\log x)^i}=O\left(\frac{1}{(\log\log x)^{\alpha+1}}\right). \tag{2.15}$$

On the other hand, again using (2.14), we have

$$\begin{aligned}\left|\int_{\varepsilon(x)}^{1}A_{\alpha+2}(t)\log^{t-1}x\,dt\right| &\le \int_{\varepsilon(x)}^{1}|A_{\alpha+2}(t)|\log^{t-1}x\,dt\\ &< M\cdot\max_{\varepsilon(x)\le t\le 1}\frac{\log^{t-1}x}{t^{\alpha+2}}\\ &= \frac{M}{\log x}\cdot\log x=M.\end{aligned} \tag{2.16}$$

Finally, observing that $A_{\alpha+1}(1)=O(1)$ and using (2.15) and (2.16), we find that (2.13) can be written as

$$\int_{\varepsilon(x)}^{1} B_1(t)x\log^{t-1}x\,dt=x\left\{\sum_{i=1}^{\alpha}\frac{A_i(1)}{(\log\log x)^i}+O\left(\frac{1}{(\log\log x)^{\alpha+1}}\right)\right\}. \tag{2.17}$$

Gathering estimates (2.12) and (2.17) completes the proof of (2.5).

We showed earlier in this section that the functions $\omega(n)$ and $\Omega(n)$ have the same average value, namely $\log\log n$. So, what can one expect for the average value of $\Omega(n)/\omega(n)$? In 1970, Duncan [16] used his inequality (2.4) to obtain the estimate

$$\sum_{n\le x}{}' \frac{\Omega(n)}{\omega(n)} = x + O\left(\frac{x}{\log\log x}\right), \tag{2.18}$$

revealing that the average value of $\Omega(n)/\omega(n)$ is 1. Can one do better than (2.18) ? More generally, is it possible to estimate $\sum_{n\le x}{}' \frac{g(n)}{f(n)}$, where $g(n)$ and $f(n)$ are two given additive functions ? Indeed, as explained in our book [9], Ivić and I were able to prove the following interesting general result.

Theorem 2.2. *Let $f(n)$ and $g(n)$ be two additive functions such that $f(p) = g(p) = 1$ for all primes p and such that for all prime powers p^r, we have $1 \le f(p^r) < c_1 r$ and $0 \le g(p^r) < c_2 r$ for some constants c_1 and c_2. Then,*

$$\sum_{n\le x}{}' \frac{g(n)}{f(n)} = x + \frac{Ax}{\log\log x} + O\left(\frac{x}{(\log\log x)^2}\right), \tag{2.19}$$

where

$$A = \sum_{r=2}^{\infty}\sum_{p}\left(1-\frac{1}{p}\right)\frac{g(p^r)-f(p^r)}{p^r}. \tag{2.20}$$

PROOF. We only provide a sketch of the proof. Let $u \in (0,1)$ and $t \in \mathbb{C}$ with $|t| \le 1$. Because of the conditions imposed on $f(p^r)$ and $g(p^r)$, one can show that the series $\sum_{n=1}^{\infty} \frac{t^{g(n)}u^{f(n)}}{n^s}$ converges uniformly and absolutely for $\Re(s) > 1$. Therefore,

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{t^{g(n)}u^{f(n)}}{n^s} &= \prod_p \left(1 + \frac{tu}{p^s} + \frac{t^{g(p^2)}u^{f(p^2)}}{p^{2s}} + \cdots\right) \\
&= (\zeta(s))^{tu} \prod_p \left(1-\frac{1}{p^s}\right)^{tu} \prod_p \left(1 + \frac{tu}{p^s} + \frac{t^{g(p^2)}u^{f(p^2)}}{p^{2s}} + \cdots\right) \\
&= (\zeta(s))^{tu} H(t,u;s),
\end{aligned}$$

where $H(t,u;s)$ is absolutely and uniformly convergent for $\Re(s) > 1/2$. Using Theorem A, we obtain that, uniformly for $|t| \le 1$ and $|u| \le 1$,

$$\begin{aligned} \sum_{n\le x} t^{g(n)}u^{f(n)} &= \frac{H(t,u;1)}{\Gamma(tu)} x\log^{tu-1} x + O(x\log^{\Re(tu-2)} x) \\ &= \frac{x}{\log x}\left(\frac{H(t,u;1)}{\Gamma(tu)}\log^{tu} x + O(1)\right), \end{aligned} \tag{2.21}$$

where, for some constants $e_2, e_3, \ldots$,

$$\frac{H(t,u;1)}{\Gamma(tu)} = tu + e_2t^2u^2 + e_3t^{g(p^2)+1}u^{f(p^2)+1} + \cdots \tag{2.22}$$

and the remaining powers of u have exponents not less than 2, since by hypothesis, $f(p^r) \ge 1$.

Differentiating both sides of (2.21) with respect to t, we obtain

$$\begin{aligned} \sum_{n\le x} g(n)t^{g(n)-1}u^{f(n)} &= \frac{x}{\log x}\Big(\log^{tu} x \cdot \frac{\partial}{\partial t}\frac{H(t,u;1)}{\Gamma(tu)} \\ &\qquad + \frac{uH(t,u;1)}{\Gamma(tu)}\log^{tu} x\cdot\log\log x + O(1)\Big). \end{aligned} \tag{2.23}$$

Setting $t = 1$ in (2.23) and dividing by u, we obtain uniformly for $u \in (0,1]$,

$$\sum_{n\le x}{}' g(n)u^{f(n)-1} = \frac{x}{\log x}\left(G(u)\log^u x + F(u)\log^u x\cdot\log\log x + O(1/u)\right), \tag{2.24}$$

where

$$F(u) = \frac{H(1,u;1)}{\Gamma(u)} \quad \text{and} \quad G(u) = \frac{1}{u}\frac{\partial}{\partial t}\left(\frac{H(t,u;1)}{\Gamma(tu)}\right)\Bigg|_{t=1}.$$

We next integrate the left-hand side of (2.24) with respect to u between $\varepsilon(x) = 1/\sqrt{\log x}$ and 1, assuming that $x \ge 3$. We then obtain

$$\begin{aligned} \int_{\varepsilon(x)}^1 \left(\sum_{n\le x}{}' g(n)u^{f(n)-1}\right) du &= \sum_{n\le x}{}' g(n)\int_{\varepsilon(x)}^1 u^{f(n)-1}\,du \\ &= \sum_{n\le x}{}' \frac{g(n)}{f(n)} - \sum_{n\le x}{}' \frac{g(n)}{f(n)}\varepsilon(x)^{f(n)} \end{aligned}$$

$$\begin{aligned} &= \sum_{n\le x}{}' \frac{g(n)}{f(n)} + O\left(\varepsilon(x)\sum_{n\le x}{}' g(n)\right) \\ &= \sum_{n\le x}{}' \frac{g(n)}{f(n)} + O\left(\frac{x\log\log x}{\sqrt{\log x}}\right), \qquad (2.25) \end{aligned}$$

since $f(n) \ge 1$ for $n \ge 1$ and since $\sum_{n\le x} g(n) \ll x\log\log x$.

From (2.22), it is easily seen that $F(u) \in C^2[0,1]$ and that $G(u) \in C^1[0,1]$. Using this, we may integrate the individual terms on the right hand side of (2.24). We then obtain

$$\begin{aligned} \int_{\varepsilon(x)}^1 F(u)\log^u x\cdot\log\log x\,du &= F(1)\log x - F(\varepsilon(x))\log^{\varepsilon(x)} x - \frac{F'(1)}{\log\log x}\log x \\ &\quad + \frac{F'(\varepsilon(x))}{\log\log x}\log^{\varepsilon(x)} x + \int_{\varepsilon(x)}^1 \frac{F''(u)\log^u x}{(\log\log x)^2}\,du \\ &= F(1)\log x - \frac{F'(1)}{\log\log x}\log x \\ &\quad + O\left(\frac{\log x}{(\log\log x)^2}\right) \qquad (2.26) \end{aligned}$$

and similarly

$$\begin{aligned} \int_{\varepsilon(x)}^1 G(u)\log^u x\,du &= G(1)\frac{\log x}{\log\log x} - G(\varepsilon(x))\frac{\log^{\varepsilon(x)} x}{\log\log x} - \int_{\varepsilon(x)}^1 \frac{G'(u)\log^u x}{\log\log x}\,du \\ &= G(1)\frac{\log x}{\log\log x} + O\left(\frac{\log x}{(\log\log x)^2}\right), \qquad (2.27) \end{aligned}$$

where we used the fact that both $F''(u)$ and $G''(u)$ are bounded on $[0,1]$.

Gathering relations (2.24) to (2.27) and using the fact that $F(1) = 1$, estimate (2.19) follows immediately along with the explicit value of the constant A given in (2.20), thus completing the proof of Theorem 2.2.

Applying Theorem 2.2 with $g(n) = \Omega(n)$ and $f(n) = \omega(n)$ leads to an improvement of estimate (2.18), namely

$$\sum_{2\le n\le x} \frac{\Omega(n)}{\omega(n)} = x + A\frac{x}{\log\log x} + O\left(\frac{x}{(\log\log x)^2}\right),$$

with $A = \displaystyle\sum_p \frac{1}{p(p-1)} \approx 0.773$.

3. *The intriguing largest prime factor function*

Given an integer $n \geq 2$, let $P(n)$ stand for its largest prime factor. For convenience, set $P(1) = 1$. Establishing an asymptotic estimate for the sum $\sum_{n\leq x} P(n)$ is not a hard task. In fact, in 1997, K. Alladi and P. Erdős [1] showed that

$$\sum_{n\leq x} P(n) = \frac{\pi^2}{12}\frac{x^2}{\log x} + O\left(\frac{x^2}{\log^2 x}\right).$$

In 1984, Ivić and I [10] improved this estimate by showing the following.

Theorem 3.1. *For every fixed $k \in \mathbb{N}$, there exist constants $c_1 = \pi^2/12$, $c_2,\dots,$ c_k such that*

$$\sum_{n\leq x} P(n) = x^2\left(\frac{c_1}{\log x} + \frac{c_2}{\log^2 x} + \cdots + \frac{c_k}{\log^k x} + O\left(\frac{1}{\log^{k+1} x}\right)\right) \tag{3.1}$$

and moreover the same formula holds when one replaces $P(n)$ by any of the two functions

$$\beta(n) := \sum_{p|n} p \quad \textit{and} \quad B(n) := \sum_{p^\alpha \| n} \alpha p.$$

PROOF. We first prove (3.1). Consider the function

$$\Psi(x,y) := \#\{n \leq x : P(n) \leq y\} \qquad (2 \leq y \leq x),$$

a function which has been the focus of much research since the 1950's. We can then write

$$\begin{aligned}\sum_{n\leq x} P(n) &= \sum_{p\leq x} p \sum_{\substack{n\leq x\\ P(n)=p}} 1 = \sum_{p\leq x} p \sum_{\substack{m\leq x/p\\ P(m\leq p}} 1 = \sum_{p\leq x} p\Psi\left(\frac{x}{p}, p\right)\\ &= \sum_{p\leq\sqrt{x}} p\Psi\left(\frac{x}{p}, p\right) + \sum_{\sqrt{x}<p\leq x} p\Psi\left(\frac{x}{p}, p\right)\\ &= \Sigma_1 + \Sigma_2,\end{aligned}$$

say. Since $\Sigma_1 \leq \sum_{p\leq\sqrt{x}} p\frac{x}{p} = x\pi(\sqrt{x}) \leq x^{3/2}$, we may ignore Σ_1 and focus our attention on the evaluation of Σ_2. Observe that the sum in Σ_2 runs over primes $p > \sqrt{x}$, so that in that range, we have $\Psi(x/p, p) = \lfloor x/p \rfloor$. This means that

$$\Sigma_2 = \sum_{\sqrt{x}<p\leq x} p\left\lfloor\frac{x}{p}\right\rfloor = \sum_{p\leq x} p\left\lfloor\frac{x}{p}\right\rfloor - \sum_{p\leq\sqrt{x}} p\left\lfloor\frac{x}{p}\right\rfloor$$

$$\begin{aligned} &= \sum_{p\le x}\sum_{n\le x/p} 1 + O(x^{3/2}) = \sum_{n\le x}\sum_{p\le x/n} p + O(x^{3/2}) \\ &= \Sigma_3 + O(x^{3/2}), \end{aligned}$$

say. Hence, from here on, we only need to estimate Σ_3. We split this sum as

$$\Sigma_3 = \sum_{n\le x^{1/4}}\sum_{p\le x/n} p + \sum_{x^{1/4}<n\le x}\sum_{p\le x/n} p = \Sigma_4 + \Sigma_5,$$

say. Since

$$\Sigma_5 \le \sum_{x^{1/4}<n\le x} \frac{x}{n} \sum_{p\le x^{3/4}} 1 \le x^{7/4} \sum_{x^{1/4}<n\le x} \frac{1}{n} \ll x^{7/4}\log x,$$

we only need to estimate Σ_4. Now observe that using partial summation and the prime number theorem, we easily establish that there exist constants $d_1 = 1/2$, $d_2, \ldots, d_k$ such that

$$\sum_{p\le y} p = y^2\left(\frac{d_1}{\log y} + \frac{d_2}{\log^2 y} + \cdots + \frac{d_k}{\log^k y} + O\left(\frac{1}{\log^{k+1} y}\right)\right). \tag{3.2}$$

Observe also that using standard techniques in real analysis, we easily obtain that for any positive integer i, there exist constants $d_{0,i}, d_{1,i}, \ldots, d_{k,i}$ such that

$$\sum_{n\le x^{1/4}} \frac{1}{n^2\log^i(x/n)} = \frac{d_{0,i}}{\log^i x} + \frac{d_{1,i}}{\log^{i+1} x} + \cdots + \frac{d_{k,i}}{\log^{i+k} x} + O\left(\frac{1}{\log^{i+k+1} x}\right). \tag{3.3}$$

Using estimate (3.2), we obtain

$$\begin{aligned} \Sigma_4 &= \sum_{n\le x^{1/4}} \left\{ \left(\frac{x}{n}\right)^2 \left(\frac{d_1}{\log(x/n)} + \frac{d_2}{\log^2(x/n)} \right.\right. \\ &\qquad \left.\left. + \cdots + \frac{d_k}{\log^k(x/n)} + O\left(\frac{1}{\log^{k+1}(x/n)}\right)\right)\right\} \\ &= x^2\left\{ d_1 \sum_{n\le x^{1/4}} \frac{1}{n^2\log(x/n)} + d_2 \sum_{n\le x^{1/4}} \frac{1}{n^2\log^2(x/n)} \right. \\ &\qquad \left. + \cdots + d_k \sum_{n\le x^{1/4}} \frac{1}{n^2\log^k(x/n)} + O\left(\sum_{n\le x^{1/4}} \frac{1}{n^2\log^{k+1}(x/n)}\right)\right\}. \end{aligned}$$

Then, making good use of estimate (3.3) in each of the above $k+1$ sums, we finally obtain the desired estimate (3.1).

To see that (3.1) also holds when $P(n)$ is replaced by $\beta(n)$ or $B(n)$, we only need to observe that, since $\omega(n) \le 2\log n$ and $\Omega(n) \le 2\log n$ for all integers $n \ge 2$,

$$\beta(n) = P(n) + O((\omega(n)-1)\cdot\sqrt{n}) = P(n) + O(\sqrt{n}\,\log n)$$

and that

$$B(n) = P(n) + O((\Omega(n)-1)\cdot\sqrt{n}) = P(n) + O(\sqrt{n}\,\log n).$$

Estimate (3.1) tells us that on average the order of the size of $P(n)$ is $n/\log n$. From the above proof, one may have noticed that the sum $\sum_{n\le x} P(n)$ is dominated by those integers n with a large prime factor. What about the average value of $\log P(n)$? In that case, we will see that most integers n contribute to the sum $\sum_{n\le x}\log P(n)$. But first, we need to better understand the behaviour of the function $\Psi(x,y)$. For this, we move back in time, more precisely to 1930, that is when the Swedish actuary Karl Dickman (1861-1947), in studying the distribution of those integers having no large prime factors, introduced [15] a function which would turn out to be extremely useful for describing the asymptotic behavior of $\Psi(x,y)$. This function now called the *Dickman function* is defined as the unique continuous function $\rho : [0,\infty) \to (0,1]$ which is differentiable on $[1,\infty)$ and satisfies

$$\begin{array}{lll} \rho(u) = 1 & \text{for} & 0 \le u \le 1, \\ u\rho'(u) + \rho(u-1) = 0 & \text{for} & u \ge 1. \end{array}$$

A key result connecting the Dickman function with the $\Psi(x,y)$ function is the estimate

$$\Psi(x,y) = x\rho(u) + O\left(\frac{x}{\log y}\right) \quad \text{uniformly for } 2 \le y \le x, \tag{3.4}$$

where $u = \log x/\log y$. For a proof of (3.4), see Theorem 9.14 in my book with Florian Luca [13]. Using this estimate, we find that

$$\begin{aligned} \sum_{n\le x}\log P(n) &= \sum_{p\le x}\log p\,\Psi(x/p,p) \\ &= \sum_{p\le x}\log p\left\{\frac{x}{p}\rho\left(\frac{\log x}{\log p}-1\right) + O\left(\frac{x}{p\log p}\right)\right\} \end{aligned}$$

$$\begin{aligned} &= x \sum_{p \le x} \frac{\log p}{p} \rho\left(\frac{\log x}{\log p} - 1\right) + O\left(x \sum_{p \le x} \frac{1}{p}\right) \\ &= x \int_2^x \frac{1}{t} \rho\left(\frac{\log x}{\log t} - 1\right) dt + O\left(x \log\log x\right) \\ &= x \log x \int_1^{\log x/\log 2} \rho\left(v-1\right) \frac{dv}{v^2} + O\left(x \log\log x\right). \end{aligned} \tag{3.5}$$

On the one hand,

$$\int_{\log x/\log 2}^{\infty} \frac{\rho(v-1)}{v^2}\, dv \le \int_{\log x/\log 2}^{\infty} \frac{1}{v^2}\, dv \ll \frac{1}{\log x}.$$

We can therefore replace $\int_1^{\log x/\log 2} \frac{\rho(v-1)}{v^2}\, dv$ in (3.5) by $\int_1^{\infty} \frac{\rho(v-1)}{v^2}\, dv$. On the other hand, using the relation $\rho(v-1) = -v\rho'(v)$, we have

$$\int_1^{\infty} \frac{\rho(v-1)}{v^2}\, dv = -\int_1^{\infty} \rho'(v) \frac{dv}{v} = -\frac{\rho(v)}{v}\Big|_1^{\infty} - \int_1^{\infty} \frac{\rho(v)}{v^2}\, dv = 1 - \int_1^{\infty} \frac{\rho(v)}{v^2}\, dv.$$

Summing up we proved that

$$\sum_{n \le x} \log P(n) = C_3 x \log x + O(x \log\log x),$$

where $C_3 = 1 - \int_1^{\infty} \frac{\rho(v)}{v^2}\, dv \approx 0.62433$.

We have thus established that the average value of $\log P(n)$ is $C_3 \log n$. What about the average size of $1/P(n)$? As we will see, finding an accurate estimate for $\sum_{n \le x} 1/P(n)$ is much harder and it has been the focus of many papers in the end of the 1970's and through the 1980's. The study of this sum is closely related to the behaviour of the $\Psi(x,y)$ function since

$$\sum_{n \le x} \frac{1}{P(n)} = \sum_{p \le x} \frac{1}{p} \sum_{\substack{n = mp \le x \\ P(m) \le p}} 1 = \sum_{p \le x} \frac{1}{p} \sum_{\substack{m \le x/p \\ P(m) \le p}} 1 = \sum_{p \le x} \frac{1}{p} \Psi\left(\frac{x}{p}, p\right). \tag{3.6}$$

A first breakthrough occurred in 1981 as Ivić [27] proved that

$$S(x) := \sum_{n \le x} \frac{1}{P(n)} = x \exp\left\{-\sqrt{2 \log x \log\log x} + O\left(\sqrt{\log x \log\log\log x}\right)\right\}. \tag{3.7}$$

To obtain estimate (3.7), Ivić used the estimate of $\Psi(x,y)$ obtained in 1951 by Nicolaas Govert de Bruijn [3] as he proved that, given any small number $\varepsilon > 0$ and setting $u := \log x/\log y$, we have

$$\Psi(x,y) = x\rho(u)\left(1 + O\left(\frac{\log(u+1)}{\log y}\right)\right) \qquad \text{for all } y > \exp\{(\log x)^{5/8+\varepsilon}\}. \quad (3.8)$$

This estimate, having an error term smaller than the one given in (3.4) is better, even though it does not hold for all $2 \le y \le x$, but rather only for $\exp\{(\log x)^{5/8+\varepsilon}\} < y \le x$. Unfortunately, estimate (3.7) did not meet the highest expectations. The reason is that it fell short of revealing an asymptotic formula for $S(x)$. Indeed, it only provided an asymptotic formula for $\log S(x)$ but not for $S(x)$. In fact, the approximation found above by Ivić already indicated that the main contribution to $S(x)$ came from the primes $p \approx e^{\sqrt{\log x \log\log x}}$. Indeed, using (3.6), we find that

$$S(x) = \sum_{p \le x} \frac{1}{p}\Psi\left(\frac{x}{p}, p\right) = \sum_{p \le x} \frac{1}{p}\left(\frac{x}{p}\rho(u)\left(1 + O\left(\frac{\log(u+1)}{\log y}\right)\right)\right),$$

where $u = \dfrac{\log(x/p)}{\log p} = \dfrac{\log x}{\log p} - 1$, implying that for those $p \approx e^{\sqrt{\log x \log\log x}}$, we have

$$u \approx \frac{\log x}{\log p} = \frac{\log x}{\sqrt{\log x \log\log x}} = \sqrt{\frac{\log x}{\log\log x}}, \quad (3.9)$$

which is unfortunately out of the range of validity of de Bruijn's estimate (3.8) since in that case, $u \le (\log x)^{\frac{3}{8}-\varepsilon}$ which is indeed smaller that $\sqrt{\log x/\log\log x}$. Hence, in order to obtain an asymptotic formula for $S(x)$, a more accurate estimate of $\Psi(x,y)$ was required. This came in 1986 when Adolf Hildebrand [26] increased the range of validity of estimate (3.8) by showing that it holds for all $y > \exp\{(\log\log x)^{5/3+\varepsilon}\}$, that is for $u \le \frac{\log x}{(\log\log x)^{5/3}}$ which includes the necessary range of u given in (3.9).

So, the same year, Ivić in a joint paper with Erdős and Pomerance [24] used Hildebrand's estimate along with a clever argument to finally obtain a true asymptotic formula for $S(x)$, namely the following.

Theorem 3.2. *(Erdős, Ivić and Pomerance) The function*

$$\delta(x) := \int_2^x \rho\left(\frac{\log x}{\log t}\right)\frac{dt}{t^2}$$

is a slowly varying function and satisfies

$$\delta(x) = \exp\{-(1+o(1))\sqrt{2\log x \log\log x}\} \qquad (x \to \infty).$$

Moreover,

$$\sum_{n\le x}\frac{1}{P(n)} = x\delta(x)\left(1+O\left(\sqrt{\frac{\log\log x}{\log x}}\right)\right)$$

and the same estimate holds if $P(n)$ is replaced by $\beta(n)$ or $B(n)$.

In closing this section on the largest prime factor function, it is interesting to mention that the sum of the reciprocals of the second largest prime factor function $P_2(n)$ defined for those integers n with at least two prime factors, has a totally different asymptotic value. In fact, I proved [5] in 1993 that

$$\sum_{\substack{n\le x\\ \Omega(n)\ge 2}}\frac{1}{P_2(n)} = \lambda_2\frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right),$$

where $\lambda_2 = \sum_{m=1}^{\infty}\frac{1}{m}\sum_{p\ge P(m)}\frac{1}{p^2} \approx 1.254$. In the same paper, I showed that, if $P_k(n)$ stands for the k-th largest prime factor of an integer, then

$$\sum_{\substack{n\le x\\ \Omega(n)\ge k}}\frac{1}{P_k(n)} = \lambda_k\frac{x(\log\log x)^{k-2}}{\log x}\left(1+O\left(\frac{1}{\log\log x}\right)\right),$$

where $\lambda_k = \lambda_2/(k-2)!$.

4. *Arithmetic functions defined on sets of primes of positive density*

Let Q be a set of primes for which there exists some positive constant $\delta < 1$ such that

$$\pi(x,Q) := \sum_{p\le x,\ p\in Q} 1 = \delta\,\mathrm{Li}(x) + O\left(\frac{x}{\log^B x}\right),$$

where B is a constant larger than 2 and $\mathrm{Li}(x) = \int_2^x \frac{dt}{\log t}$. We define $P(n,Q)$ as

$$P(n,Q) = \begin{cases} \max\{p : p \mid n \text{ and } p\in Q\} & \text{if } (n,Q)>1,\\ 0 & \text{otherwise,}\end{cases} \tag{4.1}$$

where $(n,Q) > 1$ (resp. $(n,Q) = 1$) means that n has a prime factor (resp. has no prime factor) from Q. Thus $P(n,Q)$ is the largest prime factor of n belonging to Q, and analogously we define the k-th largest prime factor of n belonging to Q as

$$P_k(n,Q) = \begin{cases} P\left(\frac{n}{P_1(n,Q)\cdots P_{k-1}(n,Q)}, Q\right) & \text{if } \Omega(n,Q)\ge k,\\ 0 & \text{otherwise,}\end{cases} \tag{4.2}$$

if $k \geq 2$, where $P_1(n,Q) = P(n,Q)$ and

$$\Omega(n,Q) = \sum_{p^\alpha \| n,\ p \in Q} \alpha$$

is the total number of prime factors of n (counting multiplicities) belonging to Q, while

$$\omega(n,Q) = \sum_{p|n,\ p \in Q} 1$$

is the number of distinct prime factors of n belonging to Q. The function defined in (4.1) is the analogue of the classical function $P(n)$, whereas the one defined in (4.2) is the analogue of the function $P_k(n)$ mentioned in the previous section.

Likewise, consider the large additive functions

$$\beta(n,Q) = \sum_{p|n,\ p \in Q} p \quad \text{and} \quad B(n,Q) = \sum_{p^\alpha \| n,\ p \in Q} \alpha p, \tag{4.3}$$

and $\beta(n,Q) = B(n,Q) = 0$ if $(n,Q) = 1$. The functions in (4.3) are the analogues of the large additive functions $\beta(n)$ and $B(n)$ defined in Theorem 3.1 and for which there exists an extensive literature (see for instance our monograph [9] and the more recent paper of Ivić [30]).

Ivić and I first studied the above functions separately. For instance, in [5], I proved that

$$\sum_{n \leq x}{}' \frac{1}{P(n,Q)} = \left(\eta(Q) + O\left(\frac{1}{\log\log x}\right)\right) \frac{x}{(\log x)^\delta}, \tag{4.4}$$

where $\eta(Q)$ is a positive constant depending on Q (that is, on δ) which may be written in closed form, whereas at the same time, Ivić obtained several results involving $\beta(n,Q)$ and $B(n,Q)$. For instance, in [29], he proved that

$$\sum_{n \leq x} \beta(n,Q) = \sum_{j=1}^{k} \frac{\delta A_j x^2}{\log^j x} + O\left(\frac{x^2}{\log^{k+1} x}\right) \tag{4.5}$$

with explicitly given constants A_j, and that the same formula holds if one replaces $\beta(n,Q)$ by $B(n,Q)$.

In the same paper [29], Ivić proved that

$$\sum_{n \leq x}{}' \frac{B(n,Q)}{\beta(n,Q)} = x + O\left(\frac{x \log\log x}{(\log x)^\delta}\right)$$

and went on to conjecture that

$$\sum_{n\le x}{}' \frac{1}{\beta(n,Q)} = \left(\eta_1(Q) + O\left(\frac{1}{\log\log x}\right)\right)\frac{x}{(\log x)^\delta} \tag{4.6}$$

and

$$\sum_{n\le x}{}' \frac{1}{B(n,Q)} = \left(\eta_2(Q) + O\left(\frac{1}{\log\log x}\right)\right)\frac{x}{(\log x)^\delta}, \tag{4.7}$$

with $0 < \eta_2(Q) \le \eta_1(Q) \le \eta(Q)$, where $\eta(Q)$ is the constant appearing in (4.4).

In a joint paper [12], Ivić and I addressed these two conjectures and in fact we proved the next four theorems.

Theorem 4.1. *There exist constants* $0 < D_1(\delta) < D_2(\delta)$ *such that*

$$\sum_{n\le x,\ (n,Q)>1} \frac{\beta(n,Q)}{P(n,Q)} = x + \left(D_1(\delta) + O\left(\frac{1}{\log\log x}\right)\right)\frac{x}{(\log x)^\delta}$$

and

$$\sum_{n\le x,\ (n,Q)>1} \frac{B(n,Q)}{P(n,Q)} = x + \left(D_2(\delta) + O\left(\frac{1}{\log\log x}\right)\right)\frac{x}{(\log x)^\delta}.$$

The next result establishes the conjectured asymptotic formulas (4.6) and (4.7).

Theorem 4.2. *There exist constants* $0 < \eta_2(Q) < \eta_1(Q)$ *such that estimates (4.6) and (4.7) hold.*

By comparing estimates of Theorem 3.1 (in the case of $\beta(n)$) and estimate (4.5), the sums $\sum_{n\le x}\beta(n)$ and $\sum_{n\le x}\beta(n,Q)$ are of the same order, whereas by comparing Theorem 3.2 (in the case of $\beta(n)$) and (4.6), one will notice that the asymptotic behaviours of $\sum_{n\le x} 1/\beta(n)$ and $\sum_{n\le x} 1/\beta(n,Q)$ are completely different.

Finally, the difference in behaviour between $P(n)$ and $P(n,Q)$ is also reflected in the asymptotic behaviour of two further arithmetic sums which contain the logarithms of these functions, as shown in the next two results.

Theorem 4.3. *There exists an effectively computable constant* B *such that*

$$\sum_{2\le n\le x} \frac{1}{n\log P(n)} = e^\gamma \log\log x + B + O\left(\frac{1}{\log x}\right),$$

where γ *is Euler's constant.*

Theorem 4.4. *There exists an effectively computable constant* $B(Q) > 0$ *such that*

$$\sum_{n \le x,\ (n,Q)>1} \frac{1}{n \log P(n,Q)} = \left(B(Q) + O\left(\frac{1}{\log\log x} \right) \right) \log^{1-\delta} x.$$

Theorem 4.3 sharpens the estimate

$$\sum_{2 \le n \le x} \frac{1}{n \log P(n)} = e^{\gamma} \log\log x + O(1)$$

which I had previously obtained in a joint 1987 paper written with R. Sitaramachandrarao [14].

5. *Joint papers with Paul Erdős*

From 1980 to 1995, Ivić wrote eight papers in collaboration with the legendary Paul Erdős (in chronological order, [19], [7], [20], [24], [23], [21], [22], [18]). In the second of these papers, of which I was also a co-author, we studied the functions $\beta(n)$ and $B(n)$, proving the next three results. Here, we set $\ell(x) := \sqrt{\log x \log\log x}$.

Theorem 5.1. *For any given* $\varepsilon > 0$*, there exists* $x_0 = x_0(\varepsilon)$ *such that*

$$\frac{x}{e^{(2+\varepsilon)\ell(x)}} \le \sum_{2 \le n \le x} \frac{1}{B(n)} \le \sum_{2 \le n \le x} \frac{1}{\beta(n)} \le \frac{x}{e^{(\frac{1}{2}-\varepsilon)\ell(x)}} \quad (x \ge x_0). \tag{5.1}$$

Theorem 5.2. *There exist some positive constants* D_1 *and* D_2 *such that*

$$\sum_{2 \le n \le x} \frac{B(n)}{\beta(n)} = x + O\left(\frac{x}{e^{D_1 \ell(x)}} \right) \quad \textit{and} \quad \sum_{2 \le n \le x} \frac{\beta(n)}{B(n)} = x + O\left(\frac{x}{e^{D_2 \ell(x)}} \right).$$

Theorem 5.3. *Given any small* $\eta > 0$*, there exists a positive constant* D_3 *such that*

$$\sum_{2 \le n \le x}{}' \frac{1}{B(n) - \beta(n)} = D_3 x + O\left(x^{\frac{1}{2}+\eta} \right).$$

Here, we only prove Theorems 5.1 and 5.3, referring the reader to our paper [7] for the proof of Theorem 5.2.

PROOF OF THEOREM 5.1. Let x be a large number. For each integer $k \ge 6$, consider the set

$$A_k := \{2 \le n \le x : \mu^2(n) = 1 \text{ and } P(n) \le x^{1/k}\}$$

(here $\mu(n)$ stands for the Möbius function). Clearly, if an integer n is the product of k distinct primes not exceeding $x^{1/k}$, then $n \in A_k$. Now, since

$$\pi(y) > \frac{3}{4}\frac{y}{\log y} \qquad \text{provided } y \text{ is large enough,}$$

it follows that

$$\pi(x^{1/k}) > \frac{3}{4}\frac{x^{1/k}}{\log x^{1/k}} = \frac{3}{4}\frac{kx^{1/k}}{\log x},$$

provided x is large enough. Therefore, setting

$$m := \left\lfloor \frac{3}{4}\frac{kx^{1/k}}{\log x} \right\rfloor,$$

we can say that there are at least m primes not exceeding $x^{1/k}$. This means that, since $m - k + 1 \geq \frac{2}{3}m$, we have

$$\sum_{n\in A_k} 1 \geq \binom{m}{k} = \frac{m!}{(m-k)!k!} = \frac{m(m-1)\cdots(m-k+1)}{k!} \geq \frac{\left(\frac{2}{3}m\right)^k}{k!}, \quad (5.2)$$

provided x is sufficiently large.

It is easily proven by induction that $\left(\frac{k}{2}\right)^k > k!$ for each integer $k \geq 6$. Using this inequality, it follows from (5.2) that

$$\begin{aligned}\sum_{2\leq n\in A_k} 1 &\geq \frac{1}{k!}\left(\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{kx^{1/k}}{\log x}\right)^k = \frac{1}{k!}\left(\frac{k}{2}\cdot\frac{x^{1/k}}{\log x}\right)^k \\ &= \frac{1}{k!}(k/2)^k \cdot \frac{x}{\log^k x} > \frac{x}{\log^k x}. \end{aligned} \quad (5.3)$$

On the other hand, using the fact that $\omega(n) \leq 2\log x/\log\log x$ for all $n \leq x$, it follows that for any $n \in A_k$, we have that

$$B(n) = \beta(n) \leq P(n)\omega(n) \ll \frac{x^{1/k}\log x}{\log\log x}. \quad (5.4)$$

Combining (5.3) and (5.4) and choosing $k = \sqrt{\frac{\log x}{\log\log x}}$, we find that

$$\sum_{2\leq n\in A_k} \frac{1}{B(n)} = \sum_{2\leq n\in A_k} \frac{1}{\beta(n)} \gg \frac{\log\log x}{x^{1/k}\log x}\sum_{2\leq n\in A_k} 1 > \frac{\log\log x}{x^{1/k}\log x}\cdot\frac{x}{\log^k x}$$

$$
\begin{aligned}
&= \frac{x^{1-1/k}}{\log^{k+1} x} \cdot \log\log x > \frac{x^{1-1/k}}{\log^{k+1} x} = \frac{x}{\log x} \cdot \frac{1}{\log^k x \cdot x^{1/k}} \\
&= \frac{x}{\log x} \exp\left\{-2\sqrt{\log x \log\log x}\right\} > \frac{x}{e^{(2+\varepsilon)\ell(x)}},
\end{aligned}
$$

thus proving the lower bound in (5.1).

To prove the upper bound, let $y = y(x)$ be a function which tends to infinity as x tends to infinity, which is to be determined later. Then, write

$$
\begin{aligned}
\sum_{2\le n\le x} \frac{1}{\beta(n)} &= \sum_{\substack{2\le n\le x\\ P(n)\le y}} \frac{1}{\beta(n)} + \sum_{\substack{2\le n\le x\\ P(n)> y}} \frac{1}{\beta(n)} \\
&\le \sum_{\substack{2\le n\le x\\ P(n)\le y}} 1 + \frac{1}{y} \sum_{\substack{2\le n\le x\\ P(n)> y}} 1 \\
&\le \Psi(x,y) + \frac{x}{y}. \qquad (5.5)
\end{aligned}
$$

Now, observe that in his 1951 paper [3], N.G. de Bruijn established an upper bound for the $\Psi(x,y)$ function (formula (1.6) in [3]), namely

$$\Psi(x,y) < r_1 x \log^2 y \cdot \exp\{-u(\log u + \log\log u - r_2)\}, \qquad (5.6)$$

where r_1 and r_2 are some positive absolute constants and where

$$3 < u := \frac{\log x}{\log y} < 4\frac{\sqrt{y}}{\log y}. \qquad (5.7)$$

At this point, we choose $y = e^{\ell(x)}$ so that (5.7) is satisfied for $x \ge x_0$ (for some large x_0), and therefore in light of (5.6), we obtain that

$$\Psi(x,y) \ll_\varepsilon x \exp\left\{-\left(\frac{1}{2}-\varepsilon\right)\ell(x)\right\}. \qquad (5.8)$$

Using bound (5.8) in (5.5), we find that

$$\sum_{2\le n\le x} \frac{1}{\beta(n)} \le \frac{x}{e^{(\frac{1}{2}-\varepsilon)\ell(x)}} + \frac{x}{e^{\ell(x)}} \ll \frac{x}{e^{(\frac{1}{2}-\varepsilon)\ell(x)}},$$

thus completing the proof of the upper bound in (5.1).

PROOF OF THEOREM 5.3. For any real $t \in [0,1]$, the corresponding function $t^{B(n)-\beta(n)}$ is a multiplicative function. Therefore, for $\Re(s) > 1$, we have

$$\sum_{n=1}^{\infty} \frac{t^{B(n)-\beta(n)}}{n^s} = \prod_p \left(1 + \frac{1}{p^s} + \frac{t^p}{p^{2s}} + \frac{t^{2p}}{p^{3s}} + \cdots\right)$$

$$
\begin{aligned}
&= \zeta(s)\prod_p\left(1-\frac{1}{p^s}\right)\prod_p\left(1+\frac{1}{p^s}+\frac{t^p}{p^{2s}}+\frac{t^{2p}}{p^{3s}}+\cdots\right)\\
&= \zeta(s)\prod_p\left(1+\frac{t^p-1}{p^{2s}}+\frac{t^{2p}-t^p}{p^{3s}}+\cdots\right)\\
&= \zeta(s)G(s,t), \qquad (5.9)
\end{aligned}
$$

say. Let $g(n,t)$ be the arithmetical function defined implicitly by

$$G(s,t)=\sum_{n=1}^{\infty}\frac{g(n,t)}{n^s}.$$

Because of (5.9), $t^{\Omega(n)-\omega(n)}=\sum_{d|n}g(d,t)$. Therefore, we have

$$
\begin{aligned}
\sum_{n\le x}t^{B(n)-\beta(n)} &= \sum_{n\le x}\sum_{d|n}g(d,t)=\sum_{d\le x}g(d,t)\left\lfloor\frac{x}{d}\right\rfloor\\
&= x\sum_{d\le x}\frac{g(d,t)}{d}-\sum_{d\le x}g(d,t)\left(\frac{x}{d}-\left\lfloor\frac{x}{d}\right\rfloor\right)\\
&= x\sum_{d\le x}\frac{g(d,t)}{d}+O\left(\sum_{d\le x}|g(d,t)|\right). \qquad (5.10)
\end{aligned}
$$

On the one hand,

$$\sum_{d\le x}\frac{g(d,t)}{d}=\sum_{d=1}^{\infty}\frac{g(d,t)}{d}-\sum_{d>x}\frac{g(d,t)}{d}=G(1,t)-\sum_{d>x}\frac{g(d,t)}{d}. \qquad (5.11)$$

Now, given any small number $\delta>0$, we have that, as $x\to\infty$,

$$
\begin{aligned}
\left|\sum_{d>x}\frac{g(d,t)}{d}\right| &\le \sum_{d>x}\frac{|g(d,t)|}{d}=\sum_{d>x}\frac{|g(d,t)|}{d^{\frac12-\delta}d^{\frac12+\delta}}\\
&\le \frac{1}{x^{\frac12-\delta}}\sum_{d>x}\frac{|g(d,t)|}{d^{\frac12+\delta}}=o\left(\frac{1}{x^{\frac12-\delta}}\right). \qquad (5.12)
\end{aligned}
$$

Using (5.12) in (5.11), we get

$$\sum_{d\le x}\frac{g(d,t)}{d}=G(1,t)+o(x^{-\frac12+\delta}). \qquad (5.13)$$

Finally,

$$\sum_{d\le x}|g(d,t)| = \sum_{d\le x}\frac{|g(d,t)|}{d^{\frac{1}{2}+\delta}}\cdot d^{\frac{1}{2}+\delta} \le x^{\frac{1}{2}+\delta}\cdot O(1) = O\left(x^{\frac{1}{2}+\delta}\right). \tag{5.14}$$

Substituting (5.13) and (5.14) in (5.10), we obtain

$$\sum_{n\le x} t^{B(n)-\beta(n)} = G(1,t)x + O\left(x^{\frac{1}{2}+\delta}\right). \tag{5.15}$$

Setting $F(t) = G(1,t)$ and observing that $F(0) = \prod_p(1-1/p^2) = 6/\pi^2$, it is easily seen that $(F(t)-6/\pi^2)/t$ is a continuous function for $0\le t\le 1$, implying that the expression $(F(t)-6/\pi^2)/t$ is bounded on the interval $[0,1]$. This means that there exists a positive constant D_3 such that

$$\int_0^1 \frac{F(t)-6/\pi^2}{t}\,dt = D_3. \tag{5.16}$$

Since $B(n)=\beta(n)$ if and only if n is a squarefree number and since it is well known that $\sum_{n\le x}\mu^2(n) = (6/\pi^2)x + O(x^{1/2})$, dividing both sides of (5.15) by t, we get

$$\begin{aligned}
{\sum_{n\le x}}' t^{B(n)-\beta(n)-1} &= \sum_{\substack{n\le x\\ B(n)\ne\beta(n)}} t^{B(n)-\beta(n)-1} \\
&= x\frac{F(t)}{t} + O\left(\frac{1}{t}x^{\frac{1}{2}+\delta}\right) - \frac{1}{t}\sum_{n\le x}\mu^2(n) \\
&= x\frac{F(t)-6/\pi^2}{t} + O\left(\frac{1}{t}x^{\frac{1}{2}+\delta}\right).
\end{aligned} \tag{5.17}$$

Choosing $\varepsilon(x) = x^{-2/3}$ and integrating both sides of (5.17) with respect to t between $\varepsilon(x)$ and 1, we obtain

$$\int_{\varepsilon(x)}^1 {\sum_{n\le x}}' t^{B(n)-\beta(n)-1} = x\int_{\varepsilon(x)}^1 \frac{F(t)-6/\pi^2}{t}\,dt + O\left(x^{1/2+\delta}\log t\Big|_{\varepsilon(x)}^1\right). \tag{5.18}$$

On the one hand,

$$\begin{aligned}
\int_{\varepsilon(x)}^1 {\sum_{n\le x}}' t^{B(n)-\beta(n)-1} &= {\sum_{n\le x}}' \frac{1}{B(n)-\beta(n)} - {\sum_{n\le x}}' \frac{\varepsilon(x)^{B(n)-\beta(n)}}{B(n)-\beta(n)} \\
&= {\sum_{n\le x}}' \frac{1}{B(n)-\beta(n)} + O(x^{1/3}).
\end{aligned} \tag{5.19}$$

On the other hand, in light of (5.16),

$$\int_{\varepsilon(x)}^{1} \frac{F(t)-6/\pi^2}{t}\,dt = \int_0^1 \frac{F(t)-6/\pi^2}{t}\,dt - \int_0^{\varepsilon(x)} \frac{F(t)-6/\pi^2}{t}\,dt = D_3 + O(\varepsilon(x)). \tag{5.20}$$

Using (5.19) and (5.20) in (5.18), we get

$$\sum_{n\le x}{}' \frac{1}{B(n)-\beta(n)} = D_3 x + O(x^{\frac{1}{2}+\delta}\log x) = D_3 x + O(x^{\frac{1}{2}+\eta}),$$

thus completing the proof of Theorem 5.3. Incidently, numerical evidence seems to indicate that $D_3 \approx 0.1039$.

6. *On a sum involving the prime counting function*

In my book with Ivić [9], we obtained several estimates which left room for improvements. Not surprisingly, many of such estimates caught the attention of other mathematicians. One of these was the quest for an accurate estimate for the sum $\sum_{2\le n\le x} 1/\pi(n)$. In our book, we established that

$$\sum_{2\le n\le x} \frac{1}{\pi(n)} = \frac{1}{2}\log^2 x + O(\log x). \tag{6.1}$$

To obtain (6.1), we simply used the prime number theorem in the form

$$\pi(x) = \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right)$$

and the approximation

$$\sum_{n\le x} \frac{\log n}{n} = \frac{1}{2}\log^2 x + O(1).$$

In 2000, Panaitopol [35] improved (6.1) by establishing that

$$\sum_{2\le n\le x} \frac{1}{\pi(n)} = \frac{1}{2}\log^2 x - \log x - \log\log x + O(1).$$

In 2002, Ivić [31] tackled the problem once more by proving a much more accurate estimate, namely that given any integer $m\ge 2$,

$$\sum_{2\le n\le x} \frac{1}{\pi(n)} = \frac{1}{2}\log^2 x - \log x - \log\log x + C + \sum_{j=1}^{m-1} \frac{k_{j+1}}{j\log^j x} + O\left(\frac{1}{\log^m x}\right),$$

where C is an absolute constant with no explicit numerical value and where the constants k_i are effectively computable constants.

Due mostly to the unspecified value of the constant C, the story of the quest for a precise estimate for the sum $\sum_{2\leq n\leq x} 1/\pi(n)$ did not end with Ivić's 2002 estimate. In fact, in 2008, Hassani and Moshtagh [25] revisited the problem by providing a rough estimate of the constant C, and finally, in 2016, Berkane and Dusart [2] used sharp estimates on the zero-free region of the Riemann zeta-function to obtain a more precise value of C, namely $6.6840 < C < 6.7830$.

7. *The Riemann zeta-function*

The study of arithmetical functions represents only one area of Ivić's contributions to analytic number theory. In fact, throughout his academic life, Ivić worked relentlessly on the Riemann zeta-function $\zeta(s)$.

Besides his popular book [28] on this function (or its 2003 reprint [32]), incidently considered by many as the number one reference in the literature on the Riemann zeta-function, Ivić's research work on the zeta-function is extensive. For instance, his numerous papers on the Dirichlet divisor problem, the mean values of $\zeta(s)$ on the critical line $\Re(s) = 1/2$, the fourth power moment of $\zeta(s)$, the Hardy Z-function (in particular through his book [33] on this function), to only name a few, are remarkable.

As many who worked on the $\zeta(s)$ function, Ivić paid special attention to the Riemann Hypothesis. A word on this conjecture. From the functional equation (given above in (2.9)), one can establish that $\zeta(s)$ has zeros at $s = -2, -4, -6, \ldots$, which are traditionally called the *trivial zeros* of $\zeta(s)$. It has been proved that all the other zeros of $\zeta(s)$ are located in the *critical strip* $0 < \sigma = \Re(s) < 1$. In his epoch-making memoir [36], Bernhard Riemann (1826-1866) wrote that "very likely all complex zeros of $\zeta(s)$ have real parts equal to $1/2$". It is precisely this statement that is called the Riemann Hypothesis (or for short RH). This conjecture is undoubtedly one of the most celebrated and difficult open problems in mathematics, since its proof (or disproof) would have far reaching consequences in the distribution of primes and on the analytic behaviour of various sums involving key arithmetical functions (for more on this, see Ivić's book [28]).

Now, those who followed closely Ivić's work have certainly recognized his constant preoccupation for rigour. This may in part explain why, whereas most mathematicians take for granted the truth of RH, Ivić is one of the few (among which we find Paul Turán and John Edensor Littlewood) who expressed doubts about the veracity of RH. As Ivić wrote in a 2003 paper [34] on the matter, "Inasmuch the Riemann Hypothesis is commonly believed to be true, and for several valid reasons, I feel that the arguments that disfavour it should also be pointed out." In fact, in [34], Ivić lists several arguments against the truth of RH, some of which are closely connected with his own research.

Let us briefly present here one of Ivić's arguments against RH, namely one that goes back to D.H. Lehmer (1905-1991) and is called the *Lehmer phenomenon*. First, some background. Consider the Hardy Z-function defined by

$$Z(t) := \frac{\zeta(1/2+it)}{\sqrt{\chi(1/2+it)}},$$

where $\chi(s)$ is the function appearing in the functional equation (2.9). Since one can check that $\chi(s)\chi(1-s) = 1$ and $\overline{\Gamma(s)} = \Gamma(\overline{s})$, it follows that $|Z(t)| = |\zeta(1/2+it)|$, $Z(t)$ is even and

$$\overline{Z(t)} = \chi^{-1/2}(\frac{1}{2}-it)\zeta(\frac{1}{2}-it) = \chi^{1/2}(\frac{1}{2}-it)\zeta(\frac{1}{2}+it) = Z(t),$$

implying that $Z(t)$ is real for real values of t and that the zeros of the function $Z(t)$ coincide with the zeros of $\zeta(1/2+it)$. Hence, one of the nice things about the Z-function is that it turns out to be computable in fairly efficient ways (the Riemann-Siegel formula is one of them), and therefore it reduces the problem of finding zeros of the zeta-function on the critical line to finding sign changes of the Z-function.

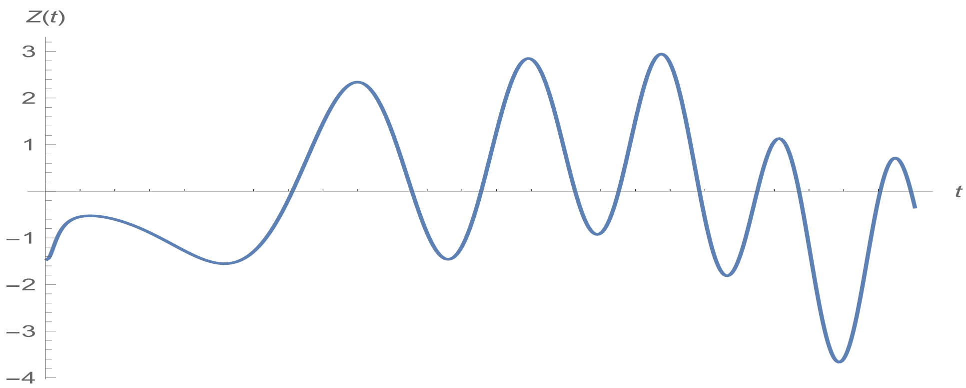


Figure 1. Graph of $Z(t)$ for $0 \leq t \leq 50$

As one can see in the graph of $Z(t)$ shown in Figure 1, the function $Z(t)$ has a negative local maximum $-0.52625\ldots$ at the point $t = 2.47575\ldots$. This is the only known occurrence of a negative local maximum for $Z(t)$, while no positive local minimum is known.

Lehmer's phenomenon is the fact that the graph of $Z(t)$ sometimes barely crosses the t-axis, implying in such cases that the absolute value of the maximum or minimum of $Z(t)$ between its two consecutive zeros is small.

In his paper [34], Ivić provides a detailed proof of the following result.

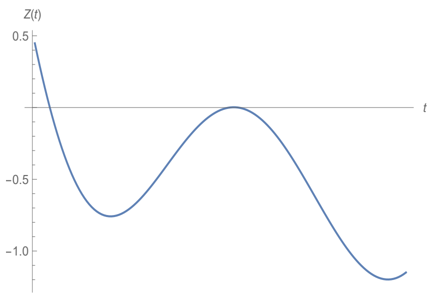
Theorem B. *If RH holds, then there exists a real number t_0 such that the graph of $Z'(t)/Z(t)$ is monotonically decreasing between the zeros of $Z(t)$ for all $t \geq t_0$.*

So, let $0 < \gamma_1 \leq \gamma_2 \leq \cdots$ denote the positive zeros of $Z(t)$ with multiplicities counted (note that all known zeros are simple) and assume that $Z(t)$ has a negative local maximum or a positive local minimum between two of its consecutive zeros γ_n and γ_{n+1}. In that case, $Z'(t)$ would have at least two distinct zeros x_1 and x_2 in the interval (γ_n, γ_{n+1}), that is with $Z'(x_1) = Z'(x_2) = 0$, in which case

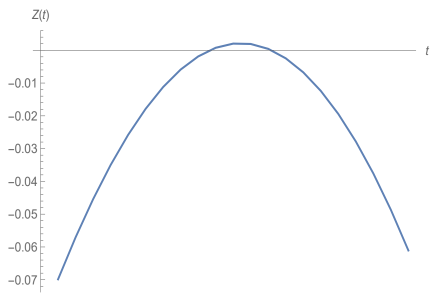
$$\frac{Z'(x_1)}{Z(x_1)} = \frac{Z'(x_2)}{Z(x_2)} = 0,$$

contradicting the fact that $Z'(t)/Z(t)$ is strictly decreasing between γ_n and γ_{n+1}. Therefore in light of Theorem B, it is impossible for $Z(t)$ to have a negative local maximum or a positive local minimum for a "large" t.

D.H. Lehmer did in fact find several values of $Z(t)$ which failed only by very little to provide a negative local maximum or a positive local minimum for some large values of t. For example, H.M. Edwards reports in his book [17] that Lehmer noticed that the numbers $\gamma_{6707} = 17143.786536$ and $\gamma_{6708} = 17143.821844$ are two consecutive zeros of $Z(t)$ which give rise to a local maximum of $Z(t)$ at the point $(t_0, Z(t_0))$, with $t_0 = 17143.803905$ and $Z(t_0) = 0.002153$, slightly above the t axis, as shown in the two figures below, the second one being simply a zoom on the local maximum $Z(t_0)$.



Graph of $Z(t)$ for $17143 \leq t \leq 17144.5$



Graph of $Z(t)$ for $17143.7 \leq t \leq 17143.9$

To summarize, if one could find a negative local maximum (other than the one occurring at $t = 2.47575\ldots$) or a positive local minimum for $Z(t)$, one would have proved that RH is false, thus making legitimate one of the doubts Aleksandar Ivić had about the truth of the Riemann hypothesis.

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