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PARTITIONING THE SET OF PRIMES TO CREATE r-DIMENSIONAL SEQUENCES WHICH ARE UNIFORMLY DISTRIBUTED MODULO $[0, 1)^r$

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ABSTRACT. Expanding on our previous results, we show that by partitioning the set of primes into a finite number of subsets of roughly the same size, we can create r-dimensional sequences of real numbers which are uniformly distributed modulo $[0, 1)^r$.

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1. Introduction

In previous papers, we used the factorization of integers to generate large families of normal numbers; see for instance [1] and [2]. Along the same lines, letting $q \geq 3$ be a prime number, we showed in a recent paper [3] how one can create an infinite sequence $\alpha_1, \alpha_2, \ldots$ of normal numbers in base $q-1$ such that, for any fixed integer $r \geq 1$, the r-dimensional sequence $(\{\alpha_1(q-1)^n\}, \ldots, \{\alpha_r(q-1)^n\})$ is uniformly distributed on $[0, 1)^r$, where $\{y\}$ stands for the fractional part of y. Here, given an appropriate partition of the primes, we create an r-dimensional sequence of real numbers which is uniformly distributed modulo $[0, 1)^r$.

First, we introduce some basic notation. Given an integer $q \geq 3$, let $A_q := \{0, 1, \ldots, q-1\}$. Given an integer $t \geq 1$, an expression of the form $i_1 i_2 \ldots i_t$, where each $i_j \in A_q$, is called a *finite word* of length t. The symbol Λ will denote the *empty word*, so that if we concatenate the words α, Λ, β, then, instead of writing $\alpha\Lambda\beta$, we simply write $\alpha\beta$.

Let $\wp$ stand for the set of all primes. Given an integer $q \geq 3$, a partition $\mathcal{T}$ of $\wp$ into sets of primes $\wp_0, \wp_1, \ldots, \wp_{q-1}, \mathcal{R}$, noted $(\mathcal{T}, q)$, is said to be a *regular partition* if $\wp_0 \cup \wp_1 \cup \ldots \cup \wp_{q-1} \cup \mathcal{R} = \wp$, where $\mathcal{R}$ is finite (possibly empty) and

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where the sets $\wp_i$'s are roughly of the same size in the sense that, for every fixed $\varepsilon > 0$,

$$\max_{\substack{j=0,1,\ldots,q-1\\ \varepsilon\le y/x\le 1}} \left| \frac{q\,\pi([x,x+y]\cap\wp_j)}{\pi([x,x+y])} - 1 \right| \to 0 \qquad \text{as } x\to\infty.$$

The following provide examples of *regular partitions* of primes:

(1) Given an arbitrary integer $k \ge 2$, let $\ell_0, \ell_1, \ldots, \ell_{\varphi(k)-1}$ be the reduced residues mod k (here φ stands for the Euler totient function). Setting

$$\begin{aligned} \wp_\nu &:= \{p\in\wp : p\equiv \ell_\nu \pmod k\} \qquad (\nu = 0,1,\ldots,\varphi(k)-1),\\ \mathcal{R} &:= \{p\in\wp : p \mid k\}, \end{aligned}$$

Using the prime number theorem for arithmetic progressions, one can easily show that $\wp_0, \wp_1, \ldots, \wp_{\varphi(k)-1}, \mathcal{R}$ represents a regular partition of the primes.

(2) Let $\alpha \in \mathbb{R}\setminus\mathbb{Q}$. Given an integer $q \ge 2$, let $I_0, I_1, \ldots, I_{q-1}$ be disjoint intervals each of length $1/q$ such that $[0,1) = I_0 \cup I_1 \cup \cdots \cup I_{q-1}$. Setting $\wp_\nu = \{p\in\wp : \{\alpha p\}\in I_\nu\}$ for $\nu = 0,1,\ldots,q-1$ and $\mathcal{R} = \emptyset$, we easily obtain a regular partition of the primes.

(3) Let $p_1 < p_2 < \cdots$ represent the sequence of all primes. Given a fixed integer k, then for each integer $\ell = 0,1,\ldots,k-1$, consider the set of primes $\wp_\ell := \{p_{mk+\ell} : m = 0,1,2,\ldots\}$ and let $\mathcal{R} = \emptyset$, then one can show that $\wp_0, \wp_1, \ldots, \wp_{k-1}, \mathcal{R}$ constitutes a regular partition of the primes.

Let λ_N be a function such that $\lambda_N \to \infty$ as $N\to\infty$. Then, for each integer $N > e^e$, we introduce the intervals $J_N = [e^N, e^{N+1})$ and $K_N = [N, N^{\lambda_N}]$, as well as the particular product of primes $Q_N := \prod_{p\in K_N} p$. Now, we write each integer $n \in J_N$ as

$$n = \pi_1(n)\pi_2(n)\cdots\pi_{h(n)}(n)\,\nu(n), \tag{1}$$

where $\pi_1(n) \le \pi_2(n) \le \cdots \le \pi_{h(n)}$ are those prime factors of n located in the interval K_N and $\nu(n)$ stands for the product of the other prime factors of n, namely those which are relatively prime to Q_N.

2. Main result

Theorem 1. *Let $(\mathcal{T}^{(j)}, q_j)_{j\ge1}$ be an arbitrary sequence of regular partitions of the primes, with corresponding partitions*

$$\wp_0^{(j)}, \wp_1^{(j)}, \ldots, \wp_{q_j-1}^{(j)}, \mathcal{R}^{(j)} \qquad (j = 1,2,\ldots).$$

Given $j, n \in \mathbb{N}$, let

$$a_{j,n} = \begin{cases} \ell & \text{if} \quad \pi_j(n) \in \wp_\ell^{(j)} \text{ with } j \le h(n), \\ 0 & \text{if} \quad \pi_j(n) \in \mathcal{R}^{(j)} \text{ or if } j > h(n), \end{cases}$$

where $h(n)$ is as in (1). For each integer $j \ge 1$, consider the number β_j whose q_j-ary expansion is

$$\beta_j = 0.a_{j,1}a_{j,2}\ldots$$

and further consider the sequence

$$u_{n,j} = \{\beta_j q_j^n\} \qquad (n = 1, 2, \ldots).$$

Then, for each fixed integer $r \ge 1$, the r-dimensional sequence

$$(u_{n,1}, u_{n,2}, \ldots, u_{n,r})_{n \ge 1}$$

is uniformly distributed modulo $[0,1)^r$.

3. The approach and some preliminary lemmas

Fix positive integers r and k. Then, consider the real $r \times k$ matrix

$$S = \begin{pmatrix} b_{1,1} & \cdots & b_{1,k} \\ b_{2,1} & \cdots & b_{2,k} \\ \vdots & & \vdots \\ b_{r,1} & \cdots & b_{r,k} \end{pmatrix},$$

where each $b_{i,j}$ belongs to A_{q_j}, and, moreover, for each positive integer n, consider the real $r \times k$ matrix

$$\kappa(n) = \begin{pmatrix} a_{1,n+1} & \cdots & a_{1,n+k} \\ a_{2,n+1} & \cdots & a_{2,n+k} \\ \vdots & & \vdots \\ a_{r,n+1} & \cdots & a_{r,n+k} \end{pmatrix},$$

where the elements $a_{j,m}$ are those appearing in the q_j-ary expansion of β_j. Then, set $T := \left(\prod_{j=1}^{r} q_j\right)^k$.

In order to prove Theorem 1, it is sufficient to prove that, given any small number $\varepsilon > 0$, for every matrix S (as the one above), we have

$$\limsup_{x \to \infty} \left| \frac{T}{x} \#\{n \le x : \kappa(n) = S\} - 1 \right| \le \varepsilon. \tag{2}$$

Indeed, fixing $\varepsilon > 0$ and assuming that we can establish that for $e^N \le x \le e^{N+1}$, we have

$$\left|T\#\{n \in [e^N, x) : \kappa(n) = S\} - (x - e^N)\right| \le \varepsilon\, x + O(1), \tag{3}$$

and similarly also with $[e^\nu, e^{\nu+1})$ instead of $[e^N, x)$ for $\nu = k, k+1, \dots, N-1$, we have

$$\begin{aligned}\#\{n \le x : \kappa(n) = S\} \quad &= \quad \sum_{\nu=k}^{N-1} \#\{n \in J_\nu : \kappa(n) = S\} \\ &\qquad + \#\{n \in [e^N, x) : \kappa(n) = S\} + O(1),\end{aligned}$$

then we easily see that (2) follows (3).

Lemma 1. *Let y_x be a function of x which tends to infinity with x. Then, the number of those positive integers $n \le x$ which have two prime divisors p_1, p_2 such that $y_x \le p_1 < p_2 < 2p_1$ and p_2 divides* $| \prod_{-k \le j \le k} (n+j)$ *is $o(x)$ as $x \to \infty$.*

P r o o f. It is clear that amongst those positive integers $n \le x$, the situation $p_1 \mid n$, $p_2 \mid n+j$ for some $j \in [-k, k]$ and $y_x \le p_1 < p_2 < 2p_1$ occurs at most $x \sum_{y_x \le p_1 < p_2 < 2p_1 < x} \frac{1}{p_1 p_2} = o(x)$ times, thus establishing our claim. □

Lemma 2. *With y_x as in Lemma 1, we have*

$$\lim_{x\to\infty} \frac{1}{x} \#\{n \le x : p^2 \mid n,\ p \ge y_x\} = 0.$$

P r o o f. This is immediate if one observes that

$$\#\{n \le x : p^2 \mid n,\ p \ge y_x\} \le \sum_{p \ge y_x} \frac{x}{p^2} \ll \frac{x}{y_x}.$$

□

Lemma 3. *As $N \to \infty$,*

$$\frac{1}{e^N} \#\{n \in J_N : \min_{-k \le \ell \le k} h(n+\ell) \le r\} \to 0.$$

P r o o f. This is an immediate consequence of the Turán-Kubilius inequality. □

Lemma 4. *Let $J = [e^N, x]$, where $e^N < x \le e^{N+1}$. Let r and k be fixed positive integers. Let $Q_{i,\ell}$, for $i = 1, \dots, r$ and $\ell = 1, \dots, k$ be distinct primes belonging to K_N such that $Q_{1,\ell} < Q_{2,\ell} < \cdots < Q_{r,\ell}$. Moreover, let $S_J(Q_{i,\ell} \mid i = 1, \dots, r,\ \ell = 1, \dots, k)$ be the number of those integers $n \in J$ for which $\pi_i(n+\ell) = Q_{i,\ell}$. Also,* *for each integer $r \geq 1$, let $\sigma(1), \dots, \sigma(k)$ be the permutation of the set $\{1, \dots, k\}$ which allows us to write*

$$Q_{r,\sigma(1)} < Q_{r,\sigma(2)} < \cdots < Q_{r,\sigma(k)}.$$

Then, given any small $\varepsilon > 0$ and provided $x - e^N \geq \varepsilon e^N$, we have, as $N \to \infty$,

$$\begin{aligned} & S_J(Q_{i,\ell} \mid i = 1, \dots, r,\ \ell = 1, \dots, k) \\ & \quad = (1 + o(1)) \frac{x - e^N}{\prod_{\substack{1 \leq i \leq r \\ 1 \leq \ell \leq k}} Q_{i,\ell}} \cdot \prod_{N \leq \pi < Q_{r,\sigma(k)}} \left(1 - \frac{\rho(\pi)}{\pi}\right), \end{aligned}$$

where

$$\rho(\pi) = \begin{cases} k & \text{if} \quad N \leq \pi < Q_{r,\sigma(1)}, \\ k-1 & \text{if} \quad Q_{r,\sigma(1)} < \pi < Q_{r,\sigma(2)}, \\ \vdots & \qquad \vdots \\ 1 & \text{if} \quad Q_{r,\sigma(k-1)} < \pi < Q_{r,\sigma(k)}, \\ 0 & \text{if} \quad \pi \in \{Q_{i,\ell} : i = 1, \dots, r,\ \ell = 1, \dots, k\}. \end{cases}$$

P r o o f. This is relation (2.1) in our paper [3]. □

4. Proof of Theorem 1

Let $\delta \in (0, 1/4)$ be fixed and set $\eta = 1 + \delta$. Further let N be a large number and let $\nu_0 = \nu_0(N)$ be an integer satisfying

$$\nu_0 \delta > \frac{1}{\varepsilon} \log N.$$

Then, for each $m \in \{0, 1, \dots, \nu_0\}$, consider the interval

$$L_m = [\eta^m N, \eta^{m+1} N].$$

Further consider the set $\mathcal{M}$ of matrices

$$M = \begin{pmatrix} m_{1,1} & \cdots & m_{1,k} \\ m_{2,1} & \cdots & m_{2,k} \\ \vdots & & \vdots \\ m_{r,1} & \cdots & m_{r,k} \end{pmatrix},$$

where $0 \leq m_{1,\ell} < m_{2,\ell} < \cdots < m_{r,\ell} \leq \nu_0(N)$ for $\ell = 1, 2, \dots, k$ and $m_{u_1,v_1} \neq m_{u_2,v_2}$ if $(u_1, v_1) \neq (u_2, v_2)$.

With J as in the statement of Lemma 4 and given $n \in J$, write

$$\tau(n) = M \quad \text{if } \pi_i(n+\ell) \in L_{m_{i,\ell}} \quad (\ell = 1, \ldots, k,\ i = 1, \ldots, r).$$

Let us drop all those integers $n \in J$ which can be neglected in light of of Lemmas 1, 2 or 3. Then, the size of the set of those $n \le x$ thus dropped is $o(x)$ as $x \to \infty$. Hence, for those $n \in J$ which are not dropped, we have that $\tau(n) = M$ holds for one and only one element of M. Let us fix a matrix $M \in \mathcal{M}$. Observe that if $Q_{i,\ell}$ is a prime in the interval $L_{m_{i,\ell}}$, then, according to Lemma 4, the expression $S_J(Q_{i,\ell} \mid i = 1, \ldots, r, \ell = 1, \ldots, k)$ tends to a constant as $N \to \infty$, since in fact it is

$$(1 + o(1)) \frac{x - e^N}{N^{rk} \prod \eta^{\sum m_{i,\ell}}} \prod_{N < \pi < N\eta^{\max m_{i,\ell}}} \left(1 - \frac{\rho^*(\pi)}{\pi}\right),$$

where $\rho^*(\pi)$ is defined by

$$\rho^*(\pi) = \begin{cases} k & \text{if} \quad N \le \pi < N\eta^{m_{r,\sigma(1)}}, \\ k-1 & \text{if} \quad N\eta^{m_{r,\sigma(1)}} \le \pi < N\eta^{m_{r,\sigma(2)}}, \\ \vdots & \qquad \vdots \\ 1 & \text{if} \quad N\eta^{m_{r,\sigma(k-1)}} \le \pi < N\eta^{m_{r,\sigma(k)}}. \end{cases}$$

The size of the collection of those primes $Q_{i,\ell} \in L_{m_{i,\ell}}$ is equal to $\displaystyle\prod_{\substack{i=1,\ldots,r \\ \ell=1,\ldots,k}} \pi(L_{m_{i,\ell}})$.

On the other hand, the size of the collection of those $Q_{i,\ell}$ which also belong to $\wp^{(i)}_{b_{i,\ell}}$ is equal to $\displaystyle\prod_{\substack{i=1,\ldots,r \\ \ell=1,\ldots,k}} \pi(L_{m_{i,\ell}} \cap \wp^{(i)}_{b_{i,\ell}})$.

Now, since

$$\prod_{\substack{i=1,\ldots,r \\ \ell=1,\ldots,k}} \frac{q_i \pi(L_{m_{i,\ell}} \cap \wp^{(i)}_{b_{i,\ell}})}{\pi(L_{m_{i,\ell}})} = (1 + O(\xi(N))), \tag{4}$$

where $\xi(N) \to 0$ as $N \to \infty$, and since (4) holds uniformly for every $M \in \mathcal{M}$, we have thus established that (3) holds. Since we can perform the same argument with the interval $[e^\nu, e^{\nu+1})$ instead of $[e^\nu, x)$, the proof of Theorem 1 is complete.

5. Final remarks

Finally, we may expand on an analogous result.

Given two integers $Q \ge 2$ and $r \ge 1$ such that $(Q, r) = 1$, let

$$\wp_{Q,r} := \{p \in \wp : p \equiv r \pmod{Q}\}.$$

Moreover, let $(\mathcal{T}^{(j)}, q_j)_{j\ge 1}$ be a sequence of regular partitions of the set of primes $\wp_{Q,r}$, with corresponding partitions

$$\wp_0^{(j)}, \wp_1^{(j)}, \ldots, \wp_{q_j-1}^{(j)}, \mathcal{R}^{(j)} \qquad (j = 1, 2, \ldots),$$

with $\#\mathcal{R}^{(j)} < \infty$, that is, such that

$$\wp_0^{(j)} \cup \wp_1^{(j)} \cup \ldots \cup \wp_{q_j-1}^{(j)} \cup \mathcal{R}^{(j)} = \wp_{Q,r} \qquad (j = 1, 2, \ldots)$$

and, for each $j \in \mathbb{N}$,

$$\max_{\substack{\ell=0,1,\ldots,q_j-1\\ \varepsilon\le y/x\le 1}} \left| \frac{q_j \pi([x, x+y] \cap \wp_\ell^{(j)})}{\pi([x, x+y] \cap \wp_{Q,r})} - 1 \right| \to 0 \qquad \text{as } x \to \infty,$$

where $\varepsilon > 0$ is any preassigned small number.

Moreover, let x_N, J_N, λ_N and K_N be defined as in Section 1, and further set

$$S_N := \prod_{\pi \in K_N \cap \wp_{Q,r}} \pi \qquad \text{for each integer } N > e^e.$$

Then, let

$$N \le \pi_1(n) \le \cdots \le \pi_{h(n)}(n) \le N^{\lambda_N}$$

be all the prime divisors $\pi_j(n) \equiv r \pmod Q$ belonging to the interval K_N, that is such that

$$n = \pi_1(n) \cdots \pi_{h(n)}(n)\nu(n), \qquad \text{where } (\nu(n), S_N) = 1.$$

Finally, for each $j \in \mathbb{N}$, consider the integers

$$a_{j,n} := \begin{cases} \ell & \text{if} \quad \pi_j(n) \in \wp_\ell^{(j)} \text{ and } j \le h(n), \\ 0 & \text{if} \quad \pi_j(n) \in \mathcal{R}^{(j)} \text{ or if } j > h(n), \end{cases}$$

the real numbers

$$\beta_j = 0.a_{j,1}a_{j,2}\ldots \qquad (q_j - \text{ary expansion}),$$

and the corresponding sequence of real numbers

$$u_{n,j} = \{\beta_j q_j^n\} \qquad (n = 1, 2, \ldots).$$

With this set up, one can prove that, for each fixed positive integer r, the r-dimensional sequence $(u_{n,1}, u_{n,2}, \ldots, u_{n,r})_{n\ge 1}$ is uniformly distributed modulo $[0, 1)^r$.

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