On the values of the Euler function around shifted primes

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Abstract

Let φ stand for the Euler totient function. Garcia and Luca have proved that, given any positive integer ℓ, the set of those primes p such that $\varphi(p+\ell)/\varphi(p-\ell) > 1$ has the same density as the set of those primes p for which $\varphi(p+\ell)/\varphi(p-\ell) < 1$. Here we prove this result using classical results from probabilistic and analytic number theory. We then establish similar results for the sum of divisors function and for the k-fold iterate of the Euler function. We also examine the modulus of continuity of some arithmetical functions. Finally, we provide a general result regarding the existence of the distribution function for the function $s(p) := f(p+\ell) - f(p-\ell)$ for any fixed positive integer ℓ provided the additive function f satisfies certain conditions.

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1 Introduction

Let φ stand for the Euler totient function and $\pi(x)$ for the number primes not exceeding x. The distribution function of $\varphi(n)/n$ as n runs through shifted primes has been widely studied in the literature; see for instance the work of Deshouillers and Hassani [1]. Recently, Garcia and Luca [6] proved that

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \# \left\{ p \le x : \frac{\varphi(p+\ell)}{\varphi(p-\ell)} > 1 \right\} = \frac{1}{2}, \tag{1.1}$$

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \# \left\{ p \le x : \frac{\varphi(p+\ell)}{\varphi(p-\ell)} < 1 \right\} = \frac{1}{2}, \tag{1.2}$$

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \# \left\{ p \le x : \frac{\varphi(p+\ell)}{\varphi(p-\ell)} = 1 \right\} = 0. \tag{1.3}$$

Actually, in their paper [6], the authors obtain other results, including a proof that

$$\#\{p \le x : \varphi(p-\ell) = \varphi(p+\ell)\} \ll \begin{cases} \frac{x}{\log^3 x} & \text{if } \ell = 4^n - 1, \\ \frac{x}{e^{\log^{1/3} x}} & \text{otherwise.} \end{cases}$$

Here, we start by showing how one can obtain their main result (1.1)–(1.3) in the case $\ell = 1$ using classical results in probabilistic and analytic number theory. We then establish similar results for the sum of divisors function and for the k-fold iterate of the Euler function. We also examine the modulus of continuity of some arithmetical functions. Finally, we provide a general result regarding the existence of the distribution function for the function $s(p) := f(p+\ell) - f(p-\ell)$ for any fixed positive integer ℓ provided the additive function f satisfies certain conditions, a consequence of which are estimates (1.1), (1.2) and (1.3).

2 Preliminary results

2.1 Limiting distribution and independent random variables

Let $\mathcal{A}$ be the set of real-valued additive functions. In what follows, the letters p and q with or without subscript always stand for primes.

In 1939, Erdős and Wintner [5] proved that if $f \in \mathcal{A}$ and if the three series

$$\sum_{|f(p)|\le 1} \frac{f(p)}{p}, \qquad \sum_{|f(p)|\le 1} \frac{f^2(p)}{p} \quad \text{and} \quad \sum_{|f(p)|>1} \frac{1}{p} \qquad \text{converge,} \tag{2.1}$$

then f has a limiting distribution, or in other words, there exists a distribution function $F : \mathbb{R} \to [0,1]$ such that

$$\frac{1}{x}\#\{n \le x : f(n) \le u\} \to F(u) \quad \text{as } x \to \infty$$

for every real number u which is a point of continuity of F.

To this distribution function F, we associate its characteristic function $\Psi(\tau)$ defined by

$$\Psi(\tau) = \Psi_F(\tau) = \prod_p \left(1 - \frac{1}{p}\right)\left(1 + \sum_{a=1}^{\infty} \frac{e^{i\tau f(p^a)}}{p^a}\right).$$

Observe that F can be interpreted as the distribution function of the random variable $\eta := \sum_p \xi_p$ where ξ_p are independent random variables with purely discrete distribution and

$$\Psi_{\xi_p} := \left(1 - \frac{1}{p}\right)\left(1 + \sum_{a=1}^{\infty} \frac{e^{i\tau f(p^a)}}{p^a}\right).$$

Note that in 1931, Lévy [11] proved that if η is the convergent sum of the independent random variables ξ_p, that is $\eta = \sum_p \xi_p$, then its distribution function F_η is continuous (everywhere) if and only if

$$\sum_p P(\xi_p \neq 0) = \infty. \tag{2.2}$$

In 1960, Lukács [12] proved that if condition (2.2) holds, then F_η is of pure type, either absolutely continuous or singular.

2.2 Continuity modulus

Let ξ be a random variable with continuous distribution function F. We now set

$$Q_\xi(h) := Q_F(h) = \sup_{x\in\mathbb{R}}(F(x+h) - F(x))$$

and call it the *continuity modulus* of ξ. We will say that the continuity modulus of ξ_1 and ξ_2 are *equivalent* if

$$c_1 < \frac{Q_{\xi_1}(h)}{Q_{\xi_2}(h)} < c_2$$

for some suitable positive constants c_1 and c_2.

2.3 Limiting distribution on the set of shifted primes

In 1968, the second author [10] proved that if f is a function in $\mathcal{A}$ for which the three series condition (2.1) holds, then f has a limiting distribution on the set of shifted primes $p+1$. In particular, the characteristic function $\Psi_F(\tau)$ of $f(p+1)$ can then be written as

$$\Psi_F(\tau) = \left(\sum_{a=1}^{\infty} \frac{e^{i\tau f(2^a)}}{2^a}\right) \prod_{q\geq 3} \left(1 - \frac{1}{q-1} + \sum_{a=1}^{\infty} \frac{e^{i\tau f(q^a)}}{q^a}\right).$$

Observe that in 1989, Hildebrand [7] proved that if $f \in \mathcal{A}$ has a limiting distribution on the set of shifted primes, then the three series condition (2.1) holds.

2.4 Equivalence of continuity modulus

As an immediate consequence of Wiener's theorem, one can prove that if Ψ_1 and Ψ_2 are two characteristic functions and if $\kappa(\tau) := \Psi_1(\tau)/\Psi_2(\tau) \neq 0$, $|\kappa(\tau)| > c$ for some positive constant c and $\kappa(\tau)$ is an almost periodic function with absolutely convergent series of Fourier coefficients, then the continuity modulus of F_{Ψ_1} and F_{Ψ_2} are equivalent.

Moreover, as already observed by Indlekofer and Kátai [9], if γ_1 and γ_2 are independent random variables, then, setting $\gamma := \gamma_1 + \gamma_2$, we have

$$Q_\gamma(h) \leq \min(Q_{\gamma_1}(h), Q_{\gamma_2}(h)),$$

and, if the distribution function of γ_1 is purely discrete, then Q_{γ_2} and Q_γ are equivalent.

2.5 Multiplicative functions evaluated at shifted primes

Let $\mathcal{M}$ be the set of multiplicative functions. The following results follow using the methods used by the second author in [10].

Theorem A. *Let $g_1, g_2 \in \mathcal{M}$ be such that $|g_i(n)| \le 1$ for $i = 1, 2$ and assume that*

$$\sum_p \frac{1 - g_i(p)}{p} \qquad \text{converges for } i = 1, 2. \tag{2.3}$$

Then,

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p\le x} g_1(p+1) g_2(p-1) = D(2) \prod_{q\ge 3} \left(1 - \frac{2}{q-1} + \sum_{a=1}^{\infty} \frac{g_1(q^a) + g_2(q^a)}{q^a}\right),$$

where

$$D(2) = g_2(2) \sum_{a=2}^{\infty} \frac{g_1(2^a)}{2^a} + g_1(2) \sum_{a=2}^{\infty} \frac{g_2(2^a)}{2^a}.$$

In particular, given $g \in \mathcal{M}$, let $g_1(n) = g(n)$ and $g_2(n) = \overline{g(n)}$, we then have

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p\le x} g(p+1)\overline{g(p-1)} = D^*(2) \prod_{q\ge 3} \left(1 - \frac{2}{q-1} + \sum_{a=1}^{\infty} \frac{g(q^a) + \overline{g(q^a)}}{q^a}\right),$$

where

$$D^*(2) = \overline{g(2)} \sum_{a=2}^{\infty} \frac{g(2^a)}{2^a} + g(2) \sum_{a=2}^{\infty} \frac{\overline{g(2^a)}}{2^a}.$$

One can even prove a more general result, namely the following.

Theorem B. *Let $g_1, \ldots, g_k \in \mathcal{M}$. Consider a set of k pairs of integers $\{(a_i, b_i) : a_i > 0, GCD(a_i, b_i) = 1,\ i = 1, \ldots, k\}$ which is such that $\dfrac{a_i n + b_i}{a_j n + b_j} \ne$ constant for $b_i \ne 0$, $i = 1, \ldots, k$. Then, assuming that (2.3) holds for $i = 1, \ldots, k$, the limit*

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p\le x} \prod_{j=1}^{k} g_j(a_j p + b_j) =: M(g_1, \ldots, g_k)$$

exists and, if we set $R := \{q : q \mid a_i b_j - a_j b_i \text{ for some } i \ne j\}$, we have

$$M(g_1, \ldots, g_k) = E \prod_{q\nmid R} \left(1 - \frac{1}{q-1} + \sum_{a=1}^{\infty} \frac{1}{q^a} \sum_{j=1}^{k} g_j(q^a)\right),$$

where the constant E depends only on the values $g_1(q^a), \ldots, g_k(q^a)$, with $q \in R$ and $a \in \mathbb{N}$.

2.6 Primes in arithmetical progressions and sieve results for shifted primes

The proofs of our theorems are based on three well known inequalities in prime number theory and one sieve estimate regarding shifted primes, which we state as follows.

Theorem C. (Brun-Titchmarsh inequality) *Given a fixed real number $\delta \in (0,1)$, there exists a positive constant $c = c(\delta)$ such that the estimate*

$$\pi(x;q,\ell) := \#\{p \le x : p \equiv \ell \pmod q\} \le \frac{c\,x}{\varphi(q)\log x}$$

holds uniformly for all integers $1 \le \ell < q$ with $(q,\ell) = 1$ and $q < x^{1-\delta}$.

Theorem D. (Siegel-Walfisz theorem) *Let $A > 0$ be a fixed constant. Then, there exists a positive constant $B = B(A)$ such that for large x the estimate*

$$\pi(x;q,\ell) = \frac{\pi(x)}{\varphi(q)} + O\left(\frac{x}{\exp(B\sqrt{\log x})}\right)$$

holds uniformly for $1 \le \ell < q$ with $(q,\ell) = 1$ and $q < \log^A x$.

Theorem E. (Bombieri-Vinogradov theorem) *Let $A > 0$ be a fixed constant. Then, there exists a positive constant $B = B(A)$ such that for large x,*

$$\sum_{q < \sqrt{x}/\log^B x} \max_{\substack{1 \le \ell < q \\ (q,\ell)=1 \\ y \le x}} \left| \pi(y;q,\ell) - \frac{li(y)}{\varphi(q)} \right| < \frac{x}{\log^A x},$$

where $li(x) := \displaystyle\int_2^x \frac{dt}{\log t}$.

Theorem F. *Let $P(n)$ stand for the largest prime factor of n. Given any $\varepsilon > 0$ and a fixed positive integer ℓ, there exists a positive number $c(\varepsilon)$ satisfying $\lim_{\varepsilon \to 0} c(\varepsilon) = 0$ and such that*

$$\#\left\{p \le x : \frac{\log P(p \pm \ell)}{\log x} < \varepsilon\right\} + \#\left\{p \le x : \frac{\log P(p \pm \ell)}{\log x} > 1 - \varepsilon\right\} \le c(\varepsilon)\pi(x).$$

3 An alternative proof of the Garcia and Luca result

Let us set

$$\varphi_0(n) := \frac{\varphi(n)}{n} \quad \text{and} \quad f(n) := \log \varphi_0(n) \qquad (n = 1, 2, \ldots).$$

Clearly, $f \in \mathcal{A}$ and $f(q) = \log(1 - 1/q)$ for each prime q. Let us further set

$$g(n) := e^{i\tau f(n)} \quad (n = 1, 2, \ldots) \quad \text{and} \quad \ell_p := f(p+1) - f(p-1) \text{ for each prime } p.$$

In light of the above Theorem A, it follows that

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p\le x} g(p+1)\overline{g(p-1)} = \prod_{q\ge 3} \left(1 - \frac{2}{q-1} + \frac{g(q) + \overline{g(q)}}{q-1}\right) =: \Psi(\tau).$$

Now let $\eta := \sum_{q\ge 3} X_q$, where X_q are independent random variables defined by

$$\begin{aligned} P(X_q = 0) &= 1 - 2/(q-1), \\ P(X_q = f(q)) &= 1/(q-1), \\ P(X_q = -f(q)) &= 1/(q-1). \end{aligned}$$

Set $F(u) := P(\eta < u)$ and let $\Psi(\tau)$ be the characteristic function of F. We know that F is continuous since $\sum_{q\ge 3} P(X_q \ne 0) = \infty$. Moreover, $\Psi(-\tau) = \Psi(\tau)$, which clearly implies that $F(u) = 1 - F(1-u)$, from which we may conclude that $F(0) = 1/2$. We have thus established that

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \#\{p \le x : \ell_p < 0\} = \lim_{x\to\infty} \frac{1}{\pi(x)} \# \left\{p \le x : \frac{\varphi_0(p+1)}{\varphi_0(p-1)} < 1\right\} = \frac{1}{2}.$$

Taking into account that

$$\frac{\varphi(p+1)}{\varphi(p-1)} = \frac{\varphi_0(p+1)}{\varphi_0(p-1)} \cdot \frac{p+1}{p-1},$$

which in turn implies that

$$\log \frac{\varphi(p+1)}{\varphi(p-1)} = \ell_p + O\left(\frac{1}{p}\right),$$

we have thus completed the proof of Garcia and Luca's estimates (1.1), (1.2) and (1.3) in the case $\ell = 1$. The general case will be examined in Section 7.

4 The k-fold iterate of the Euler function at shifted primes

Let $\varphi_0(n) = n$, $\varphi_1(n) = \varphi(n)$ and $\varphi_k(n) = \varphi(\varphi_{k-1}(n))$ for each integer $k \ge 2$. The behaviour of the quotient $\dfrac{\varphi_{k+1}(n)}{\varphi_k(n)}$ as n becomes large was widely studied; in particular, see Indlekofer and Kátai [8]. In fact, one can prove that, given any fixed positive integer k, for almost all primes $p \le x$,

$$\frac{\varphi_{k+1}(p\pm 1)}{\varphi_k(p\pm 1)} = (1 + o(1)) \prod_{q<(\log\log x)^k} \left(1 - \frac{1}{q}\right) \qquad (x \to \infty).$$

Using this, the following result follows.

Theorem 1. *Given any fixed $k \in \mathbb{N}$, for almost all primes p,*

$$\frac{\varphi_{k+1}(p+1)}{\varphi_{k+1}(p-1)} = (1+o(1))\frac{\varphi(p+1)}{\varphi(p-1)} \qquad (p \to \infty),$$

thereby implying that, given any fixed $k \in \mathbb{N}$,

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \# \left\{ p \le x : \frac{\varphi_k(p+1)}{\varphi_k(p-1)} < e^u \right\} = F(u)$$

for every $u \in \mathbb{R}$.

5 Analogous results for the sum of divisors function

Let $\sigma(n)$ stand for the sum of the positive divisors of n. Consider the following three functions:

$$\sigma_0(n) = \frac{\sigma(n)}{n} = \prod_{q^a \| n} \left(1 + \frac{1}{q} + \cdots + \frac{1}{q^a}\right), \qquad f(n) = \log \sigma_0(n), \qquad g(n) = e^{i\tau f(n)}.$$

The distribution function of $\sigma(n)/n$ was extensively studied, in particular by Erdős [2]. Here, we are interested in the values of $\sigma(n)/n$ as n runs through shifted primes $p+1$ and $p-1$. First of all, setting $S(x) := \frac{1}{\pi(x)} \sum_{p \le x} g(p+1)\overline{g(p-1)}$, one can prove that, as $x \to \infty$,

$$S(x) = (1+o(1)) \left(\overline{g(2)} \sum_{a=2}^{\infty} \frac{g(2^a)}{2^a} + g(2) \sum_{a=2}^{\infty} \frac{\overline{g(2^a)}}{2^a} \right) \cdot \prod_{q \ge 3} \left(1 - \frac{2}{q-1} + \sum_{a=1}^{\infty} \frac{g(q^a) + \overline{g(q^a)}}{q^a} \right).$$

It follows from this that $\lim_{x\to\infty} S(x) = \Psi(\tau)$ with $\Psi(\tau) = \Psi(-\tau)$. Now let $\eta = \sum_{q \ge 2} X_q$, where the X_q are independent random variables defined by

$$\begin{aligned} P(X_2 = f(2^a) - f(2)) &= \frac{1}{2^a} \quad (a = 2, 3, \ldots) \\ P(X_2 = f(2) - f(2^a)) &= \frac{1}{2^a} \quad (a = 2, 3, \ldots) \end{aligned}$$

and, for each prime $q \ge 3$, by

$$\begin{aligned} P(X_q = 0) &= 1 - 2/(q-1) \\ P(X_q = f(q^a)) &= \frac{1}{q^a} \quad (a = 1, 2, \ldots) \end{aligned}$$

$$P(X_q = -f(q^a)) \;=\; \frac{1}{q^a} \quad (a = 1, 2, \ldots).$$

In light of the above, the following result can be obtained.

Theorem 2. *We have*

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \# \left\{ p \le x : \frac{\sigma_0(p+1)}{\sigma_0(p-1)} < e^u \right\} = F(u),$$

where $F(u) := P(\eta < u)$ is a continuous distribution function with $F(0) = 1/2$. Moreover, since

$$\frac{\sigma(p+1)}{\sigma(p-1)} = (1 + o(1)) \frac{\sigma_0(p+1)}{\sigma_0(p-1)} \qquad (p \to \infty),$$

we have

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \# \left\{ p \le x : \frac{\sigma(p+1)}{\sigma(p-1)} < e^u \right\} = F(u).$$

Remark. *By setting $\sigma_2(n) := \sigma(\sigma(n))$, one can show that the above result also holds for σ_2 in place of σ. Indeed, first observe that*

$$\frac{\sigma_2(p+1)}{\sigma_2(p-1)} = \frac{\sigma_2(p+1)}{\sigma(p+1)} \cdot \frac{\sigma(p-1)}{\sigma_2(p-1)} \cdot \frac{\sigma(p+1)}{\sigma(p-1)} = A_p \cdot B_p \cdot C_p,$$

say. Since, for almost all primes $p \le x$, we have

$$\frac{\sigma_2(p \pm 1)}{\sigma(p \pm 1)} = (1 + o(1)) \prod_{q < \log\log x} \frac{1}{1 - 1/q} \qquad (x \to \infty),$$

it follows that $A_p \cdot B_p = 1 + o(1)$ as $p \to \infty$, implying that

$$\frac{\sigma_2(p+1)}{\sigma_2(p-1)} = (1 + o(1)) \frac{\sigma(p+1)}{\sigma(p-1)} \qquad (p \to \infty),$$

thereby implying that these last two quotients have the same distribution function, thus proving our claim.

6 Remarks on the modulus of continuity of some arithmetical functions

Recall the definitions $\varphi_0(n) = \varphi(n)/n$ and $\sigma_0(n) = \sigma(n)/n$. Tjan [13] proved that there exists a positive constant c_1 such that

$$Q_{\varphi_0}\left(\frac{1}{t}\right) < \frac{c_1}{\log t},$$

whereas Erdős [3] proved that there exists a positive constant c_2 such that

$$Q_{\sigma_0}\left(\frac{1}{t}\right) < \frac{c_2}{\log t}.$$

On the other hand, Erdős and Kátai [4] proved that if g is a strongly additive function satisfying $\sum_p \frac{|g(p)|}{p} < \infty$ and such that, for suitable positive constants A and δ, we have $\sum_{p>t^A} \frac{|g(p)|}{p} < \frac{1}{t}$ and $|g(p_1)-g(p_2)| > 1/t$ for all $p_1 \neq p_2$ with $p_1, p_2 < t^\delta$, then

$$\frac{1}{\log t} \ll Q_g(1/t) \ll \frac{1}{\log t}.$$

Observe that one can drop the "strongly" condition by simply assuming that $g(p^a) = O(1)$ for all primes p and integers $a \geq 2$.

Now, using Theorem A in Indlekofer and Kátai [9], we can deduce the following.

Theorem 3. *Let F be the distribution function for which*

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \#\left\{p \leq x : \frac{\varphi(p+1)}{\varphi(p-1)} < e^u\right\} = F(u).$$

Then, there exist some positive constants c_3 and c_4 such that

$$\frac{c_3}{\log^2 t} \leq Q_F(1/t) \leq \frac{c_4}{\log^2 t}.$$

Similarly, if G is defined by

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \#\left\{p \leq x : \frac{\sigma(p+1)}{\sigma(p-1)} < e^u\right\} = G(u),$$

then, there exist some positive constants c_5 and c_6 such that

$$\frac{c_5}{\log^2 t} \leq Q_G(1/t) \leq \frac{c_6}{\log^2 t}.$$

On the other hand, repeating the argument used in Section 5 of Indlekofer and Kátai [9], one can prove the following.

Theorem 4. *Let $\varphi_0(n) = \varphi(n)/n$ and $\sigma_0(n) = \sigma(n)/n$. For each prime p, let*

$$a_p := \sigma_0(p-1)+\sigma_0(p+1), \quad b_p := \varphi_0(p-1)+\varphi_0(p+1), \quad c_p := \sigma_0(p-1)+\varphi_0(p+1),$$

with F_a, F_b and F_c standing for their respective distribution functions. Then,

$$Q_{F_a}(1/t) \asymp \frac{1}{\log t},$$

whereas

$$Q_{F_b}(1/t) \asymp Q_{F_c}(1/t) \asymp \frac{1}{\log^2 t}.$$

7 The general case

In our next result regarding functions $f \in \mathcal{A}$, the condition of the convergence of the first series in the three series condition (2.1) is not required. Indeed, we shall prove the following result.

Theorem 5. *Let $f \in \mathcal{A}$ be such that*

$$\sum_{|f(p)|\le 1} \frac{f^2(p)}{p} \qquad \text{and} \qquad \sum_{|f(p)|>1} \frac{1}{p} \qquad \text{converge.} \tag{7.1}$$

Let $\ell > 0$ be a fixed integer and, for each prime p, set $s(p) := f(p+\ell) - f(p-\ell)$. Then, the limit

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \#\{p \le x : s(p) < u\} =: F_\ell(u)$$

exists and its characteristic function $\Psi_\ell(\tau)$ is given by

$$\Psi_\ell(\tau) := D_\ell(\tau) \prod_{\substack{q\ge 3\\ q\nmid \ell}} \left(1 - \frac{2}{q-1} + \sum_{a=1}^{\infty} \frac{e^{i\tau f(q^a)} + e^{-i\tau f(q^a)}}{q^a}\right),$$

where

$$D_\ell(\tau) = \begin{cases} 1 & \text{if } \ell \text{ is even,} \\ 2\Re\left(e^{i\tau f(2)} \sum_{a=2}^{\infty} \frac{e^{-i\tau f(2^a)}}{2^a}\right) & \text{if } \ell \text{ is odd.} \end{cases}$$

In other words, the distribution function $F_\ell(u)$ is defined as follows:

$$F_\ell(u) = \begin{cases} P\left(\sum_{\substack{q\ge 3\\ q\nmid \ell}} \xi_q < u\right) & \text{if } \ell \text{ is even,} \\ P\left(\xi_2 + \sum_{\substack{q\ge 3\\ q\nmid \ell}} \xi_q < u\right) & \text{if } \ell \text{ is odd,} \end{cases}$$

where the ξ_q's are independent random variables defined by

$$\begin{aligned} P(\xi_2 = f(2^a) - f(2)) &= \frac{1}{2^a} \quad (a = 2, 3, \ldots), \\ P(\xi_2 = f(2) - f(2^a)) &= \frac{1}{2^a} \quad (a = 2, 3, \ldots) \end{aligned}$$

and, for each prime $q \ge 3$, by

$$\begin{aligned} P(\xi_q = 0) &= 1 - 2/(q-1), \\ P(\xi_q = f(q^a)) &= \frac{1}{q^a} \quad (a = 1, 2, \ldots), \\ P(\xi_q = -f(q^a)) &= \frac{1}{q^a} \quad (a = 1, 2, \ldots). \end{aligned}$$

Moreover, if $\sum_{f(p)\neq 0} \frac{1}{p} = \infty$, *then* $F_\ell(u)$ *is everywhere continuous and* $F_\ell(0) = 1/2$.

As a consequence of Theorem 5, it is clear that the general Garcia and Luca estimates (1.1)–(1.3) follow immediately.

Proof of Theorem 5. Let Y_x be a function which goes to infinity with x but slowly enough so that $Y_x = O(\log\log\log x)$. Let $\varepsilon > 0$ be a small but fixed real number and consider the following subsets of $\{p \le x\}$:

$$\begin{aligned}
E_1 &= \{p \le x : \exists q^K \mid p+\ell \text{ or } q^K \mid p-\ell \text{ where } q \le Y_x,\ K > Y_x\},\\
E_2 &= \{p \le x : P(p+\ell) > x^{1-\varepsilon} \text{ or } P(p-\ell) > x^{1-\varepsilon}\},\\
E_3 &= \{p \le x : \exists q \in (Y_x, x^{1-\varepsilon}) \text{ such that } q \mid p+\ell \text{ or } q \mid p-\ell \text{ and } |f(q)| > \varepsilon\},\\
E_4 &= \{p \le x : \exists q > Y_x \text{ such that } q^2 \mid p+\ell \text{ or } q^2 \mid p-\ell\}.
\end{aligned}$$

In light of conditions (7.1) (to estimate the size of E_3) and of Theorem F (to estimate the size of E_2), we easily find that there exists an absolute constant $c_7 > 0$ such that

$$\#E_1 + \#E_2 + \#E_3 + \#E_4 < c_7\, \varepsilon\, \pi(x). \tag{7.2}$$

We now let

$$f(n) = f_1(n) + f_2(n),$$

where $f_1 \in \mathcal{A}$ is defined on prime powers q^k by $f_1(q^k) = f(q^k)$ if $q \le Y_x$ and 0 otherwise, and where $f_2 \in \mathcal{A}$ is defined implicitly. Now consider the function $g(n) = g_1(n)g_2(n)$, where $g_j(n) = e^{i\tau f_j(n)}$ for $j = 1, 2$. From here on, we shall assume that ℓ is an even integer, since the case where ℓ is odd can be handled in a similar manner.

First observe that we can deduce from Theorem D that, as $x \to \infty$,

$$\begin{aligned}
\frac{1}{\pi(x)} \sum_{p \le x} g_1(p+\ell)\overline{g_1(p-\ell)} &= (1+o(1)) \prod_{\substack{3 \le q \le Y_x \\ q \nmid \ell}} \left(1 - \frac{2}{q-1} + \sum_{a=1}^{\infty} \frac{g(q^a) + \overline{g(q^a)}}{q^a}\right)\\
&\quad + O\left(\frac{\#E_1}{\pi(x)}\right),
\end{aligned} \tag{7.3}$$

where we have taken into account the fact that

$$\sum_{q < Y_x} \sum_{a > Y_x} \frac{|g(q^a) + \overline{g(q^a)}|}{q^a} \ll \sum_q \frac{1}{q^{Y_x}} \ll \frac{1}{2^{Y_x}}.$$

The convergence as $Y_x \to \infty$ of the product appearing on the right hand side of (7.3) is guaranteed by the following argument. First observe that in the case where $|f(q)| \le 1$, we have

$$e^{i\tau f(q)} = 1 + i\tau f(q) + \frac{1}{2}\tau^2 f^2(q) + \cdots,$$

$$e^{-i\tau f(q)} \quad = \quad 1 - i\tau f(q) + \frac{1}{2}\tau^2 f^2(q) - \cdots ,$$

which implies in particular that, for some positive absolute constant c_8,

$$\begin{aligned} \left|e^{i\tau f(q)} - (1 + i\tau f(q))\right| &\leq c_8|\tau| f^2(q), \\ \left|e^{-i\tau f(q)} - (1 - i\tau f(q))\right| &\leq c_8|\tau| f^2(q), \end{aligned}$$

from which it follows that, if $|f(q)| \leq 1$,

$$\left|e^{i\tau f(q)} + e^{-i\tau f(q)} - 2\right| \leq 2c_8|\tau| f^2(q),$$

whereas, in the case where $a \geq 2$ or $a = 1$ and $|f(q)| > 1$, we have the trivial inequality

$$\left|e^{i\tau f(q^a)} + e^{-i\tau f(q^a)} - 2\right| \leq 4.$$

Therefore, we may write that

$$\left|\left(1 - \frac{2}{q-1} + \sum_{a=1}^{\infty} \frac{g(q^a) + \overline{g(q^a)}}{q^a}\right) - 1\right| \leq \frac{\min\left(2c_1|\tau| f^2(q), 4\right)}{q} + \frac{4}{q^2}.$$

In light of the two conditions stated in (7.1), this last estimate guarantees the convergence of the product appearing in (7.3) as $Y_x \to \infty$. From this observation and of the fact that the error term in (7.3) is no larger than $c_7\varepsilon$ (by (7.2)), we may conclude that, as $x \to \infty$,

$$\text{(7.4)} \quad \frac{1}{\pi(x)} \sum_{p \leq x} g_1(p+\ell)\overline{g_1(p-\ell)} = (1 + o(1)) \prod_{\substack{q \geq 3 \\ q \nmid \ell}} \left(1 - \frac{2}{q-1} + \sum_{a=1}^{\infty} \frac{g(q^a) + \overline{g(q^a)}}{q^a}\right).$$

Now recall that our ultimate goal is to show that, as $x \to \infty$,

$$\text{(7.5)} \quad \frac{1}{\pi(x)} \sum_{p \leq x} g(p+\ell)\overline{g(p-\ell)} = (1 + o(1)) \prod_{\substack{3 \leq q \leq Y_x \\ q \nmid \ell}} \left(1 - \frac{2}{q-1} + \sum_{a=1}^{\infty} \frac{g(q^a) + \overline{g(q^a)}}{q^a}\right).$$

But if we can prove that there exists some positive constant c_9 such that

$$\text{(7.6)} \quad \frac{1}{\pi(x)} \sum_{p \leq x} \left|g_2(p+\ell)\overline{g_2(p-\ell)} - 1\right|^2 < c_9\,\varepsilon,$$

provided that $x \geq x(\varepsilon)$, then (7.5) will follow. Indeed, if (7.6) holds, then

$$\frac{1}{\pi(x)} \left|\sum_{p \leq x} g(p+\ell)\overline{g(p-\ell)} - \sum_{p \leq x} g_1(p+\ell)\overline{g_1(p-\ell)}\right|$$

$$\leq \frac{1}{\pi(x)} \sum_{p\leq x} \left| g_2(p+\ell)\overline{g_2(p-\ell)} - 1 \right| \leq \sqrt{c_{10}\varepsilon}$$

for some positive constant c_{10}, thereby implying, in light of (7.4) that (7.5) holds as well, as required.

Therefore, it remains to prove (7.6). To do so, it is enough to consider the above sum only for those primes $p \leq x$ which do not belong to any of the sets E_1, E_2, E_3 or E_4. So, let us now introduce the additive function $h(n)$ defined by $h(q^k) = 0$ if q satisfies anyone of the four conditions: (1) $q \leq Y_x$, (2) $|f(q)| > \varepsilon$ when $q \in (Y_x, x^{1-\varepsilon})$, (3) $q > x^{1-\varepsilon}$, (4) $k \geq 2$, while otherwise we set $h(q^k) = f(q)$.

Furthermore, let us introduce the strongly additive function $h_0(n)$ defined by

$$h_0(q) := \begin{cases} h(q) & \text{if } q \leq x^{1/5}, \\ 0 & \text{otherwise.} \end{cases}$$

Then, consider the two functions defined on primes p by

$$t_p := h(p+\ell) - h(p-\ell) \quad \text{and} \quad t_p^* := h_0(p+\ell) - h_0(p-\ell),$$

so that

$$t_p = t_p^* + O(\varepsilon),$$

thereby implying that

$$t_p^2 \leq (t_p^*)^2 + O(\varepsilon^2).$$

Further set

$$S_x := \sum_{p\leq x} t_p^2 \quad \text{and} \quad S_x^* := \sum_{p\leq x} (t_p^*)^2,$$

so that, for some absolute constant $c_{11} > 0$,

$$S_x \leq S_x^* + c_{11}\,\varepsilon\,\pi(x). \tag{7.7}$$

By the definition of t_p^*, we have

$$S_x^* \leq \sum_{p\leq x} \left(h_0^2(p+\ell) + h_0^2(p-\ell) \right) - 2\sum_{p\leq x} h_0(p+\ell)h_0(p-\ell). \tag{7.8}$$

On the other hand,

$$\sum_{p\leq x} h_0^2(p+\ell) = \sum_{\substack{q_1\neq q_2 \leq x^{1/5} \\ q_1 | p+\ell \\ q_2 | p+\ell}} h_0(q_1)h_0(q_2) + \sum_{\substack{q\leq x^{1/5} \\ q | p+\ell}} h_0^2(q), \tag{7.9}$$

$$\sum_{p\leq x} h_0^2(p-\ell) = \sum_{\substack{q_1\neq q_2 \leq x^{1/5} \\ q_1 | p-\ell \\ q_2 | p-\ell}} h_0(q_1)h_0(q_2) + \sum_{\substack{q\leq x^{1/5} \\ q | p-\ell}} h_0^2(q), \tag{7.10}$$

whereas

$$\sum_{p\le x} h_0(p+\ell)h_0(p-\ell) = \sum_{\substack{q_1\ne q_2\le x^{1/5}\\ q_1|p+\ell\\ q_2|p-\ell}} h_0(q_1)h_0(q_2). \tag{7.11}$$

Gathering estimates (7.9), (7.10) and (7.11) in (7.8), using (7.7) and taking into account (7.2), we then obtain, in light of Theorem C, that, for some positive constant c_{12},

$$\begin{aligned} S_x &\le \sum_{\substack{q_1\ne q_2\le x^{1/5}\\ q_1q_2|p+\ell}} h_0(q_1)h_0(q_2)\left(\pi(x;q_1q_2,-\ell)+\pi(x;q_1q_2,\ell)-2\pi(x;q_1q_2,\rho(\ell))\right)\\ &\qquad + \sum_{q\le x^{1/5}} h_0^2(q)\left(\pi(x;q,-\ell)+\pi(x;q,\ell)\right)+c_{12}\,\varepsilon\,\pi(x)\\ &\le \sum_{q_1\ne q_2\le x^{1/5}} \max_{r \pmod{q_1q_2}} \left|\pi(x;q_1q_2,r)-\frac{\mathrm{li}(x)}{\varphi(q_1q_2)}\right| + \sum_{q>Y_x}\frac{h_0^2(q)}{\varphi(q)}\mathrm{li}(x)+c_{12}\,\varepsilon\,\pi(x), \end{aligned} \tag{7.12}$$

where the expression $\rho(\ell)$ is the integer solution $p\equiv\rho(\ell) \pmod{q_1q_2}$ of the system of congruences $p\equiv-\ell \pmod{q_1}$ and $p\equiv\ell \pmod{q_2}$.

Using Theorem E, it follows from (7.12) that

$$S_x \le \left(\sum_{\substack{q>Y_x\\ |f(q)|\le 1}} \frac{f^2(q)}{q}\right)\cdot \mathrm{li}(x)+c_{12}\,\varepsilon\pi(x).$$

Hence, in light of (7.1),

$$\lim_{x\to\infty}\frac{S_x}{\pi(x)}\le c_{12}\,\varepsilon,$$

thereby establishing (7.6), and the theorem follows. □

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