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Characterising some triplets of completely multiplicative functions

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Abstract. Let $\mathcal{M}_1^*$ stand for the set of completely multiplicative functions f such that $|f(n)| = 1$ for all positive integers n and let c_0, c_1, c_2 be three complex numbers such that $(c_0, c_1, c_2) \neq (0, 0, 0)$. Given $f \in \mathcal{M}_1^*$ and setting $s(n) := c_0 f(n-1) + c_1 f(n) + c_2 f(n+1)$, we prove that if $\lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} |s(n)| = 0$, then $c_0 + c_1 + c_2 = 0$ and there exists a real number τ such that $f(n) = n^{i\tau}$ for all positive integers n. Moreover, let $f_0, f_1, f_2 \in \mathcal{M}_1^*$ and consider the sum $s(n) := c_0 f_0(n-1) + c_1 f_1(n) + c_2 f_2(n+1)$. Assuming that $\lim_{n\to\infty} s(n) = 0$ and assuming also that either $f_0(n) = f_1(n)$ or $f_0(n) = f_2(n)$ or $f_1(n) = f_2(n)$, then $c_0 + c_1 + c_2 = 0$ and there exists $\tau \in \mathbb{R}$ such that $f_0(n) = f_1(n) = f_2(n) = n^{i\tau}$ for all positive integers n. Further similar results are also proved.

1. Introduction

Let, as usual, $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{C}$ stand for the sets of positive integers, the set of integers and the set of complex numbers, respectively. Let also

$$T := \{z \in \mathbb{C} : |z| = 1\}, \quad U := \{z \in \mathbb{C} : |z| \le 1\}.$$

Moreover, let $\mathcal{M}^*$ be the set of completely multiplicative functions and let $\mathcal{M}_1^*$ be the subset of $\mathcal{M}^*$ containing those functions $f \in \mathcal{M}^*$ for which $|f(n)| = 1$ for every $n \in \mathbb{N}$. Finally, given a sequence $(a_n)_{n\in\mathbb{N}}$ of complex numbers, we denote by $\overline{\{a_n : n \in \mathbb{N}\}}$ the set of limit points of sequence $(a_n)_{n\in\mathbb{N}}$.

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It is clear that if $f(n) = n^{i\tau}$ for some $\tau \in \mathbb{R}$, then $f \in \mathcal{M}_1^*$ and $\Delta f(n) := f(n+1) - f(n) \to 0$ as $n \to \infty$. I. Kátai conjectured [1] that given any $f \in \mathcal{M}_1^*$ such that $\Delta f(n) \to 0$ as $n \to \infty$, then $f(n) = n^{i\tau}$ for all $n \in \mathbb{N}$ for some $\tau \in \mathbb{R}$. This conjecture was proved by E. Wirsing and D. Zagier [12], and independently by Shao Pin-Tung and Tang Yuan Sheng (see [11]).

Another conjecture of I. Kátai and M. V. Subbarao [3] is the following.

Conjecture 1. *Let $f \in \mathcal{M}_1^*$ and set $S(f) := \overline{\{f(n+1)\overline{f(n)} : n \in \mathbb{N}\}}$. Assume that*

$$\#S(f) = k < \infty.$$

Then, for all $n \in \mathbb{N}$, $f(n) = n^{i\tau}F(n)$ for some $\tau \in \mathbb{R}$ and some function $F(n)$ satisfying $F(n)^k = 1$ for all $n \in \mathbb{N}$.

Partial results regarding Conjecture 1 were proved by Kátai and Subbarao [3], [4], and by Phong [7]. On the other hand, Wirsing [10] proved the following.

Theorem A. *If $f \in \mathcal{M}_1^*$ and $\#S(f) = k < \infty$, then there exists a real number τ such that $f(n) = n^{i\tau}F(n)$ for some function $F(n)$ satisfying $F^\ell(n) = 1$ for some positive integer ℓ.*

The following result is essentially a consequence of Theorem A.

Theorem 1.1. *Let $f, g \in \mathcal{M}_1^*$. Set $S(f,g) := \overline{\{g(n+1)\overline{f(n)} : n \in \mathbb{N}\}}$, $S(f) := \overline{\{f(n+1)\overline{f(n)} : n \in \mathbb{N}\}}$ and $S(g) := \overline{\{g(n+1)\overline{g(n)} : n \in \mathbb{N}\}}$. Assume that $\#S(f,g) < \infty$. Then, $\#S(f) < \infty$ and $\#S(g) < \infty$. Moreover, $f(n) = n^{i\tau}F(n)$, $g(n) = n^{i\tau}G(n)$, $F(n)^k = 1$ and $G(n)^\ell = 1$ for some functions $F(n)$ and $G(n)$ and positive integers k and ℓ.*

Proof. Let $\alpha \in S(f)$. Then there exists a sequence of positive integers $(n_\nu)_{\nu\in\mathbb{N}}$ with $n_\nu \to \infty$ and such that $f(n_\nu)\overline{f(n_\nu - 1)} \to \alpha$ as $\nu \to \infty$. Then, $f(2n_\nu)\overline{f(2n_\nu - 2)} \to \alpha$ as $\nu \to \infty$. Let $(m_\ell)_{\ell\in\mathbb{N}}$ be a subsequence of $(n_\nu)_{\nu\in\mathbb{N}}$ for which the sets $\{g(2m_\ell - 1)\overline{f(2m_\ell - 2)} : \ell \in \mathbb{N}\}$ and $\{g((2m_\ell - 1)^2)\overline{f(4m_\ell^2 - 4m_\ell)} : \ell \in \mathbb{N}\}$ have some limit points $\beta_1, \beta_2 \in S(f,g)$. Since

$$\begin{aligned}\beta_2 &= \lim_{\ell\to\infty} g((2m_\ell - 1)^2)\overline{f(4m_\ell^2 - 4m_\ell)} \\ &= \lim_{\ell\to\infty} g(2m_\ell - 1)\overline{f(2m_\ell - 2)} \times \lim_{\ell\to\infty} g(2m_\ell - 1)\overline{f(2m_\ell)},\end{aligned}$$

we have

$$\lim_{\ell\to\infty} g(2m_\ell - 1)\overline{f(2m_\ell)} = \frac{\beta_2}{\beta_1}.$$

The left hand side can be rewritten as

$$g(2m_\ell - 1)\overline{f(2m_\ell - 2)} \times f(2m_\ell - 2)\overline{f(2m_\ell)} \to \beta_1\overline{\alpha} \qquad (\ell \to \infty),$$

and therefore,

$$\beta_1\overline{\alpha} = \frac{\beta_2}{\beta_1}, \text{ so that } \alpha = \beta_1^2\overline{\beta_2}.$$

This means that $\alpha = \beta_1^2\overline{\beta_2} \in S(f,g)$, which, since $\#S(f,g) < \infty$, proves that $\#S(f) < \infty$. Therefore, it follows from Theorem A that, for all $n \in \mathbb{N}$, we have

$f(n) = n^{i\tau}F(n)$ for some $\tau \in \mathbb{R}$ and some function $F(n)$ for which $F(n)^k = 1$.

Similarly, we can prove that $\#S(g) < \infty$ and therefore that there exists $\tau_2 \in \mathbb{R}$ and $G(n)$ such that $g(n) = e^{i\tau_2}G(n)$ with $G(n)^\ell = 1$ for some positive integer ℓ. Now, since the set

$$S(f,g) = \overline{\left\{ \frac{(n+1)^{i\tau_2}}{n^{i\tau_1}} \frac{G(n+1)}{G(n)} : n \in \mathbb{N} \right\}}$$

contains only finite many points, we may conclude that $\tau_1 = \tau_2$, thereby completing the proof of Theorem 1.1.

2. The linear expansion of completely multiplicative functions

Another conjecture of I. Kátai is the following.

If $f \in \mathcal{M}_1^*$ and

$$\frac{1}{x}\sum_{n \le x} |\Delta f(n)| \to 0 \quad \text{or} \quad \frac{1}{\log x}\sum_{n \le x} \frac{|\Delta f(n)|}{n} \to 0,$$

then $f(n) = n^{i\tau}$ for some $\tau \in \mathbb{R}$ for every $n \in \mathbb{N}$.

This conjecture was recently proved by O. Klurman [5] and O. Klurman and A. Mangerel [6].

It is obvious that if $f(n) = n^\ell$ for some ($\ell \in \mathbb{N}$), then with suitable constants $c_0, c_1, \ldots, c_\ell$, $(c_0, c_1, \ldots, c_\ell) \ne (0, 0, \ldots, 0)$, we have

$$\sum_{k=0}^{\ell} c_k f(n+k) = 0 \qquad (n \in \mathbb{N}).$$

Interestingly, A. Sárközy [9] solved the above equation for multiplicative functions f.

Conjecture 2.1. *Let* $(c_0, c_1, \ldots, c_\ell) \neq (0, 0, \ldots, 0)$. *Then*

$$\sum_{k=0}^{\ell} c_k f(n+k) \to 0 \quad \text{as} \quad n \to \infty$$

has a solution $f \in \mathcal{M}_1^*$ *only if* $c_0 + c_1 + \cdots + c_\ell = 0$, *and all solutions are of the form* $f(n) = n^{i\tau}$ *for some* $\tau \in \mathbb{N}$.

Conjecture 2.2. *Let* $(c_0, c_1, \ldots, c_\ell) \neq (0, 0, \ldots, 0)$, $f_0, f_1, \ldots, f_\ell \in \mathcal{M}_1^*$ *and assume that*

$$\sum_{k=0}^{\ell} c_k f_k(n+k) \to 0 \quad \text{as} \quad n \to \infty. \tag{2.1}$$

Then, (2.1) has a solution only if $c_0 + c_1 + \cdots + c_\ell = 0$, *and in that case all the solutions are of the form*

$$f_j(n) = n^{i\tau} F_j(n) \qquad \text{for some } \tau \in \mathbb{N},$$

where $F_j^{k_j}(n) = 1$ *for all* $n \in \mathbb{N}$, $j = 0, 1, \ldots, \ell$, *and*

$$\sum_{k=0}^{\ell} c_k F_k(n+k) = 0 \qquad (n \in \mathbb{N}). \tag{2.2}$$

Observe that for $\ell = 1$ and $F_0, F_1 \in \mathcal{M}$, all the solutions of (2.2) have been obtained by Kátai and Phong [2].

3. Characterising triplets of completely multiplicative functions

Theorem 3.1. *Let* c_0, c_1, c_2 *be three complex numbers such that* $(c_0, c_1, c_2) \neq (0, 0, 0)$ *and let* $f \in \mathcal{M}_1^*$. *Consider the sum* $s(n) := c_0 f(n-1) + c_1 f(n) + c_2 f(n+1)$.

(i) Assuming that

$$\lim_{x \to \infty} \frac{1}{x} \sum_{n \le x} |s(n)| = 0, \tag{3.1}$$

then $c_0 + c_1 + c_2 = 0$ *and there exists a real number* τ *such that* $f(n) = n^{i\tau}$ *for all* $n \in \mathbb{N}$.

(ii) Assuming that $c_0+c_1+c_2=0$ and that there exists a real number τ such that $f(n)=n^{i\tau}$ for all $n\in\mathbb{N}$, then $\lim_{n\to\infty} s(n)=0$.

Proof. First consider the case when $c_1\neq 0$ and set

$$D_0=\frac{c_0}{c_1},\ D_1=\frac{c_2}{c_1},\ S(n)=\frac{s(n)}{c_1 f(n)}=D_0\frac{f(n-1)}{f(n)}+D_1\frac{f(n+1)}{f(n)}+1,\ \alpha(n)=\frac{f(n+1)}{f(n)},$$

these last two definitions implying that

$$S(n)=D_0\overline{\alpha(n-1)}+D_1\alpha(n)+1.$$

Then, let (γ_1,δ_1) and (γ_2,δ_2) be the two couples of numbers located on the unit circle for which

$$\begin{cases} D_0\gamma_1+D_1\delta_1+1=0,\\ D_0\gamma_2+D_1\delta_2+1=0.\end{cases}$$

It is clear that if $|S(n)|<\varepsilon$, then

$$\min(|\delta_1-\alpha(n)|,|\delta_2-\alpha(n)|)+\min(|\gamma_1-\overline{\alpha(n-1)}|,|\gamma_2-\overline{\alpha(n-1)}|)<c\varepsilon$$

for some positive constant c. Now, observe that, given any positive integer d,

$$\frac{n+1}{n}=\frac{dn+1}{dn}\cdot\frac{dn+2}{dn+1}\cdots\frac{dn+d}{dn+d-1}\qquad (n=1,2,\ldots).$$

Written otherwise, this means that

$$\alpha(n)=\alpha(dn)\alpha(dn+1)\cdots\alpha(dn+d-1).$$

On the other hand, it follows from (3.1) that for every $\varepsilon>0$, we have

$$\frac{1}{x}\sum_{\min(|\alpha(n)-\delta_1|,|\alpha(n)-\delta_2|)>\varepsilon} 1\to 0\quad \text{as } x\to\infty.$$

This motivates the definition

$$\kappa(n):=\begin{cases}\delta_1 & \text{if}\quad |\alpha(n)-\delta_1|\le|\alpha(n)-\delta_2|,\\ \delta_2 & \text{if}\quad |\alpha(n)-\delta_1|>|\alpha(n)-\delta_2|.\end{cases}$$

Consequently,

$$\kappa(n)=\kappa(dn)\kappa(dn+1)\cdots\kappa(dn+d-1)$$

holds for every $d\in\mathbb{N}$, provided n is large enough.

For the rest of the proof, we consider four distinct cases, namely the following.

Case I: $\delta_1^2 \neq 1$ and $\delta_2^2 \neq 1$. In this case, each of the two relations $\delta_1 = \delta_1\delta_2$ and $\delta_2 = \delta_1\delta_2$ does not hold, which implies that

$$\frac{1}{x}\#\{n \le x : \kappa(2n) \neq \kappa(2n+1)\} \to 0 \quad \text{as } x \to \infty.$$

It follows from this observation that with the exception of $o(x)$ integers $n \le x$, if $\kappa(n) = \delta_1$, then $\kappa(2n) = \kappa(2n+1) = \delta_2$ and therefore $\delta_1 = \kappa(n) = \delta_1^4$. Hence, as $x \to \infty$,

$$\frac{1}{x}\#\{n \le x : \kappa(n) = \delta_1 \text{ and } \kappa(4n)\kappa(4n+1)\kappa(4n+2)\kappa(4n+3) \neq \delta_1^4\} \to 0$$

and similarly

$$\frac{1}{x}\#\{n \le x : \kappa(n) = \delta_2 \text{ and } \kappa(4n)\kappa(4n+1)\kappa(4n+2)\kappa(4n+3) \neq \delta_2^4\} \to 0.$$

In light of these observations, we may conclude that $\delta_1^3 = 1$ and $\delta_2^3 = 1$. Therefore, $\delta_2 = \overline{\delta_1}$. In this case, we have $\gamma_1 = \overline{\delta_1}$ and $\gamma_2 = \overline{\delta_2} = \delta_1$. Now, if $\alpha(n) \to \delta_1$, then $\overline{\alpha(n-1)} \to \overline{\delta_1}$, whereas if $\alpha(n) \to \delta_2$, then $\overline{\alpha(n-1)} \to \overline{\delta_2}$ for almost all n. Hence, in light of a result of Klurman [5], we may conclude that $f(n) = n^{i\tau}$ for some $\tau \in \mathbb{R}$.

Case II: $\delta_1 = 1$ and $\delta_2 \neq -1$. For this case, we introduce the quantities

$$A(x) = \#\{n \in [x/2, x) : \kappa(n) = 1\} \quad \text{and} \quad B(x) = \#\{n \in [x/2, x) : \kappa(n) = \delta_2\},$$

so that in light of (3.1), we have that $A(x) + B(x) = x/2 + o(x)$ as $x \to \infty$. Hence (in this case), with the exception of no more than $o(x)$ of those integers $n \in [x/2, x)$, we have that

$$\kappa(n) = 1 \quad \text{implies} \quad \kappa(2n) = \kappa(2n+1) = 1$$

and

$$\kappa(n) = \delta_2 \quad \text{implies} \quad \kappa(2n) \cdot \kappa(2n+1) = \delta_2 \cdot 1 \text{ or } 1 \cdot \delta_2.$$

From this, it follows that

$$A(x) \ge 2A(x/2) + B(x/2) - o(x) \quad \text{and} \quad B(x) \le B(x/2) + o(x)$$

and therefore,

$$\frac{B(x)}{x/2} \le \frac{1}{2}\frac{B(x/2)}{x/4} + o(x)/x. \tag{3.2}$$

Letting $\xi := \limsup_{x\to\infty} \frac{B(x)}{x/2}$, it follows from (3.2) that $\xi \le \frac{\xi}{2}$ and therefore that $\xi = 0$, implying that

$$\frac{f(n+1)}{f(n)} \to 1 \quad \text{for almost all } n.$$

With these conditions, we may conclude from Klurman's theorem [5] that $f(n) = n^{i\tau}$ for some $\tau \in \mathbb{R}$.

Case III: $\delta_1 \neq 1$ and $\delta_2 = -1$. In this case, $\kappa(2n) \neq \kappa(2n+1)$ and $\kappa(n) = \kappa(2n)\kappa(2n+1)$ would imply that $\kappa(n)$ is equal to $\kappa(2n)$ or to $\kappa(2n+1)$. Therefore, $1 \in \{\kappa(2n), \kappa(2n+1)\}$, implying that $\delta_1 = 1$. The case $\kappa(2n) = \kappa(2n+1) = -1$ implies that $\kappa(n) = 1 (= \delta_1)$, and similarly, $\kappa(2n) = \kappa(2n+1) = \delta_1$ implies that $\delta_1 = \delta_1^2$, and so $\delta_1 = 1$. This allows us to conclude that Case III cannot occur.

Case IV: $\delta_1 = 1$ and $\delta_2 = -1$. In this case, we have that

$$\frac{f^2(n+1)}{f^2(n)} \to 1 \qquad \text{for almost all } n,$$

and therefore, by Klurman's theorem [5], we may conclude that $f(n) = n^{2i\tau}$. Hence, $f(n) = n^{i\tau}F(n)$ with $F^2(n) = 1$. Consequently, the relation

$$D_0 F(n-1) + D_1 F(n+1) + F(n) = 0 \tag{3.3}$$

holds for every integer $n \geq n_0$ for some $n_0 \in \mathbb{N}$. Now, relation (3.3) implies that $F(n+1)/F(n)$ is a constant and that this constant is either 1 or -1. There are two possibilities. If $F(n+1) = F(n)$ for all $n \geq n_0$, then $F(n) = 1$ for all $n \in \mathbb{N}$. On the other hand, if $F(n+1) = -F(n)$ for all $n \geq n_0$, then $F(2n) = -F(2n+1) = F(2n+2)$, implying that $F(n) = F(n+1)$, which is impossible.

In conclusion, we have covered all the possible cases, except when $f(n) = n^{i\tau}$ and $D_0 + D_1 + 1 = 0$, which occurs if and only if $c_0 + c_1 + c_2 = 0$, which proves item (ii) of the Theorem.

It remains to consider the case when $c_1 = 0$, that is when

$$s(n) = c_0 f(n-1) + c_2 f(n+1),$$

which is equivalent to

$$\frac{s(2n+1)}{c_2 f(2n)} = \frac{c_0}{c_2} + \frac{f(n+1)}{f(n)},$$

which, in light of (3.1), implies that, with $A = c_0/c_2$,

$$\frac{1}{x} \sum_{n \leq x} \left| A + \frac{f(n+1)}{f(n)} \right| \to 0 \quad \text{as } x \to \infty.$$

Therefore, since $\alpha(n) = \alpha(2n)\alpha(2n+1)$, we have that $\alpha(n) \to -A$ for almost all n. This implies that $-A = A^2$ and therefore that $A = -1$. We can then again apply Klurman's theorem [5]. This completes the proof of Theorem 3.1.

Using the same kind of technique, we could also prove the following result.

Theorem 3.2. *Let c_0, c_1, c_2 be three complex numbers such that $(c_0, c_1, c_2) \neq (0,0,0)$, let $f \in \mathcal{M}_1^*$ and consider the sum $s(n) := c_0 f(n-1) + c_1 f(n) + c_2 f(n+1)$. Assuming that*

$$\lim_{x\to\infty} \frac{1}{\log x} \sum_{n\le x} \frac{|s(n)|}{n} = 0,$$

then $c_0 + c_1 + c_2 = 0$ and there exists $\tau \in \mathbb{R}$ such that $f(n) = n^{i\tau}$ for all $n \in \mathbb{N}$.

We also have the following.

Theorem 3.3. *Let c_0, c_1, c_2 be three complex numbers such that $(c_0, c_1, c_2) \neq (0,0,0)$, let $f_0, f_1, f_2 \in \mathcal{M}_1^*$ and consider the sum $s(n) := c_0 f_0(n-1) + c_1 f_1(n) + c_2 f_2(n+1)$. Assume that $\lim_{n\to\infty} s(n) = 0$ and assume also that either $f_0(n) = f_1(n)$ or $f_0(n) = f_2(n)$ or $f_1(n) = f_2(n)$. Then $c_0 + c_1 + c_2 = 0$ and there exists $\tau \in \mathbb{R}$ such that $f_0(n) = f_1(n) = f_2(n) = n^{i\tau}$ for all $n \in \mathbb{N}$.*

Proof. Without any loss of generality, we can assume that $f_1(n) = f_2(n)$ $(=: f(n))$. We will first assume that $c_1 \neq 0$. Then,

$$\frac{s(n)}{c_1 f(n)} = \frac{c_0}{c_1}\frac{f_0(n-1)}{f(n)} + \frac{c_2}{c_1}\frac{f(n+1)}{f(n)} + 1 \to 0 \quad \text{as } n \to \infty.$$

Further set

$$\gamma(n) := \frac{f_0(n-1)}{f(n)}, \quad \delta(n) := \frac{f(n+1)}{f(n)}, \quad D_0 := \frac{c_0}{c_1}, \quad D_1 := \frac{c_2}{c_1}.$$

Recalling that T stands for the unit circle, $D_0 T + D_1 T + 1 = 0$ has no more than two solutions, say (γ_1, δ_1) and (γ_2, δ_2), that is, satisfying

$$D_0\gamma_1 + D_1\delta_1 + 1 = 0 \quad \text{and} \quad D_0\gamma_2 + D_1\delta_2 + 1 = 0.$$

Now, assume that the sequence $(\delta(n))_{n\in\mathbb{N}}$ has two limits points, say δ_1 and δ_2. Then, by the theorem of Kátai and Subbarao [3], we obtain that there exists a real number τ such that $f(n) = n^{i\tau} F(n)$ with $F^2(n) = 1$ for all $n \in \mathbb{N}$. Consequently, $\delta_1 = 1$ and $\delta_2 = -1$, so that we obtain successively

$$\begin{aligned} D_0\gamma_1 + D_1 + 1 &= 0, & D_0\gamma_2 - D_1 + 1 &= 0,\\ D_0\gamma_1 &= -D_1 - 1, & D_0\gamma_2 &= D_1 - 1,\\ |D_0|^2 = |D_1|^2 + 1 &+ (D_1 + \overline{D_1}), & |D_0|^2 = |D_1|^2 + 1 &- (D_1 + \overline{D_1}), \end{aligned}$$

from which we may conclude that $\Re(D_1) = 0$ and therefore that $D_1 = iV$ for some $V \in \mathbb{R}$.

Now, since

$$\frac{f_0(n+1)}{f_0(n-1)} = \frac{\gamma(n^2)}{\gamma(n)^2} \quad \text{and} \quad \frac{f_0(n+1)}{f_0(n)} = \frac{\gamma((2n+1)^2)}{\gamma(2n+1)^2},$$

we obtain that $S(f_0) \subseteq \{1/\gamma_1, 1/\gamma_2, \gamma_1/\gamma_2^2, \gamma_2/\gamma_1^2\}$, implying that $\#S(f_0) \le 4$. Consequently,

$$f_0(n) = n^{i\tau} F_0(n) \text{ with } F_0^k(n) = 1, \text{ where } k \le 4.$$

We may then conclude that

$$D_0 \frac{F_0(n-1)}{F(n)} + iV \frac{F(n+1)}{F(n)} + 1 = 0,$$

that is,

$$D_0 F_0(n-1) + iVF(n+1) + F(n) = 0,$$

so that

$$F_0(n-1) = \frac{-F(n) - iVF(n+1)}{D_0}. \tag{3.4}$$

If $F(n)$ takes on two values, that is, $F(n) = -1$ holds once, then $F(n) = -1$ for infinitely many n's, implying that the right hand side of (3.4) takes on four distinct values, namely the values 1, -1, i and $-i$. Writing $D_0 = U + iQ$, relation (3.4) can be written as

$$(U + iQ)F_0(n-1) = -F(n) - iVF(n+1).$$

Hence, if $F_0(n-1) = 1$, we have that $U \in \{1, -1\}$, whereas if $F_0(n-1) = i$, we have that $V \in \{1, -1\}$.

Since $D_0^2 = \pm 2i$, it follows that either

$$F_0^2(n-1) = F(n)F(n+1) \tag{3.5}$$

or

$$F_0^2(n-1) = -F(n)F(n+1). \tag{3.6}$$

Now, let $G(n) := F_0^2(n)$. Observe that $G(n) \in \{1, -1\}$ for all $n \in \mathbb{N}$.

We first consider the case when (3.5) holds. In light of (3.4), we have that

$$G(n) = F(n+1)F(n+2) \quad \text{and} \quad G(2n) = F(2n+1)F(2n+2),$$

from which it follows that

$$G(2) = \frac{F(2n+1)F(2n+2)}{F(n+1)F(n+2)} = F(2)\frac{F(2n+1)}{F(n+2)}.$$

Hence,

$$\frac{G(2)}{F(2)} = \frac{F(2m-3)}{F(m)}, \tag{3.7}$$

so that

$$\frac{G(2)}{F(2)} = \frac{F(6m-3)}{F(3m)} = \frac{F(2m-1)}{F(m)},$$

implying that

$$G(2) = \frac{F(2m-1)}{F(2m)},$$

which in turn implies that

$$G(2) = \frac{F(4m^2-1)}{F(4m^2)} = \frac{F(2m-1)}{F(2m)} \cdot \frac{F(2m+1)}{F(2m)} = G(2)\frac{F(2m+1)}{F(2m)},$$

which itself implies that

$$\frac{F(2m+1)}{F(2m)} = 1 \quad \text{and therefore that} \quad F(2m+1) = G(2)F(2m-1).$$

Now, since $G^2(2) = 1$, it follows that $F(n+4) = F(n)$ if n is odd. Since there obviously exist infinitely many odd integers m for which $2m-3 \equiv m \pmod 4$, it follows from (3.7) that

$$\frac{F(2m+3)}{F(2m+1)} = \frac{F(2m-1)}{F(2m-2)},$$

so that $F(2m-2) = F(2m+2)$, implying that $F(m-1) = F(m+1)$ and therefore that $F(n)$ is a constant for n sufficiently large and that this constant is 1. From this, we easily see that $F(n) = 1$ for all positive integers n. Consequently, $f(n) = n^{i\tau}$ for all $n \in \mathbb{N}$.

It follows that $D_0F_0(n-1) + D_1 + 1 = 0$, implying that $F_0(n)$ is constant and therefore that $F_0(n) = 1$ for all positive integers n.

It remains to consider the case when (3.6) holds. Since $G(n) = F_0^2(n)$, we have that $G(n) = -F(n+1)F(n+2)$ and $G(2n) = -F(2n+1)F(2n+2)$. Hence,

$$G(2) = \frac{F(2n+1)F(2n+2)}{F(n+1)F(n+2)} = F(2)\frac{F(2n+1)}{F(n+2)} = \frac{F(2n+1)}{F(2n+4)} = \frac{F(2m-3)}{F(2m)}.$$

Then, the rest of the proof in the case when (3.6) is similar to the one for the case when (3.5) holds. We will therefore skip it.

It remains to consider the case when one of the c_i's is 0, say $c_1 = 0$. In fact, this case is much easier to handle. By hypothesis, we have that

$$c_0 f_0(n) + c_2 f(n+2) \to 0 \qquad \text{as } n \to \infty,$$

which implies that

$$c_0 f_0(2) f_0(n) + c_2 f(2) f(n+1) \to 0 \qquad \text{as } n \to \infty,$$

from which It follows that

$$\frac{f(n+1)}{f_0(n)} \to 1 \quad \text{as } n \to \infty, \quad \text{where } A = -\frac{c_2 f(2)}{c_0 f(2)}.$$

Now,

$$\frac{f(n^2)}{f_0(n-1)} = \frac{f(n)}{f_0(n-1)} \cdot \frac{f(n)}{f_0(n+1)}. \tag{3.8}$$

Since each of the quotients $\dfrac{f(n^2)}{f_0(n-1)}$ and $\dfrac{f(n)}{f_0(n-1)}$ tend to A as $n \to \infty$, it follows from (3.8) that

$$\frac{f(n)}{f_0(n+1)} \to 1 \quad \text{and} \quad \frac{f(n-1)}{f_0(n)} \to 1 \qquad (n \to \infty),$$

and therefore that

$$\frac{f(n+1)}{f_0(n-1)} \to 1 \text{ and } \frac{f(2n+2)}{f(2n)} = \frac{f(n+1)}{f(n)} \to 1 \quad (n \to \infty).$$

From this, we may conclude that there exists $\tau \in \mathbb{R}$ such that $f(n) = n^{i\tau}$ for all positive integers n.

We can also prove an even more general result, namely the following.

Theorem 3.4. *Let $a, b \in \mathbb{N}$, $f \in \mathcal{M}_1^*$ and $S(n) = c_0 f(n-a) + c_1 f(n) + c_2 f(n+b)$. Assume that $\lim_{n\to\infty} S(n) = 0$. If $(c_0, c_1, c_2) \neq (0,0,0)$, then $c_0 + c_1 + c_2 = 0$ and there exists $\tau \in \mathbb{R}$ such that $f(n) = n^{i\tau}$ for all positive integers n. On the other hand, if $c_0 + c_1 + c_2 = 0$, then $c_0(n-a)^{i\tau} + c_1 n^{i\tau} + c_2(n+b)^{i\tau} \to 0$ as $n \to \infty$.*

Proof. The proof is similar to that of the preceding theorem and we will therefore omit it.

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