On some consequences of recently proved conjectures

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Abstract

We provide some consequences of recently proved conjectures of Kátai regarding the values taken by arithmetic functions at consecutive integers.

1 Introduction

We provide an update on some consequences of some old conjectures formulated by Kátai, many of which have recently been proved by O. Klurman [2] and others by O. Klurman and A.P. Mangerel [3], [4].

2 Notation

Let $T := \{z \in \mathbb{C} : |z| = 1\}$ stand for the set of the points on the unit circle and let $\mathcal{M}_1$ stand for the set of multiplicative functions f such that $|f(n)| = 1$ for all positive integers n. Given $f \in \mathcal{M}_1$, consider the arithmetic function $\delta(n) = \delta_f(n) := f(n+1)\overline{f(n)}$. Given $x \in \mathbb{R}$, we set $\|x\| = \min_{n\in\mathbb{Z}} |x-n|$. As is common, we let $\mathcal{A}$ stand for the set of additive functions. Finally, given $h \in \mathcal{A}$, we set $\Delta h(n) := h(n+1)-h(n)$.

3 Some old conjectures of Kátai and their recent proofs

We first state some conjectures.

Conjecture 1. *(Kátai [1]) Let $f \in \mathcal{M}_1$ and consider its corresponding function $\delta = \delta_f$. If $\lim_{x\to\infty} \frac{1}{x}\sum_{n\le x} |\delta(n) - 1| = 0$, then $f(n) = n^{it}$ for some $t \in \mathbb{R}$.*

Conjecture 2. *(Kátai [1]) Let $f \in \mathcal{M}_1$ and consider its corresponding function $\delta = \delta_f$. If $\lim_{x\to\infty} \frac{1}{\log x}\sum_{n\le x} \frac{1}{n}|\delta(n) - 1| = 0$, then $f(n) = n^{it}$ for some $t \in \mathbb{R}$.*

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Conjecture 1 was proved by Klurman [2], whereas Conjecture 2 can be proved in a similar manner.

Conjecture 3. *Let $f \in \mathcal{M}_1$ and consider its corresponding function $\delta = \delta_f$. Assume that there exists some $w \in T$ and some $\varepsilon > 0$ for which $|\delta(n)w - 1| \geq \varepsilon$ for all $n \in \mathbb{N}$. Then $f(n) = g(n)n^{it}$ for some $t \in \mathbb{R}$, where $g(n)^k = 1$ for all $n \in \mathbb{N}$ and some $k \in \mathbb{N}$.*

Conjecture 3 was proved by Klurman and Mangerel [3].

Conjecture 4. *Let $f \in \mathcal{M}_1$ and consider its corresponding function $\delta = \delta_f$. Assume that there exist some $w \in T$ and some $\varepsilon > 0$ for which $\lim_{x\to\infty} \frac{1}{x} \sum_{\substack{n \leq x \\ |\delta(n)w-1|<\varepsilon}} 1 = 0$. Then $f(n) = g(n)n^{it}$ for some $t \in \mathbb{R}$, where $g(n)^k = 1$ for all $n \in \mathbb{N}$ and some $k \in \mathbb{N}$.*

Klurman and Mangerel claim (private communication) that they can prove Conjecture 4.

The above statements can be reformulated for additive functions through the following theorem.

Theorem A. *Let $h \in \mathcal{A}$ and assume that either*

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\leq x} \|\Delta h(n)\| = 0 \tag{3.1}$$

or

$$\lim_{x\to\infty} \frac{1}{\log x} \sum_{n\leq x} \frac{1}{n}\|\Delta h(n)\| = 0 \tag{3.2}$$

holds. Then there exists some $c \in \mathbb{R}$ such that $h(n) \equiv c \log n \pmod 1$ for all $n \in \mathbb{N}$.

Proof. This result is an obvious consequence of Conjectures 1 and 2. Indeed, setting $f(n) := e^{2\pi i h(n)}$, we have that $f \in \mathcal{M}_1$ and $\delta_f(n) - 1 \asymp \|\Delta h(n)\|$, implying that (3.1) is equivalent to the condition of Conjecture 1 whereas (3.2) is equivalent to the condition of Conjecture 2. □

We state our last conjecture.

Conjecture 5. *Let $h \in \mathcal{A}$, $\xi \in [0, 1)$ and $\varepsilon > 0$. Let $n_1 < n_2 < \cdots$ be a sequence of positive integers of positive density. Assume that $\lim_{x\to\infty} \frac{1}{x} \sum_{\substack{n_j \leq x \\ \|\Delta h(n_j)-\xi\|<\varepsilon}} 1 = 0$. Then, there exists $k \in \mathbb{N}$ such that $k\xi \in \mathbb{Z}$.*

One can easily see that Conjecture 5 is actually a reformulation of Conjecture 4.

4 Main result

Theorem 1. *Let $h \in \mathcal{A}$ and $\tau \in \mathbb{R} \setminus \mathbb{Q}$. Assume that*

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} \|\Delta h(n)\| = 0 \quad \text{and} \quad \lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} \|\tau\Delta h(n)\| = 0 \tag{4.1}$$

or

$$\lim_{x\to\infty} \frac{1}{\log x} \sum_{n\le x} \frac{1}{n}\|\Delta h(n)\| = 0 \quad \text{and} \quad \lim_{x\to\infty} \frac{1}{\log x} \sum_{n\le x} \frac{1}{n}\|\tau\Delta h(n)\| = 0. \tag{4.2}$$

Then, there exists $c \in \mathbb{R}$ such that $h(n) = c\log n$ for all $n \in \mathbb{N}$.

5 Proof of Theorem 1

It follows from Theorem A that there exist $c_1, c_2 \in \mathbb{R}$ and integer valued additive functions $u(n)$ and $v(n)$ such that

$$h(n) = c_1 \log n + u(n) \quad \text{and} \quad \tau h(n) = c_2 \log n + v(n) \qquad \text{for all } n \in \mathbb{N}.$$

Since $\tau h(n) = c_1\tau \log n + \tau u(n)$, we have that, for all $n \in \mathbb{N}$,

$$D \log n = v(n) - \tau u(n), \quad \text{where } D = c_1\tau - c_2. \tag{5.1}$$

If $D = 0$, then $v(n) = \tau u(n)$ for every $n \in \mathbb{N}$, implying that $u(n) = v(n) = 0$ for each integer $n \ge 1$, thus completing the proof of Theorem 1 in the case $D = 0$.

From here on, we can therefore assume that $D \ne 0$. From (5.1), we have that

$$\log n = \frac{v(n)}{D} - \frac{\tau u(n)}{D},$$

so that, for arbitrary positive integers p and q, we have

$$\begin{aligned} Du(q) \log p &= u(q)v(p) - \tau u(p)u(q), \\ Du(p) \log q &= u(p)v(q) - \tau u(p)u(q), \end{aligned}$$

from which we obtain that

$$D \log \left(\frac{p^{u(q)}}{q^{u(p)}} \right) = u(q)v(p) - u(p)u(q) =: L(p, q). \tag{5.2}$$

So, let us first assume that there exist distinct primes p, q and co-prime prime powers P, Q for which $L(p, q) \ne 0$ and $L(P, Q) \ne 0$. Let A, B be such that

$$\frac{A}{B} = \frac{L(p, q)}{L(P, Q)}.$$

It follows that

$$\log\left(\frac{p^{u(q)}}{q^{u(p)}}\right)^B = \log\left(\frac{P^{u(Q)}}{Q^{u(P)}}\right)^A.$$

But, in light of the uniqueness of prime factorisation, this can hold only if $u(P) = u(Q) = 0$ and $u(p) = u(q) = 0$, which contradicts our condition $D \neq 0$.

Hence, it remains to consider the case where there exist at most three primes $\pi_1 < \pi_2 < \pi_3$ for which $u(\pi_j^{e_j}) \neq 0$ for some $e_j \in \mathbb{N}$ for $j = 1, 2, 3$. Consider the integers $n = \pi_1^{e_1}\nu$, where ν runs over those integers such that $(\nu, \pi_1\pi_2\pi_3) = 1$ and $(n+1, \pi_1\pi_2\pi_3) = 1$. In this case, we have

$$\Delta h(n) = h(n+1) - h(n) = c_1 \log\left(1 + \frac{1}{\pi_1^{e_1}\nu}\right) - u(\pi_1^{e_1}),$$

from which it follows that

$$\lim_{n=\pi_1^{e_1}\nu\to\infty} \Delta h(n) = -u(\pi_1^{e_1}),$$

which in turn implies that

$$\lim_{n=\pi_1^{e_1}\nu\to\infty} \Delta\tau h(n) = \lim_{n=\pi_1^{e_1}\nu\to\infty} (v(n+1) - v(n)) = -\tau u(\pi_1^{e_1}).$$

Now, since $v(n+1) - v(n) \in \mathbb{Z}$ and $u(\pi_1^{e_1}) \neq 0$, we have established that, for a suitable $\delta > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\|\tau u(\pi_1^{e_1}) + (v(n+1) - v(n))\| > \delta > 0 \quad \text{for all } n \geq n_0,$$

again a contradiction. This completes the proof of Theorem 1 in this particular case.

It remains to consider the case where there exist only two primes $\pi_1 < \pi_2$ for which for suitable $e_1, e_2 \in \mathbb{N}$ we have $u(\pi_1^{e_1}) \neq 0$ and $u(\pi_2^{e_2}) \neq 0$. Similarly as above, let us consider those integers $n = \pi_1^{e_1}\nu$, where $(\nu, \pi_1\pi_2) = 1$ and $(n+1, \pi_1\pi_2) = 1$. We may then argue as above and conclude that this situation also leads to a contradiction.

Therefore, it only remains to consider the case where $u(P) = 0$ for some prime power $P = p^\ell$ and $u(m) = 0$ for every m coprime to P. Let us first assume that there exist positive integers Q_1 and Q_2 such that $(Q_1, Q_2) = 1$ and $(p, Q_1Q_2) = 1$ for which $v(Q_1) \neq 0$ and $v(Q_2) \neq 0$. We then have

$$\begin{aligned} D \log Q_j &= v(Q_j) \quad \text{for } j = 1, 2, \\ \frac{\log Q_1}{\log Q_2} &= \frac{v(Q_1)}{v(Q_2)}, \end{aligned}$$

which implies that $Q_1^{v(Q_2)} = Q_2^{v(Q_1)}$, which is clearly impossible. If $u(n) = 0$ for all $n \in \mathbb{N}$ or if $v(n) = 0$ for all $n \in \mathbb{N}$, we are done.

So, consider those integers $n = p^\ell\nu$, where ν runs over those positive integers satisfying $(\nu, p) = 1$. In this case, we have $u(n+1) = 0$ and $u(n) = u(p^\ell)$. Consequently,

$$\lim_{n=p^\ell\nu\to\infty} \Delta h(n) = -u(p^\ell) \quad \text{and} \quad \lim_{n=p^\ell\nu\to\infty} \Delta\tau h(n) = \lim_{n=p^\ell\nu\to\infty} (v(n+1) - v(n)) = -\tau u(p^\ell),$$

which is also impossible, thus completing the proof of Theorem 1.

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