CONSECUTIVE INTEGERS WITH CLOSE KERNELS

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ABSTRACT. Let k be an arbitrary positive integer and let $\gamma(n)$ stand for the product of the distinct prime factors of n. For each integer $n \geq 2$, let a_n and b_n stand respectively for the maximum and the minimum of the k integers $\gamma(n+1), \gamma(n+2), \ldots, \gamma(n+k)$. We show that $\liminf_{n\to\infty} a_n/b_n = 1$. We also prove that the same result holds in the case of the Euler function, the sum of the divisors function as well as the functions $\omega(n)$ and $\Omega(n)$ which stand respectively for the number of distinct prime factors of n and the total number of prime factors of n counting their multiplicity.

1. INTRODUCTION

The local behavior of arithmetic functions has been the focus of various studies. One of these involves comparing the values of an arithmetic function at its consecutive arguments. For instance, we were able to show (see our recent book [3], Proposition 8.9) that, given any integer $k \geq 2$ and letting ϕ stand for the Euler function, $\phi(n+1) < \phi(n+2) < \cdots < \phi(n+k)$ holds for infinitely many positive integers n. The same type of statement can be made for the sum of divisors function $\sigma(n)$. Besides these and other multiplicative functions, similar statements can be made for additive functions. For instance, De Koninck, Friedlander and Luca [2] proved that, given any integer $k \geq 2$ and setting $g(n) = \omega(n) := \sum_{p|n} 1$ or $g(n) = \Omega(n) := \sum_{p^\alpha \| n} \alpha$, then

$$g(n+1) < g(n+2) < \cdots < g(n+k) \qquad \text{holds infinitely often.} \tag{1.1}$$

See also [1]. However, such results do not provide sufficient information to conclude that, in the above string of inequalities, $g(n+k)/g(n+1)$ can be arbitrarily close to 1 on an infinite sequence of integers n. Here, we fill this gap for several arithmetic functions, in particular for the *kernel function* $\gamma(n) := \prod_{p|n} p$. More precisely, let $f : \mathbb{N} \to \mathbb{R}_+$ be an arithmetic function with values in the positive reals. For each positive integer k, let

$$f_k = \liminf_{n\to\infty} \frac{\max\{f(n+1), \ldots, f(n+k)\}}{\min\{f(n+1), \ldots, f(n+k)\}}.$$

We show that $f_k = 1$ for all $k \geq 1$ for a variety of classical arithmetic functions like $f(n) = \gamma(n),\ \phi(n),\ \sigma(n),\ \omega(n),\ \Omega(n)$.

1991 *Mathematics Subject Classification.* Primary 11A25, Secondary 11N64.

Key words and phrases. arithmetic function, local behavior.

The work of the first author was supported in part by a grant from NSERC of Canada.

The work of the second author was supported in part by grant CPRR160325161141 and an A-rated scientist award both from the NRF of South Africa and by grant no. 17-02804S of the Czech Granting Agency.

2. The construction for $\omega(n)$ and $\Omega(n)$

For $f(n) = \omega(n),\ \Omega(n)$, this is easy. By the Turán-Kubilius inequality, for each $\varepsilon > 0$, the number of positive integers $n \le x$ with the property that $\omega(n) \notin ((1-\varepsilon)\log\log x, (1+\varepsilon)\log\log x)$ is $O(x/\log\log x)$. Doing this for $n+1,\ \ldots,\ n+k$, it follows that for

$$x + O(kx/\log\log x) = (1+o(1))x$$

positive integers $n \le x$ as $x \to \infty$, we have that

$$\omega(n+i) \in ((1-\varepsilon)\log\log x, (1+\varepsilon)\log\log x)), \qquad i = 1, \ldots, k.$$

Thus, for such n we have that

$$\frac{\max\{f(n+1), \ldots, f(n+k)\}}{\min\{f(n+1), \ldots, f(n+k)\}} \in \left[1, \frac{1+\varepsilon}{1-\varepsilon}\right].$$

Making ε tend to zero we get the desired assertion. Of course, the same works for $\Omega(n)$.

3. The construction for $\gamma(n)$

Let $k \ge 2$, $K = (2k+1)!$. Let $n \equiv K \pmod{K^2}$. Then $n = K + K^2 m$ for some nonnegative integer m. Write

$$n + i = i n_i, \qquad \text{where} \qquad n_i = \left(1 + \frac{K}{i}\right) + \left(\frac{K^2}{i}\right) m \qquad \text{for} \quad i = 1, \ldots, k.$$

Since $i^2 \mid (k!)^2 \mid K$, it follows that n_i is coprime to all primes $p \le k$ for $i = 1, \ldots, k$. Moreover, since K/i is a multiple of all primes $p \in [k+1, 2k+1]$ for all $i \le k$, it follows that n_i is coprime to all primes $p \in [k+1, 2k+1]$ as well. Thus, n_i is coprime to all primes $p \le 2k+1$. By multiplicativity, $f(n+i) = f(i)f(n_i)$ for all $i = 1, \ldots, k$. Let $\varepsilon > 0$ be fixed. Let us put $g(i) = f(i)/i$. Choose a prime $p_1 > 2k+1$ sufficiently large so that each of the intervals

$$\left(\frac{g(i)}{g(1)} p_1, \frac{g(i)}{g(1)} p_1 (1+\varepsilon)\right) \qquad i = 1, \ldots, k$$

contains a prime $p_i > 2k+1$ such that $p_1, \ldots, p_k$ are distinct primes. This is possible if

$$p_1 > (2k+1) g(1) \max\{g(i)^{-1} : 1 \le i \le k\},$$

and if

$$\pi\left(\frac{g(i)}{g(1)} p_1 (1+\varepsilon)\right) - \pi\left(\frac{g(i)}{g(1)} p_1\right) > k \qquad \text{for all} \qquad i = 1, \ldots, k,$$

which holds for large p_1 by the Prime Number Theorem. Impose that

$$n + i \equiv p_i^2 \pmod{p_i^3} \qquad i = 1, \ldots, k.$$

This puts n into a certain progression modulo $M := K^2(\prod_{i=1}^k p_i)^3$. Say the progression is $n = N_0 + M\ell$, where N_0 is the smallest positive integer in that progression. Let x be large such that

$$\log x > 12 P \log P, \quad \text{where} \quad P := \max\{p_1, p_2, \ldots, p_k\}.$$

Note that

$$M = K^2 \left(\prod_{i=1}^k p_i\right)^3 < (2k+1)^{4k+2} (p_1 \cdots p_k)^3 < P^{2P} \cdot P^{3k} < P^{4P} < x^{1/3}.$$

Thus, the number of such $n \le x$ is $\ge \lfloor x/M \rfloor - 1$. We claim that a positive proportion of them have n_i/p_i^2 square-free. Indeed, if not, n_i cannot be divisible by squares of primes $p \le 2k+1$, so it must be the case that $p^2 \mid n_i$ for some $p > 2k+1$ and $p \ne p_i$. Clearly, $p \ne p_j$ for some $j \ne i$, otherwise p divides both $n+i$ and $n+j$, so their difference $0 < |j-i| < k < p$, a contradiction. If $p \le \sqrt{x/M}$, this puts n into an arithmetic progression of ratio $Mp^2 < x$, so the number of such $n \le x$ is at most

$$\frac{x}{Mp^2} + O(1). \tag{3.1}$$

If $p > \sqrt{x/M}$, then this puts $N_0 + i + M\ell$ into an arithmetic progression modulo p^2, and the number of such possibilities is $O(1)$. Thus, the number of such possibilities is at most what is shown in (3.1) independently of p, and only $p > 2k+1$ is possible. Summing this up over all $p \le x^{1/2}$, and over all $i = 1, \ldots, k$, we get that number of such possibilities is

$$\le \frac{kx}{M} \sum_{p \ge 2k+3} \frac{1}{p^2} + O(k\sqrt{x}).$$

The first sum is at most

$$\frac{kx}{M} \sum_{m \ge 2k+3} \frac{1}{m^2} < \frac{kx}{M} \sum_{m \ge 2k+3} \frac{1}{m(m-1)} = \frac{kx}{2M(k+1)}.$$

Since $M \ll x^{1/3}$, it follows that $x/M \gg x^{2/3}$, so that $k\sqrt{x} = o(x/M)$. Thus, for large x the number of such $n \le x$ is at most

$$\frac{x}{M}\left(\frac{k}{2k+2} + o(1)\right) < \frac{x}{2M}.$$

It follows that for large x there are

$$\left\lfloor \frac{x}{M} \right\rfloor - 1 - \frac{x}{2M} > \frac{x}{3M}$$

such positive integers n for which n_i/p_i^2 is squarefree. Now if $f = \gamma$, we have that

$$f(n+i) = f(i)f(n_i) = f(i)p_i\left(\frac{n+i}{ip_i^2}\right) = \frac{f(i)}{ip_i} n(1+o(1)) = \frac{g(i)}{p_i} n(1+o(1))$$

as $x \to \infty$ for $i = 1, \ldots, k$. Since

$$\frac{g(i)}{p_i} \in \left[\frac{g(1)}{p_1}(1+\varepsilon)^{-1}, \frac{g(1)}{p_1}\right],$$

it follows that

$$\frac{\max\{f(n+1), \ldots, f(n+k)\}}{\min\{f(n+1), \ldots, f(n+k)\}} \in [1+o(1), 1+\varepsilon+o(1)]$$

as $x \to \infty$. Now we make ε go to zero and x go to infinity and get the desired result.

4. The construction for $\phi(n)$, $\sigma(n)$

For this, we use the fact that $\phi(a)/a$ is dense in $[0,1]$ and the same is true for $a/\sigma(a)$. To adapt the previous construction we choose again $n \equiv K \pmod{K^2}$ and such that additionally $n+i \equiv a_i \pmod{a_i^2}$, where $a_1, \ldots, a_k$ are mutually coprime positive integers, divisible only with primes $> 2k+1$ and such that, for each $i = 1, \ldots, k$,

$$g(i)\frac{f(a_i)}{a_i} \in \left(g(1)\frac{f(a_1)}{a_1}, g(1)\left(\frac{f(a_1)}{a_1}\right)(1+\varepsilon)\right). \tag{4.1}$$

This is possible by the denseness of $\phi(a)/a$ and $a/\sigma(a)$ in $[0,1]$ even if a is required to be coprime to primes from a fixed finite set. To find such numbers we can start with $a_1 = 1$, then use the denseness of $\phi(a)/a$ or $a/\sigma(a)$ to find a suitable a_2 coprime to K with the property (4.1), then use the denseness of $\phi(a)/a$ or $a/\sigma(a)$ to find a_3 coprime to Ka_2 with property (4.1), and so on. Then the proof goes as in the preceding case except that instead of taking $n+i \equiv p_i^2 \pmod{p_i^3}$, we take $n+i \equiv a_i \pmod{a_i^2}$. Furthermore, take $Q = \max\{P(a_i) : 1 \le i \le k\}$. We set $P = 0.1 \log x/\log\log x$ and ask of x to be such that $P > Q$. Then the inequality $x > 9P\log P$ is satisfied for large x. We also let $\mathcal{Q}$ be the set of primes $\le P$ not dividing $K\prod_{i=1}^k a_i$ and ask of all $p \in \mathcal{Q}$ to divide n. Thus, the progression for our Chinese Remainder Theorem has modulus

$$M = K^2\left(\prod_{i=1}^k a_i\right)^2 \prod_{q\in\mathcal{Q}} q \le K^2\left(\prod_{2k+1\le q\le P} q\right)^2 < P^{2P}e^{2.5P} < P^{3P} < x$$

for large x. Here, we used the Prime Number Theorem under the form

$$\prod_{p\le y} p = e^{(1+o(1))y} \qquad \text{as} \qquad y \to \infty. \tag{4.2}$$

By multiplicativity,

$$f(n+i) = f(i)f(a_i)f\left(\frac{n+i}{ia_i}\right).$$

We can show that

$$f\left(\frac{n+i}{ia_i}\right) = \frac{n}{ia_i}(1+o(1)) \tag{4.3}$$

as $x \to \infty$. Indeed, this is due to the fact that $(n+i)/(ia_i) := m_i$ is a number of size $\le x$ which has no prime factors below $P = 0.1\log x/\log\log x$. Since $n \le x$, n has at most $2\log x/\log\log x$ distinct prime primes in total for large x (again by the Prime Number Theorem (4.2)) and so

$$\begin{aligned}\frac{f(m_i)}{m_i} &= \prod_{p\mid m_i}\left(1+O\left(\frac{1}{p}\right)\right) = \exp\left(O\left(\sum_{p\mid m_i}\frac{1}{p}\right)\right)\\ &= \exp\left(O\left(\sum_{c_1\log x/\log\log x<p<c_2\log x/\log\log x}\frac{1}{p}\right)\right)\\ &= \exp(o(1)) = 1+o(1),\end{aligned}$$

as

$$\sum_{c_1 \log x/\log\log x < p < c_2 \log x/\log\log x} \frac{1}{p} = o(1) \qquad \text{for} \qquad x \to \infty.$$

Here, $c_1 = 0.1 < c_2 = 2$. We have thus proved (4.3) and therefore established that

$$\frac{\max\{f(n), \ldots, f(n+k-1)\}}{\min\{f(n), \ldots, f(n+k-1)\}} \in [1 + o(1), 1 + \varepsilon + o(1)],$$

which completes the proof of this case by making ε tend to zero and x tend to infinity.

Added after acceptance. We have just realized that in the case $f(n) = \gamma(n)$, the fact that the associated f_k satisfies $f_k = 1$ for all $k \geq 1$ is actually a consequence of the main result in a 12-year-old paper by F. Luca and I. Shparlinski "Approximating positive reals by ratios of consecutive integers" in Diophantine analysis and related fields 2006, 141–149, Sem. Math. Sci., 35, Keio Univ., Yokohama, 2006. Their proof follows a different approach and applies only to the $\gamma(n)$ function.

Acknowledgements. This research was done while the second author was visiting Université Laval in the summer of 2018.

References

[1] J.M. De Koninck and F. Luca, *Arithmetic functions monotonic at consecutive arguments*, Studia Scientiarum Mathematicarum Hungarica, **51** (2014), No.2, 155–164.

[2] J.M. De Koninck, J. Friedlander and F. Luca, *On Strings of Consecutive Integers with a Distinct Number of Prime Factors*, Proc. Amer. Math. Soc. **137** (2009), no. 5, 1585–1592.

[3] J.M. De Koninck and F. Luca, *Analytic Number Theory: Exploring the Anatomy of Integers*, Graduate Studies in Mathematics, Vol. 134, American Mathematical Society, 2012

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