On the divisors of shifted primes

Jean-Marie DE KONINCK
Departement of mathematics
Université Laval
Quebec G1V 0A6
CANADA
email: jmdk@mat.ulaval.ca

Imre KÁTAI
Computer Algebra Department
Eötvös Loránd University
1117 Budapest
Pázmány Péter Sétány I/C
HUNGARY

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Abstract

Let $\tau(n)$ stand for the number of positive divisors of n. Given an additive function f and a real number $\alpha \in [0,1)$, let $h_n(\alpha) := \frac{1}{\tau(n)} \sum_{\substack{d|n \\ \{f(d)\}<\alpha}} 1$, where $\{y\}$ stands for the fractional part of y, and consider the discrepancy $\Delta(n) := \sup_{0\le\alpha<\beta<1} |h_n(\beta)-h_n(\alpha)-(\beta-\alpha)|$. We show that $\Delta(p+1) \to 0$ for almost all primes p if and only if $\sum_{q\in\wp} \frac{\|mf(q)\|^2}{q} = \infty$ for every positive integer m, where $\|x\|$ stands for the distance between x and its nearest integer and where the sum runs over all primes q.

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1 Introduction and notation

Let $\tau(n)$ stand for the number of positive divisors of n. Given an additive function f and a real number $\alpha \in [0,1)$, let $h_n(\alpha) := \frac{1}{\tau(n)} \sum_{\substack{d|n \\ \{f(d)\}<\alpha}} 1$, where $\{y\}$ stands for the fractional part of y, and consider the discrepancy $\Delta(n) := \sup_{0\le\alpha<\beta<1} |h_n(\beta) - h_n(\alpha) - (\beta-\alpha)|$. It is well known that $h_n(\alpha) \to \alpha$ as $n \to \infty$ uniformly for $\alpha \in [0,1)$ if and only if $\lim_{n\to\infty} \Delta(n) = 0$.

Let $\|x\|$ stand for the distance between x and its nearest integer and let $\wp$ stand for the set of all primes. From here on, the letters p and q will be used exclusively to denote primes. In 1976, the second author [5] proved that $\Delta(n) \to 0$ for almost all n if and only if $\sum_{q\in\wp} \frac{\|mf(q)\|^2}{q} = \infty$ for every positive integer m (see Theorem A below). Observe that there is a small error in the original paper of Kátai [5]: in relation (5), the number 2 should be removed.

Here, we consider the case of shifted primes $p+1$ and show that $\Delta(p+1) \to 0$ for almost all primes p if and only if $\sum_{q\in\wp} \frac{\|mf(q)\|^2}{q} = \infty$ for every positive integer m.

Finally, we examine an interesting outcome in the particular case $f(n) = \log n$.

2 Main result

Theorem 1. *Let f be an additive function and $\alpha \in [0,1)$. Set $h_n(\alpha) := \frac{1}{\tau(n)} \sum_{\substack{d\mid n \\ \{f(d)\}<\alpha}} 1$ and $\Delta(n) := \sup_{0\le\alpha<\beta<1} |h_n(\beta) - h_n(\alpha) - (\beta-\alpha)|$. Then, $\Delta(p+1) \to 0$ for almost all primes p if and only if $\sum_{q\in\wp} \frac{\|mf(q)\|^2}{q} = \infty$ for every positive integer m.*

3 Preliminary results

Let $P(n)$ stand for the largest prime factor of n and $\pi(x)$ for the number of primes not exceeding x.

Lemma 1. *Given $\delta \in (0,1/2)$ and a large number x, set $\wp_{x,\delta} := \{p \le x : P(p+1) \notin [x^\delta, x^{1-\delta}]\}$. Then, for some absolute constant $C_1 > 0$,*

$$\#\wp_{x,\delta} < C_1\, \delta\, \pi(x).$$

Proof. The fact that there exists an absolute constant $c_1 > 0$ such that

$$\#\{p \le x : P(p+1) > x^{1-\delta}\} < c_1\, \delta\, \pi(x)$$

is essentially a direct application of Theorem 3.8 in the book of Halberstam and Richert [2]. Therefore, it remains to prove that there exists an absolute constant $c_2 > 0$ such that

$$\#\{p \le x : P(p+1) < x^\delta\} < c_2\, \delta\, \pi(x). \tag{3.1}$$

To do so, we shall first obtain an upper bound for the sum $T_\delta(x) := \sum_{\substack{p\le x\\ P(p+1)<x^\delta}} \log(p+1)$.

Letting as usual $\pi(x;a,b)$ stand for $\#\{p \le x : p \equiv b \pmod a\}$, then, for some absolute constants $c_3 > 0$, $c_4 > 0$ and $c_5 > 0$, we have that

$$\begin{aligned} T_\delta(x) &= \sum_{\substack{q^k\le x,\ k\ge 1\\ q<x^\delta}} (\log q)\,\pi(x;q^k,1) \\ &\le c_3 \frac{x}{\log x} \sum_{q<x^\delta} (\log q) \left(\frac{1}{q-1} + \frac{1}{q(q-1)} + \cdots \right) + O\left(x \sum_{\substack{\sqrt{x}<q^k<x,\ k\ge 1\\ q<x^\delta}} \frac{1}{q^k} \right) \\ &\le c_4 \frac{x}{\log x} \sum_{q<x^\delta} \frac{\log q}{q} \le c_5 \frac{x}{\log x} \delta \log x = c_5\,\delta\, x. \end{aligned}$$

It follows from this last estimate that, provided $x > x_0(\delta)$, we have

$$\#\{p \in [x/2,x] : P(p+1) < x^\delta\} < \frac{c_5\,\delta\,x}{\log\sqrt{x}} + \sqrt{x} \le \frac{3c_5\,\delta\,x}{\log x}.$$

Replacing successively in the above the value of x by $x/2, x/4, x/8, \ldots$, we obtain that, for some absolute constant $c_6 > 0$,

$$\#\{p \le x : P(p+1) < x^\delta\} = \sum_{1\le j\le \log x/\log 2} \sum_{\substack{\frac{x}{2^j}<p\le\frac{x}{2^{j-1}}\\ P(p+1)<x^\delta}} 1 \le \frac{c_6\,\delta\,x}{\log x},$$

thus proving (3.1) and thereby completing the proof of Lemma 1. □

Now, assume that $0 < \delta < 1/2$ and set

$$\wp_x^* := \{p \in [x/2,x] : x^\delta \le P(p+1) \le x^{1-\delta}\}.$$

Given a prime $p \in \wp_x^*$ with $P(p+1) = q$, then

$$p + 1 = mq \quad \text{for some positive integer } m. \tag{3.2}$$

Let $R_m(x)$ be the number of solutions of (3.2) with $p \in \wp_x^*$. Then, if we let ϕ stand for the Euler totient function, we have the following result.

Lemma 2. *There exists an absolute constant $C_2 > 0$ such that*

$$R_m(x) < C_2 \frac{x}{\log^2(x/m)\ \phi(m)} < C_2 \frac{x}{\delta^2\,(\log x)^2\,\phi(m)}.$$

Proof. For a proof, see Theorem 4.6 in the book of Prachar [7]. □

Lemma 3. *Given any real number $\kappa \in (0,1)$, there exists an absolute constant $C_3 > 0$ such that, for all integers $u \geq 1$,*

$$S_u := \sum_{\substack{u \leq m \leq 2u \\ \phi(m)/m < \kappa}} \frac{1}{\phi(m)} < C_3\, \kappa. \tag{3.3}$$

Moreover, there exists an absolute constant $C_4 > 0$ such that

$$\sum_{\substack{m \leq x \\ \phi(m)/m < \kappa}} \frac{1}{\phi(m)} < C_4\, \kappa\, \log x. \tag{3.4}$$

Proof. Clearly,

$$S_u \leq \sum_{u \leq m \leq 2u} \frac{\kappa\, m}{\phi(m)} \cdot \frac{m}{u} \cdot \frac{1}{\phi(m)} = \frac{\kappa}{u} \sum_{u \leq m \leq 2u} \left(\frac{m}{\phi(m)} \right)^2. \tag{3.5}$$

Since one can easily establish that there exists a computable constant $c_7 > 0$ such that

$$\sum_{m \leq x} \left(\frac{m}{\phi(m)} \right)^2 = (1 + o(1)) c_7\, x \qquad (x \to \infty),$$

it follows from (3.5) that, for some absolute constant c_8, we have

$$S_u \leq \frac{\kappa}{u} \cdot c_8\, u \qquad (u \geq 1),$$

thus proving (3.3). Estimate (3.4) is a direct consequence of (3.3). □

Given real numbers $z_1, \ldots, z_M \in [0,1)$, let

$$D(z_1, \ldots, z_M) := \frac{1}{M} \sup_{0 \leq \alpha < \beta < 1} \left| \sum_{z_\nu \in [\alpha, \beta)} 1 - M(\beta - \alpha) \right|$$

stand for the discrepancy of the sequence of numbers $z_1, \ldots, z_M$. We have the following result.

Lemma 4. *Let $x_1, \ldots, x_M \in [0,1)$ and, for $\ell = 1, \ldots, M$, let $x_{M+\ell} = x_\ell + a$, where $a \in [0,1)$. Then*

$$D(x_1, \ldots, x_{2M}) \leq D(x_1, \ldots, x_M).$$

Proof. The proof follows easily from the definition of the discrepancy and will therefore be omitted. □

Lemma 5. *Let $x_1, \ldots, x_M \in [0,1)$ and let m be an arbitrary integer. Then,*

$$\frac{1}{M} \left| \sum_{j=1}^{M} e(m x_j) \right| \leq 2\pi m D(x_1, \ldots, x_M).$$

Proof. Even though this is a well known inequality, let us only mention that it can be obtained by the relation

$$\frac{1}{M}\sum_{j=1}^{M} e(mx_j) = -\int_0^1 \left(\left(\frac{1}{M}\sum_{x_\nu<u} 1\right) - u\right) 2\pi i m\, e(mu)\, du$$

and partial integration. $\square$

Theorem A. (Kátai [5]) *Let f be an additive function and $\alpha \in [0,1)$. Further set $h_n(\alpha) := \frac{1}{\tau(n)} \sum_{\substack{d\mid n \\ \{f(d)\}<\alpha}} 1$ and $\Delta(n) := \sup_{0\le\alpha<\beta<1} |h_n(\beta) - h_n(\alpha) - (\beta-\alpha)|$. Then, $\Delta(n) \to 0$ for almost all n if and only if $\sum_{q\in\wp} \frac{\|mf(q)\|^2}{q} = \infty$ for every positive integer m.*

4 Proof of the main result

Let κ, δ and ε be arbitrarily small positive numbers. We shall find an upper bound for the number of primes $p \in [x/2, x]$ for which $\Delta(p+1) \ge \varepsilon$.

First of all, we know from Lemma 1 that

$$\#\{p \in [x/2,x] : p \notin \wp_x^*\} < C_1\,\delta\,\pi(x). \tag{4.1}$$

On the other hand, it is clear that

$$\#\{p \in [x/2,x] : P^2(p+1) \mid p+1\} < c_9\,\delta\,\pi(x) \tag{4.2}$$

for some constant $c_9 > 0$. Hence, we are left to consider the contribution of the other primes.

It follows from Lemma 4 that if (3.2) holds, then $\Delta(p+1) > \varepsilon$ only if $\Delta(m) > \varepsilon$. Now, according to Lemma 2, we may write that

$$\#\{p \in [x/2,x] : \Delta(p+1) > \varepsilon\} \le C_2 \frac{x}{\log^2 x} \sum_{\substack{\frac{x^\delta}{2} \le m < x^{1-\delta} \\ \Delta(m)>\varepsilon}} \frac{1}{\phi(m)} = C_2 \frac{x}{\log^2 x} S(x), \tag{4.3}$$

say. Let us write $S(x) = S_1(x) + S_2(x)$, where the sum in $S_1(x)$ runs over those m for which $\phi(m)/m \ge \kappa$, whereas in $S_2(x)$ it runs over those m for which $\phi(m)/m < \kappa$. As an easy consequence of Theorem A, we have that

$$S_1(x) = o(\log x) \qquad (x \to \infty). \tag{4.4}$$

On the other hand, it follows from inequality (3.4) in Lemma 3 that

$$S_2(x) \le C_4 \kappa \log x. \tag{4.5}$$

Therefore, gathering (4.1), (4.2), (4.4) and (4.5), it follows from (4.3) that, for some absolute constant $c_{10} > 0$,

$$\#\{p \in [x/2, x] : \Delta(p+1) > \varepsilon\} \le c_{10}\delta\pi(x) + C_4\kappa\pi(x) + o(\pi(x)) \qquad (x \to \infty) \tag{4.6}$$

Applying this very same inequality with x replaced by $x/2^j$ as $j = 0, 1, \ldots, \lfloor \log x/\log 2 \rfloor$, we easily obtain that

$$\frac{1}{\pi(x)}\#\{p \le x : \Delta(p+1) > \varepsilon\} \le c_{10}\delta + C_4\kappa + o(1) \qquad (x \to \infty),$$

from which it follows that

$$\limsup_{x\to\infty} \frac{1}{\pi(x)}\#\{p \le x : \Delta(p+1) > \varepsilon\} \le c_{10}\delta + C_4\kappa.$$

Since κ and δ can be chosen arbitrarily small, this completes the proof of the sufficient part of Theorem 1.

We will now show the necessity of the divergence of the series $\displaystyle\sum_{q\in\wp} \frac{\|mf(q)\|^2}{q}$. To do so, let us assume the contrary, that is, that there exists some positive integer m such that

$$\sum_{q\in\wp} \frac{\|mf(q)\|^2}{q} < \infty. \tag{4.7}$$

Now, consider the multiplicative function g_m defined by

$$g_m(n) = \frac{1}{\tau(n)} \left| \prod_{p^a\|n} \left(1 + e^{2\pi i m f(p)} + e^{2\pi i m f(p^2)} + \cdots + e^{2\pi i m f(p^a)}\right) \right|.$$

Observe that $0 \le g_m(n) \le 1$ for all integers $n \ge 1$ and that, at primes p,

$$g_m(p) = \frac{|1 + e^{2\pi i m f(p)}|}{2},$$

so that

$$|g_m(p)|^2 = \frac{2 + 2\cos 2\pi m f(p)}{4} = \cos^2 \pi m f(p),$$

which implies that

$$g_m(p) = |\cos \pi m f(p)|.$$

From this it follows that there exists an absolute constant $c_6 > 0$ such that, for all primes p, $0 \le 1 - g_m(p) \le 1 - g_m^2(p) = \sin^2 \pi m f(p) \le c_6\|mf(p)\|^2$. Hence, (4.7) implies that

$$\sum_{p\in\wp} \frac{1 - g_m(p)}{p} < \infty. \tag{4.8}$$

On the other hand, recall that the second author [6] has proved the analogue of the famous Delange result [1] for shifted primes, namely the following.

Theorem B. *Let $g(n)$ be a complex-valued multiplicative function such that $|g(n)| \leq 1$ for all $n \in \mathbb{N}$ and such that the series*

$$\sum_{p\in\wp} \frac{g(p)-1}{p}$$

converges. Let $N(g)$ be the product

$$N(g) = \prod_{p\in\wp} \left(1 - \frac{1}{p-1} + \sum_{j=1}^{\infty} \frac{g(p^j)}{p^j}\right).$$

Then,

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p\leq x} g(p+1) = N(g).$$

In light of (4.8), we may apply Theorem B to the function g_m and get that

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p\leq x} g_m(p+1) = N(g_m). \tag{4.9}$$

Now we may assume that $N(g_m) \neq 0$, except in the case where $g_m(2^\ell) = 0$ for $\ell = 1, 2, \ldots$. But if $g_m(2) = 0$, then it is easily seen that $g_{2m}(2) \neq 0$, which implies that $N(g_{2m}) \neq 0$. This is why we can make the assumption that $N(g_m) \neq 0$.

On the other hand, since $0 \leq g_m(n) \leq 1$ for all integers $n \geq 1$, it follows from (4.9) that for a suitable constant $\lambda > 0$ there exists a real number $x_0 > 0$ such that

$$\frac{1}{\pi(x)} \#\{p \leq x : g_m(p+1) > \lambda\} > \lambda$$

for all $x > x_0$. Therefore, since $g_m(n) < c_7\Delta(n)$ for a suitable constant $c_7 > 0$ (which follows from Lemma 5), we obtain that there exists a constant $\lambda_1 > 0$ such that

$$\frac{1}{\pi(x)} \#\{p \leq x : \Delta(p+1) > \lambda_1\} > \lambda \qquad (x > x_0),$$

thereby contradicting our assumption that $\Delta(p+1) \to 0$ for almost all primes p, thus completing the proof of Theorem 1.

5 The special case $f(n) = \log n$

Consider the functions

$$h_n^*(\alpha) := \frac{1}{\tau(n)} \sum_{\substack{d\mid n \\ \{\log d\}<\alpha}} 1 \quad \text{and} \quad \Delta^*(n) := \sup_{0\leq\alpha<\beta<1} |h_n^*(\beta) - h_n^*(\alpha) - (\beta-\alpha)|.$$

Hall [3] proved that, given any positive number $\lambda < 1/2$,

$$\Delta^*(n) \le \frac{1}{\tau(n)^\lambda} \qquad \text{for almost all } n. \tag{5.1}$$

The second author [4] improved Hall's result by showing the following.

Theorem B. *Inequality (5.1) holds for any positive number* $\lambda < \dfrac{\log \pi}{\log 2} - 1 \approx 0.651.$

Interestingly, we can prove that the analogue of Theorem B also holds for shifted primes. Indeed, using Theorem B and Lemma 4, similarly as Theorem 1 was deduced from Theorem A, one can easily show the following.

Theorem 2. *Given any positive number* $\lambda < \dfrac{\log \pi}{\log 2} - 1,$

$$\Delta^*(p+1) \le \frac{1}{\tau(p+1)^\lambda} \qquad \textit{for almost all primes } p.$$

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