Further generalisations of a classical theorem of Daboussi

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Dedicated to the memory of Professor Hedi Daboussi

Abstract

According to a well known theorem of Hédi Daboussi, if $\mathcal{M}_1$ stands for the set of those complex valued multiplicative functions f such $|f(n)| \leq 1$ for all positive integers n and if α is an arbitrary irrational number, then

$$\lim_{x\to\infty} \sup_{f\in\mathcal{M}_1} \left| \frac{1}{x} \sum_{n\leq x} f(n) \exp\{2\pi i\alpha n\} \right| = 0.$$

Given an infinite set A of positive integers, let $A(x)$ stand for its counting function, and let α be an arbitrary irrational number. We examine various sets A along with an appropriate weight function $w(n)$ for which one can prove that $\lim_{x\to\infty} \frac{1}{A(x)} \sum_{\substack{n\leq x\\ n\in A}} w(n) \exp\{2\pi i\alpha n\} = 0$.

2010 *Mathematics Subject Classification*: Primary 11L07; Secondary 11N37.
Key words and phrases: exponential sums, multiplicative functions.

1 Introduction

Let $\mathcal{M}$ stand for the set of complex valued multiplicative functions and let $\mathcal{M}_1$ be the subset of those functions $f \in \mathcal{M}$ for which $|f(n)| \leq 1$ for all $n \in \mathbb{N}$. Further set $U := \{z \in \mathbb{C} : |z| \leq 1\}$. For short, we will write $e(y)$ for $e^{2\pi i y}$. Finally, let $\wp$ represent the set of all primes.

Decades ago, Daboussi [5], [6] proved that if α is an arbitrary irrational number, then

$$\lim_{x\to\infty} \sup_{f\in\mathcal{M}_1} \left| \frac{1}{x} \sum_{n\leq x} f(n)e(n\alpha) \right| = 0. \tag{1.1}$$

Later, in 1990, Daboussi [4] proved the following result.

Theorem A. (Daboussi) *Let f be a completely multiplicative function and assume that there exists a real number $\lambda \in (0,2)$ such that $|f(p)| = \lambda$ for all $p \in \wp$. Then, given any irrational number α,*

$$\left| \sum_{n\leq x} f(n)e(n\alpha) \right| = o\left(\sum_{n\leq x} |f(n)| \right) \qquad (x\to\infty).$$

Then, in 1995, Goubin [10] proved the following.

Theorem B. (Goubin) *Let f be a multiplicative function and assume that there exists a real number $\lambda > 0$ such that $|f(p)| \leq \lambda$ for all $p \in \wp$. Assume also that $\sum_p \sum_{\nu=2}^{\infty} \frac{|f(p^\nu)|}{p^\nu} (\log p^\nu)^{\max(1-\lambda,0)} < \infty$. Then, given any irrational number α,*

$$\sum_{n\leq x} f(n)e(n\alpha) = o\left(x(\log x)^{\lambda-1}\right) \qquad (x\to\infty).$$

On the other hand, in 1986, the second author [15] gave a proof of (1.1) using the Turán-Kubilius inequality. More precisely, he proved the following.

Theorem C. (Kátai) *Let $\wp_1 \subset \wp$ be a set of primes satisfying $\sum_{p\in\wp_1} \frac{1}{p} = \infty$. Let $\mathcal{B}$ be the set of those functions $f : \mathbb{N} \to U$ for which $f(pm) = f(p)f(m)$ whenever $p \in \wp_1$ and $(p,m) = 1$. Moreover, let $a : \mathbb{N} \to U$ be a function for which*

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\leq x} a(p_1 n)\overline{a(p_2 n)} = 0 \tag{1.2}$$

for every distinct primes $p_1, p_2 \in \wp_1$. *Then,*

$$\lim_{x\to\infty} \sup_{f\in\mathcal{B}} \left| \frac{1}{x} \sum_{n\le x} f(n)a(n) \right| = 0 \tag{1.3}$$

Several generalisations of Daboussi's theorem are now part of the mathematical literature. See for instance Bassily, De Koninck and Kátai [1], Bassily and Kátai [2], De Koninck and Kátai [7], [8], [9], Indlekofer and Kátai [11], [12], [13], [14], Kátai [15], [16], [17], [18], [19], [20].

Remark 1.1. It is interesting to observe that in order to obtain (1.1) and (1.3), the multiplicative character of f was only "partially" used.

Some applications of Theorem C are given in [1] and [2].

Let us now consider a set of primes $\wp_1$ for which there exists a real number $\tau \in (0, 1]$ such that

$$\sum_{\substack{p\le x\\ p\in\wp_1}} \log p = \tau\, x + O\left(\frac{x}{\log^2 x}\right), \tag{1.4}$$

and let $\mathcal{N}(\wp_1)$ denote the semigroup generated by $\wp_1$. Also, let $N_1(x) := \#\{n \le x : n \in \mathcal{N}(\wp_1)\}$. The Turán-Kubilius inequality remains true for the set $\mathcal{N}(\wp_1)$ as well. Indeed, let $\wp_1^*$ be any subset of $\wp_1$ for which $\sum_{p\in\wp_1^*} 1/p = \infty$ and set

$$A_x := \sum_{\substack{p\le \log x\\ p\in\wp_1^*}} \frac{1}{p} \qquad \text{and} \qquad \omega_x(n) := \sum_{\substack{p\mid n\\ p\le\log x,\ p\in\wp_1^*}} 1.$$

Then, it is easy to show that there exists a positive constant C such that

$$\frac{1}{N_1(x)} \sum_{\substack{n\le x\\ n\in\mathcal{N}(\wp_1)}} (\omega_x(n) - A_x)^2 \le C\, A_x.$$

On the other hand, one can deduce the following analogue of Theorem C.

Theorem D. *Let* $\wp_1$ *and* $\wp_1^*$ *be as above and let* $\mathcal{B}$ *be the set of those functions* $f : \mathcal{N}(\wp_1) \to U$ *for which* $f(pm) = f(p)f(m)$ *whenever* $p \in \wp_1^*$ *and* $(p, m) = 1$. *Moreover, let* $a : \mathcal{N}(\wp_1) \to U$ *be a function satisfying*

$$\lim_{x\to\infty} \frac{1}{N_1(x)} \sum_{\substack{m\le x\\ m\in\mathcal{N}(\wp_1)}} a(p_1 m)\overline{a(p_2 m)} = 0 \tag{1.5}$$

for every distinct primes $p_1, p_2 \in \wp_1^*$. *Then,*

$$\lim_{x\to\infty} \sup_{f\in\mathcal{B}} \left| \frac{1}{N_1(x)} \sum_{\substack{n\le x\\ n\in\mathcal{N}(\wp_1)}} f(n)a(n) \right| = 0.$$

Remark 1.2. It is well known that the function $a(n) := e(\alpha n)$ satisfies (1.2) for every irrational number α. On the other hand, (1.2) holds as well when $a(n) = e(P(n))$, where $P(n) = a_0 + a_1 n + \cdots + a_k n^k$ and at least one of the coefficients $a_1, \ldots, a_k$ is an irrational number. In fact, there are many other functions for which (1.2) holds.

Remark 1.3. It is not known whether the function $a(n) := e(\alpha n)$ satisfies (1.5) for every irrational number α. We will however formulate the following conjecture.

Conjecture. *Let* $\wp_1$ *be a set of primes satisfying (1.4) and let* $\mathcal{N}(\wp_1)$ *be the semigroup generated by* $\wp_1$. *Then, given any irrational number* α,

$$\lim_{x\to\infty} \frac{1}{N_1(x)} \sum_{\substack{n\le x\\ n\in\mathcal{N}(\wp_1)}} e(\alpha n) = 0. \tag{1.6}$$

Observe that (1.6) does hold for almost all irrational numbers α and in fact it is known that, for any given $\varepsilon > 0$,

$$\sum_{\substack{n\le x\\ n\in\mathcal{N}(\wp_1)}} e(\alpha n) = O\left(x^{\frac{1}{2}+\varepsilon}\right).$$

This can be deduced from a 1966 theorem of Carleson [3] and a 2002 result of Murty and Sankaranarayanan [21].

In any event, we are unable to prove the above conjecture even in the case $\wp_1 = \{p \in \wp : p \equiv 1 \pmod 4\}$. However, we can manage by inserting a "weight" function inside the sum appearing in (1.6). For instance, we can proceed in the following manner. Let $\wp_3 = \{p \in \wp : p \equiv 3 \pmod 4\}$. It is clear that if $m = A^2 + 4B^2$, where $(A, 2) = 1$ and $((A, B), \mathcal{N}(\wp_3)) = 1$, then $m \in \mathcal{N}(\wp_1)$. Then, consider the two functions

$$r(n) := \#\{n = A^2 + 4B^2 : (A, 2) = 1,\ (A, B) \in \mathcal{N}(\wp_1)\}$$

and

$$\kappa(n) := \begin{cases} 1 & \text{if} \quad n = A^2 \text{ with } A \in \mathcal{N}(\wp_1), \\ 0 & \qquad\ \ \text{otherwise.} \end{cases}$$

Then, let $r_0(n) = r(n) + \kappa(n)$. Observe that $r_0(n) > 0$ if and only if $n \in \mathcal{N}(\wp_1)$ and moreover that $r_0(n) \le \tau(n)$, where $\tau(n)$ stands for the number of positive divisors of n. Finally, observe that it is clear that if $n \notin \mathcal{N}(\wp_1)$, then $r_0(n) = 0$. We can then prove the following.

Lemma 1.4. *For every irrational number α, we have*

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} r_0(n) e(\alpha n) = 0.$$

Proof. Since $\sum_{n\le x} \kappa(n) e(\alpha n) = O(\sqrt{x})$, it is sufficient to show that

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} r(n) e(\alpha n) = 0. \tag{1.7}$$

To do so, first observe that

$$\begin{aligned} S(x|\alpha) &:= \sum_{\substack{A^2+4B^2\le x \\ (A,B)\in\mathcal{N}(\wp_1) \\ (A,2)=1}} e(\alpha(A^2+4B^2)) \\ &= \sum_{\delta\in\mathcal{N}(\wp_3)} \mu(\delta) \sum_{\substack{A_1^2+4B_1^2\le x/\delta^2 \\ (A_1,2)=1}} e(\alpha\delta^2(A_1^2+4B_1^2)) \end{aligned}$$

In light of the above representation, it is clear that in order to prove (1.7), it is sufficient to show that

$$\lim_{x\to\infty} \frac{1}{x} \sum_{\substack{U^2+4V^2\le x \\ (U,2)=1}} e(\alpha(U^2+4V^2)) = 0.$$

But it is clear that, for every irrational number γ,

$$\begin{aligned} R(x|\gamma) &:= \sum_{\substack{U^2+4V^2\le x \\ (U,2)=1}} e(\gamma(U^2+4V^2)) \\ &= \sum_{\substack{U\le\sqrt{x/2} \\ (U,2)=1}} \sum_{v\le(x-U^2)/4} e(\gamma(U^2+4V^2)) \\ &\qquad + \sum_{2V\le\sqrt{x/2}} \sum_{\substack{U\le(x-4V^2)/4 \\ (U,2)=1}} e(\gamma(U^2+4V^2)) \\ &\qquad - \sum_{4V^2\le x/2} e(4V^2\gamma) \sum_{\substack{U^2\le x/2 \\ (U,2)=1}} e(U^2\gamma) \\ &= o(x) \qquad (x\to\infty). \end{aligned} \tag{1.8}$$

Observe that the functions $S(x|\alpha)$ and $R(x|\gamma)$ are connected through the relation

$$S(x|\alpha) = \sum_{\delta \in \mathcal{N}(\wp_3)} \mu(\delta) R\left(\frac{x}{\delta^2}\Big|\alpha\delta^2\right).$$

Therefore, given an arbitrarily large number (but fixed) H, we have

$$\begin{aligned} |S(x|\alpha)| &\leq \left| \sum_{\substack{\delta \in \mathcal{N}(\wp_3) \\ \delta \leq H}} \mu(\delta) R\left(\frac{x}{\delta^2}\Big|\alpha\delta^2\right) \right| + \sum_{\substack{\delta \in \mathcal{N}(\wp_3) \\ \delta > H}} \left| R\left(\frac{x}{\delta^2}\Big|\alpha\delta^2\right) \right| \\ &= o(H) + O\left(\sum_{\delta > H} \frac{x}{\delta^2} \right) \\ &= o(H) + O(x/H), \end{aligned}$$

where we used (1.8) to bound the first of the two sums. Thus, there exists an absolute constant $c > 0$ such that

$$\limsup_{x \to \infty} \frac{S(x|\alpha)}{x} \leq \frac{c}{H},$$

so that

$$\lim_{x \to \infty} \frac{S(x|\alpha)}{x} = 0,$$

thereby completing the proof of Lemma 1.4. □

Lemma 1.5. *Assume that m is not a perfect square, and let p be a prime such that $(m,p) = 1$ and $mp \in \mathcal{N}(\wp_1)$. Then, $r(pm) = 2r(m)$.*

Proof. Let $p = a^2 + 4b^2$ and $m = U^2 + 4V^2$. then,

$$\begin{aligned} mp &= (aU + 4bV)^2 + 4(aV + 2bU)^2 \\ &= (aU - 4bV)^2 + 4(aV + 2bU)^2. \end{aligned}$$

Since p is a prime, we have that $b \neq 0$. If $V \neq 0$, then $(aU + 4bV| \neq |aU - 4bV|$, and therefore,

$$U_1 = |aU + 4bV|, \quad U_2 = |aU - 4bV|, \quad K = |aV + 2bU|$$

allow for the two different representations $mp = U_1^2 + 4K^2$ and $mp = U_2^2 + 4K^2$, thus completing the proof of Lemma 1.5. □

2 Main results

Theorem 2.1. *Let $\beta(n)$ stand for the sum of the binary digits of n. Let also α be an irrational number. Fix two positive integers $p < q$ and consider the function $s(n) := \beta(pn) - \beta(qn)$. Then,*

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} e(\alpha s(n)) = 0. \tag{2.1}$$

Theorem 2.2. *Let $\wp_1 = \{p \in \wp : p \equiv 1 \pmod 4\}$ and choose a set of primes $\wp^* \subseteq \wp_1$ such that $\sum_{p\in\wp^*} 1/p = \infty$. Consider the set $\mathcal{D}$ of those functions $f : \mathcal{N}(\wp_1) \to U$ such that $f(pm) = f(p)f(m)$ for all $p \in \wp^*$ and $m \in \mathcal{N}(\wp_1)$ satisfying $(m,p) = 1$. Then,*

$$\lim_{x\to\infty} \sup_{f\in\mathcal{D}} \left| \frac{1}{x} \sum_{\substack{n\le x\\ n\in\mathcal{N}(\wp_1)}} f(n)r(n)e(n\alpha) \right| = 0.$$

3 Proof of Theorem 2.1

Let k be the smallest positive integer such that $p < q < 2^k$. Let $A = \{0,1\}$ and, for each $r \in \mathbb{N}$, we introduce the set of words of length r

$$A^r = \{\epsilon_1 \dots \epsilon_r : \epsilon_\nu \in A, \nu = 1, \dots, r\}.$$

At times, to indicate that a word γ is of length r, we will write $\lambda(\gamma) = r$. Further define $q_1 := \frac{2^{q-1}-1}{q}$. Since $s(0) = 0$ and

$$s(q_1) = \beta(pq_1) - \beta(qq_1) = \beta(pq_1) - \beta(2^{q-1}-1) = \beta(pq_1) - (q-1) < 0,$$

it follows that

$$\frac{1}{2^{q-1}} \left| \sum_{m=0}^{2^{q-1}-1} e(\alpha s(m)) \right| = 1 - \Delta \qquad \text{for some } \Delta > 0.$$

Let $\mathcal{B}_q$ be the set of words of length $2k + q - 1 =: K$

$$0^k\ \gamma\ 0^k = \underbrace{0\dots0}_{k \text{ times}}\ \gamma\ \underbrace{0\dots0}_{k \text{ times}},$$

where γ runs over the elements of A^{q-1}.

Assume that x is large and let N be the unique integer satisfying $2^{N-1} \le x < 2^N$. Given an arbitrarily small number $\varepsilon > 0$, any positive integer $n < 2^N$ can be written as $n = \sum_{\nu=0}^{N-1} \epsilon_\nu(n) 2^\nu$, where each $\epsilon_\nu(n) \in A$. Further let $\eta(n) = \epsilon_0(n)\ldots\epsilon_{N-1}(n)$, and for any given word $\gamma = \delta_0\delta_1\ldots\delta_{h-1} \in A^h$, let

$$E(\gamma) = \delta_0 + \delta_1 \cdot 2 + \cdots + \delta_{h-1} \cdot 2^{h-1}.$$

Furthermore, let $R_1 = R_1(n)$ be the smallest index for which

$$\epsilon_{R_1}(n)\ldots\epsilon_{R_1+K-1}(n) \in \mathcal{B}_q.$$

Then, let $R_2 = R_2(n)$ be the smallest index such that $R_2(n) \geq R_1(n) + K$ and

$$\epsilon_{R_2}(n)\ldots\epsilon_{R_2+K-1}(n) \in \mathcal{B}_q,$$

and so on. In the end, let $R_1(n), \ldots, R_{\nu(n)}$ be the entire sequence of these indexes.

By a simple probabilistic argument, we obtain that for every fixed $T > 0$,

$$\frac{1}{2^N}\#\{n < 2^N : \nu(n) < T\} \to 0 \qquad (N \to \infty)$$

and that for some fixed T there exists a bound B such that for every integer $N > N_0(\varepsilon)$,

$$\frac{1}{2^N}\#\{n < 2^N : R_T(n) > B\} < \frac{\varepsilon}{2}.$$

Assume that T is large enough so that both the conditions $(1-\Delta)^T \leq \varepsilon$ and $N > N_0(\varepsilon)$ are satisfied. Now, consider only those integers $n \leq x$ for which $\nu(n) \geq T$ and $R_T(n) < B$. Clearly the numbers of such integers $n < 2^N$ is less than $\varepsilon \cdot 2^N$. Writing $\eta(n)$ as

$$\eta(n) = \theta_1\beta_1\theta_2\beta_2\ldots\theta_T\beta_T\nu,$$

it is clear that

$$\begin{aligned} s(n) \quad = \quad & s(L(\theta_1)) + s(L(\beta_1)) + s(L(\theta_2)) + s(L(\beta_2)) \\ & \qquad + \cdots + s(L(\theta_T)) + s(L(\beta_T)) + s(L(\nu)). \end{aligned}$$

Further let $E(\theta_1, \ldots, \theta_T; \nu)$ be the set of those integers n for which $\beta_1, \ldots, \beta_T$ run over $\mathcal{B}_q$. If we consider those integers $n < 2^N$, then $\#E(\theta_1, \ldots, \theta_T; \nu) = 2^{(q-1)T}$. On the other hand, if we consider only those $n \leq x$, there are only some special ν's for which $0 < \#E(\theta_1, \ldots, \theta_T; \nu) < 2^{(q-1)T}$. Let $\mathcal{S}$ be that particular collection of those sets $E(\theta_1, \ldots, \theta_T; \nu)$ for which there exist such integers n_1, n_2 with $n_1 \leq x < n_2$ and $\eta(n_1), \eta(n_2) \in E(\theta_1, \ldots, \theta_T; \nu)$. Setting $M := \lambda(\theta_1\beta_1\ldots\theta_T\beta_T)$, we may write that

$$n_1 = u_1 + 2^M v, \quad n_2 = u_2 + 2^M v, \quad 0 \leq u_1 < u_2 < 2^M.$$

Consequently, $2^M v \le x < 2^M(v+1)$, and therefore no more than one v (and so only one $\nu = \eta(v)$) with this property exists. Since $M \le B+K$, the number of possible $E(\theta_1,\dots,\theta_T;\nu) \in \mathcal{S}$ is less than 2^{B+K}, and therefore

$$\sum_{E(\theta_1,\dots,\theta_T;\nu)\in\mathcal{S}} \#E(\theta_1,\dots,\theta_T;\nu) \le 2^{(q-1)T+B+K} = C$$

for some positive constant C.

On the other hand, if $E(\theta_1,\dots,\theta_T;\nu) \notin \mathcal{S}$, then

$$\begin{aligned}\left|\sum_{n\in E(\theta_1,\dots,\theta_T;\nu)} e(\alpha s(n))\right| &= \left|\sum_{m=0}^{2^{q-1}-1} e(\alpha s(m))\right|^T \\ &\le (1-\Delta)^T \#E(\theta_1,\dots,\theta_T;\nu) \le \varepsilon \#E(\theta_1,\dots,\theta_T;\nu). \end{aligned} \tag{3.1}$$

We can now estimate the size of those positive integers $n \le x$ for which either

$$\nu(n) < T \quad \text{or} \quad R_T(n) > B \quad \text{or} \quad n \in E(\theta_1,\dots,\theta_T;\nu) \in \mathcal{S}.$$

Indeed, the total number of such $n \le x$ is bounded by $2\varepsilon x$, provided x is sufficiently large.

We have thus established that

$$\left|\sum_{n\le x} e(\alpha s(n))\right| \le \left|\sum_{\substack{n\in E(\theta_1,\dots,\theta_T;\nu)\\ \lambda(\theta_1,\dots,\theta_T)\le B\\ E(\theta_1,\dots,\theta_T;\nu)\notin\mathcal{S}}} e(\alpha s(n))\right| + 2\varepsilon x. \tag{3.2}$$

As a consequence of (3.1), the sum on the right hand side of (3.2) is bounded by $\varepsilon \sum \#E(\theta_1,\dots,\theta_T;\nu)$. Consequently,

$$\limsup_{x\to\infty} \frac{1}{x}\left|\sum_{n\le x} e(\alpha s(n))\right| \le 3\varepsilon.$$

Since ε can be chosen arbitrarily small, the proof of Theorem 2.1 is complete.

4 Proof of Theorem 2.2

Consider two sequences of real numbers $(U_k)_{k\ge1}$ and $(V_k)_{k\ge1}$ satisfying $\lim_{k\to\infty} U_k = \infty$ and $U_k < V_k$ for each $k \in \mathbb{N}$. Moreover, for each integer $k \ge 1$, consider the sum $A_k := \sum_{\substack{p\in\wp_1\\ U_k<p<V_k}} \frac{1}{p}$ and assume that $\lim_{k\to\infty} A_k = \infty$.

Also, for each integer $k \geq 1$, set $\omega_k(n) := \sum_{\substack{p|n,\ p\in\wp_1 \\ U_k<p<V_k}} 1$. We can prove that, for each $k \in \mathbb{N}$,

$$\sum_{n\leq x} r(n)(\omega_k(n) - 2A_k)^2 \leq CxA_k \qquad (x \geq e^{V_k}) \tag{4.1}$$

for some positive constant C.

Indeed, first observe that it is well known that there exists a constant $C > 0$ such that

$$D(x) := \sum_{n\leq x} r(n) = Cx + O(x^\beta) \tag{4.2}$$

for some positive constant $\beta < 1$.

Setting $\wp_k^* = \{p \in \wp^* :\ U_k < p < V_k\}$, we have

$$\begin{aligned} S_1(x) &:= \sum_{n\leq x} r(n)\omega_k(n) = \sum_{\substack{pm\leq x \\ p\in\wp_k^*}} r(pm) \\ &= \sum_{\substack{pm\leq x \\ p\in\wp_k^* \\ (p,m)=1}} r(pm) + \sum_{\substack{pm\leq x \\ \alpha\geq 2 \\ p\in\wp_k^*}} r(p^\alpha m) \\ &= 2\sum_{p\in\wp_k^2} \left(D(x/p) + O\left(\frac{x}{p^2}\right)\right) \\ &= 2A_kCx + o(x), \end{aligned}$$

where we used (4.2). Similarly, we obtain that

$$\begin{aligned} S_2(x) &:= \sum_{n\leq x} r(n)\omega_k^2(n) = 2\sum_{\substack{p<q \\ p,q\in\wp_k}} r(pqm) + S_1(x) \\ &= 4A_k^2Cx + O(x\,A_k). \end{aligned}$$

Using the above estimates of $S_1(x)$ and $S_2(x)$, we have proved (4.1).

Now, by using this "Turán-Kubilius type inequality", we continue as in Kátai [15]. First we set

$$\begin{aligned} H(x) &:= \sum_{n\leq x} r(n)f(n)e(n\alpha), \\ H_1(x) &:= \sum_{n\leq x} r(n)f(n)\omega_k(n)e(n\alpha). \end{aligned}$$

In light of (4.1) and (4.2), we have, by the Cauchy-Schwarz inequality,

$$\begin{aligned} |H(x)A_k - H_1(x)| &\le \sum_{n\le x} r(n)|\omega_k(n) - A_k| \\ &\ll \left(\sum_{n\le x} r(n)\right)\left(\sum_{n\le x} r(n)|\omega_k(n) - A_k|^2\right)^{1/2} \\ &\ll x\sqrt{A_k}. \end{aligned}$$

Since, using Lemma 1.5, we have $r(pm) = r(p)r(m) = 2r(m)$, we may write that

$$\begin{aligned} H_1(x) &:= \sum_{\substack{p\in\wp_k^* \\ pm\le x,\ (p,m)=1}} r(p)r(m)f(p)f(m)e(\alpha pm) + O\left(\sum_{\substack{p\in\wp_k^* \\ mp^\alpha\le x,\ \alpha\ge 2}} r(p^\alpha m)\right) \\ &= H_2(x) + O(x), \end{aligned}$$

where

$$H_2(x) = 2\sum_{m\le x} r(m)f(m) \sum_{\substack{p\le x/m \\ p\in\wp_k^*,\ (p,m)=1}} f(p)e(\alpha pm).$$

Thus

$$\begin{aligned} |H_2(x)|^2 &\le \left\{4\sum_{m\le x} r(m)\right\} \\ &\quad\times\left\{\sum_{\substack{p_1,p_2\in\wp_k^* \\ p_1\ne p_2}} f(p_1)\overline{f(p_2)} \sum_{\substack{m\le\min(x/p_1,x/p_2) \\ (m,p_1p_2)=1}} r(m)e(\alpha(p_1-p_2)m)\right. \\ &\qquad\left. + \sum_{p\in\wp_k^*} f^2(p)\sum_{m\le x/p} r(m)\right\}. \end{aligned}$$

We have therefore established, in light of Lemma 1.5, that

$$|H_2(x)|^2 \le O(x)\cdot\{o(x) + O(xA_k)\}$$

and consequently

$$H_2(x) = O(x\sqrt{A_k}) \quad \text{and} \quad H_1(x) = O(x\sqrt{A_k}),$$

from which we can conclude that

$$\limsup_{x\to\infty} \frac{|H(x)|}{x} \le \frac{C}{\sqrt{A_k}}.$$

Since we have assumed that $\lim_{k\to\infty} A_k = \infty$, the proof of Theorem 2.2 is complete.

Acknowledgements

The research of the first author was supported in part by a grant from NSERC.

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