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ON THE DISTRIBUTION OF THE NUMBER OF PRIME FACTORS OF THE k-FOLD ITERATE OF VARIOUS ARITHMETIC FUNCTIONS

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Dedicated to the memory of Professor Antal Iványi

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Abstract. Given an arithmetic function $f : \mathbb{N} \to \mathbb{N}$, let the k-fold iterate of f be defined by $f_0(n) = n$ and $f_k(n) = f(f_{k-1}(n))$ for each integer $k \geq 1$. Let $\omega(1) = 0$ and, for each integer $n \geq 2$, let $\omega(n)$ stand for the number of distinct prime factors of n. Here, we examine the distribution of the functions $\omega(f_k(n))$ for various arithmetic functions f.

1. Introduction and notation

Given an arithmetic function $f : \mathbb{N} \to \mathbb{N}$, let us consider the k-fold iterate of the function f by setting $f_0(n) = n$ and $f_k(n) = f(f_{k-1}(n))$ for each integer $k \geq 1$. Let $\sigma(n)$ stand for the sum of the positive divisors of n, let ϕ stand for the Euler totient function, let $\psi(n)$ stand for the Dedekind function defined by $\psi(n) := n \prod_{p|n} \left(1 + \frac{1}{p}\right)$ and, for each fixed integer $\ell \neq 0$, let $\psi^{(\ell)}(n) :=$ $:= n \prod_{p|n}(p + \ell)$. Moreover, let $\omega(n)$ stand for the number of distinct prime factors of the integer $n \geq 2$ with $\omega(1) = 0$.

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We denote by $p(n)$ and $P(n)$ the smallest and largest prime factors of n, respectively. The letters p, q, π, Q, with or without subscript, will stand exclusively for primes. In fact, we let $\wp$ stand for the set of all primes. On the other hand, the letters c and C, with or without subscript, will stand for absolute constants but not necessarily the same at each occurrence. Moreover, we shall use the abbreviations $x_1 = \log x$, $x_2 = \log\log x$, and so on. Also, given any real number $x \geq 1$, we let $\mathcal{N}_x = \{1, 2, \ldots, \lfloor x \rfloor\}$. The set $\mathcal{M}$ denotes the set of multiplicative functions, while $\mathcal{M}^*$ stands for the set of strongly multiplicative functions. Finally, we let

$$\Phi(z) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u^2/2}\, du \tag{1.1}$$

stand for the standard Gaussian law.

We further set, for each integer $k \geq 0$,

$$a_k = \frac{1}{(k+1)!}, \qquad b_k = \frac{1}{k!\sqrt{2k+1}}, \qquad \text{and} \qquad s_k(n \mid x) = \frac{\omega(n) - a_k x_2^{k+1}}{b_k x_2^{k+1/2}}.$$

In [2], we proved the following.

Theorem A. *For each $k \in \mathbb{N}$ and every $z \in \mathbb{R}$,*

$$\lim_{x\to\infty} \frac{1}{x} \#\{n \leq x : s_k(\phi_k(n) \mid x) < z\} \quad = \quad \Phi(z).$$

Let $\theta \in \mathcal{M}^*$ be defined on primes p by $\theta(p) = p - 1$ and, for each integer $k \geq 0$, consider the strongly additive function $\tau_k(n)$ defined recursively by $\tau_0(p) = 1$ and $\tau_k(p) = \sum_{q|p-1} \tau_{k-1}(q)$ for each integer $k \geq 1$.

Our proof of Theorem A was essentially based on the inequalities

$$\omega(\theta_k(n)) \leq \omega(\phi_k(n)) \leq \omega(n) + \omega(\theta(n)) + \cdots + \omega(\theta_k(n))$$

and the fact that $\omega(\theta_k(n))$ can be approximated by $\tau_k(n)$. In fact, Theorem A was deduced by the following result.

Theorem B. *For each $k \in \mathbb{N}$ and every $z \in \mathbb{R}$,*

$$\lim_{x\to\infty} \frac{1}{x} \# \left\{ n \leq x : \frac{\tau_k(n) - a_k x_2^{k+1}}{b_k x_2^{k+1/2}} < z \right\} \quad = \quad \Phi(z).$$

Given a non zero integer ℓ such that $-\ell \notin \wp$, let $\theta^{(\ell)} \in \mathcal{M}^*$ be defined on the primes p by $\theta^{(\ell)}(p) = p + \ell$ and let $\theta_k^{(\ell)}(n)$ be the k-fold iterate of $\theta^{(\ell)}(n)$. Moreover, let $\tau_k^{(\ell)}$ be the strongly additive function defined recursively on primes p by $\tau_0^{(\ell)}(p) = 1$ and $\tau_k^{(\ell)}(p) = \sum_{q|p+\ell} \tau_{k-1}(q)$ for each integer $k \geq 1$.

Here, we examine how the above theorems can be generalized to the distribution of the functions $\omega(\theta_k^{(\ell)}(n))$, $\omega(\tau_k^{(\ell)}(n))$, $\omega(\psi_k^{(\ell)}(n))$ and $\omega(\sigma_k(n))$.

2. Main results

Theorem 1. *For each $k \in \mathbb{N}$, $\ell \in \mathbb{Z} \setminus \{0\}$ such that $-\ell \notin \wp$ and every $z \in \mathbb{R}$,*

$$\lim_{x\to\infty} \frac{1}{x} \# \left\{ n \leq x : \frac{\omega(\theta_k^{(\ell)}(n)) - a_k x_2^{k+1}}{b_k x_2^{k+1/2}} < z \right\} \quad = \quad \Phi(z).$$

Theorem 2. *For each $k \in \mathbb{N}$, $\ell \in \mathbb{Z} \setminus \{0\}$ such that $-\ell \notin \wp$ and every $z \in \mathbb{R}$,*

$$\lim_{x\to\infty} \frac{1}{x} \# \left\{ n \leq x : \frac{\omega(\tau_k^{(\ell)}(n)) - a_k x_2^{k+1}}{b_k x_2^{k+1/2}} < z \right\} \quad = \quad \Phi(z).$$

Theorem 3. *For each $k \in \mathbb{N}$, $\ell \in \mathbb{Z} \setminus \{0\}$ such that $-\ell \notin \wp$ and every $z \in \mathbb{R}$,*

$$\lim_{x\to\infty} \frac{1}{x} \# \left\{ n \leq x : \frac{\omega(\psi_k^{(\ell)}(n)) - a_k x_2^{k+1}}{b_k x_2^{k+1/2}} < z \right\} \quad = \quad \Phi(z).$$

Theorem 4. *For each $k \in \mathbb{N}$ and every $z \in \mathbb{R}$,*

$$\lim_{x\to\infty} \frac{1}{x} \# \left\{ n \leq x : \frac{\omega(\sigma_k(n)) - a_k x_2^{k+1}}{b_k x_2^{k+1/2}} < z \right\} \quad = \quad \Phi(z).$$

3. Preliminary lemmas

Lemma 1. *For all integers $k \geq 1$ and ℓ, let*

$$\delta(x, k, \ell) := \sum_{\substack{p \leq x \\ p \equiv \ell \pmod{k}}} \frac{1}{p}.$$

Then, for $\ell = 1$ or -1, $k \le x$, and $x \ge 3$, we have

$$\delta(x,k,\ell) \le \frac{C_1 x_2}{\phi(k)},$$

where $C_1 > 0$ is an absolute constant.

Proof. This is Lemma 2.5 in Bassily, Kátai and Wisjmuller [1]. ■

We say that a $k+1$-tuple of primes $(q_0, q_1, \ldots, q_k)$ is a *k-chain* if $q_{i-1} \mid q_i + 1$ for $i = 1, 2, \ldots, k$, in which case we write $q_0 \to q_1 \to \cdots \to q_k$. We then have the following obvious result.

Lemma 2. *For any fixed prime q_0 and integer $k \ge 1$, there exist absolute constants $c_1, c_2, \ldots, c_k$ such that*

$$\sum_{\substack{q_0 \to q_1 \\ q_1 \le x}} \frac{1}{q_1} \le \frac{c_1 x_2}{q_0}, \qquad \sum_{\substack{q_0 \to q_1 \to q_2 \\ q_2 \le x}} \frac{1}{q_2} \le \frac{c_2 x_2^2}{q_0}, \quad \ldots \quad , \sum_{\substack{q_0 \to q_1 \to \cdots \to q_k \\ q_k \le x}} \frac{1}{q_k} \le \frac{c_k x_2^k}{q_0}.$$

4. Proof of the Theorems

Using essentially the same techniques as those we used in [2] to establish Theorems A and B, it is somewhat easy to prove Theorems 1, 2 and 3. However, Theorem 4 needs more attention. Hence, here we shall provide a detailed proof of Theorem 4.

The general idea is to write, for all $n \le x$ (except possibly for at most $o(x)$ integers $n \le x$ which we can ignore),

$$\sigma_k(n) = A_k(n) B_k(n), \tag{4.1}$$

where $(A_k(n), B_k(n)) = 1$, $B_k(n)$ is squarefree and $p(B_k(n)) > x_2^{2k}$.

We first consider the cases $k = 1$ and $k = 2$.

Let $\mathcal{N}_x := \{1, 2, \ldots, \lfloor x \rfloor\}$. Let Y_x be a function which tends to infinity with x but slowly enough to satisfy $Y_x \le x_5$, say.

We then write each positive integer $n \le x$ as

$$n = A_0(n) B_0(n), \tag{4.2}$$

where $P(A_0(n)) \le Y_x$ and $p(B_0(n)) > Y_x$. Setting

$$\mathcal{U}_x^{(0)} := \{n \in \mathcal{N}_x : A_0(n) > Y_x^{Y_x} \text{ or } \mu(B_0(n)) = 0\},$$

it is clear $\#\mathcal{U}_x^{(0)} = o(x)$ as $x \to \infty$. This is why we set

$$\mathcal{N}_x^{(1)} := \mathcal{N}_x \setminus \mathcal{U}_x^{(0)}$$

and from here on we work only with $\mathcal{N}_x^{(1)}$.

In light of (4.2), we then have

$$\sigma(n) = \sigma(A_0(n))\sigma(B_0(n)). \tag{4.3}$$

To each prime number q, we associate the strongly additive function f_q defined on primes p by

$$f_q(p) = \begin{cases} k & \text{if} \quad q^k \| p+1, \\ 0 & \text{if} \quad q \nmid p+1. \end{cases}$$

Using this definition of f_q, we can write

$$\sigma(B_0(n)) = \prod_{q \le x_2^2} q^{f_q(B_0(n))} \cdot \prod_{\substack{q > x_2^2 \\ q^{\gamma_q} \| \sigma(n)}} q^{\gamma_q} = s(n) \cdot B_1(n), \tag{4.4}$$

say.

Observe that, in light of Lemma 1,

$$\begin{aligned} \sum_{n \in \mathcal{N}_x^{(1)}} \sum_{q \le x_2^2} (\log q) f_q(B_0(n)) &\le \sum_{q \le x_2^2} (\log q) \sum_{q^k \le x} \sum_{q^k | p+1} \frac{x}{p} \le \\ &\le Cxx_2 \sum_{\substack{q \le x_2^2 \\ q^k \le x}} \frac{\log q}{\phi(q^k)} \le C_1 x x_2 x_3 \end{aligned} \tag{4.5}$$

and that, from Lemma 2,

$$\begin{aligned} \sum_{n \in \mathcal{N}_x^{(1)}} \sum_{\substack{q^2 | \sigma(n) \\ q > x_2^2}} 1 \quad &\le \quad \sum_{q > x_2^2} \sum_{\substack{p_1 p_2 \le x \\ q \to p_1 \\ q \to p_2 \\ p_1 \ne p_2}} \left\lfloor \frac{x}{p_1 p_2} \right\rfloor \le \\ &\le \quad Cxx_2^2 \sum_{q > x_2^2} \frac{1}{q^2} \le cxx_2^2 \frac{1}{x_2^2 x_3} = c\frac{x}{x_3}. \end{aligned} \tag{4.6}$$

Hence, letting

$$\begin{aligned} \mathcal{U}_x^{(1)} &= \{n \in \mathcal{N}_x^{(1)} : s(n) > x_2 x_3^2\}, \\ \mathcal{U}_x^{(2)} &= \{n \in \mathcal{N}_x^{(1)} : q^2 \mid \sigma(n) \text{ for some } q > x_2^2\}, \end{aligned}$$

it follows from (4.5) and (4.6) that

$$\#\left(\mathcal{U}_x^{(1)} \cup \mathcal{U}_x^{(2)}\right) = o(x) \qquad (x \to \infty)$$

and this why we set

$$\mathcal{N}_x^{(2)} := \mathcal{N}_x^{(1)} \setminus \left(\mathcal{U}_x^{(1)} \cup \mathcal{U}_x^{(2)}\right)$$

and from here on we work only with $\mathcal{N}_x^{(2)}$.

Now, for $n \in \mathcal{N}_x^{(2)}$, in light of (4.3) and (4.4), we may write

$$\sigma(n) = A_1(n)B_1(n), \tag{4.7}$$

where $(A_1(n), B_1(n)) = 1$, $A_1(n) = \sigma(A_0(n))s(n)$ and $B_1(n)$ is squarefree.

Observe that, for $n \in \mathcal{N}_x^{(2)}$, we have

$$\omega(A_1(n)) \le \log \sigma(A_0(n)) + \log s(n) \le x_2 x_3 x_4, \tag{4.8}$$

say. Thus it follows from (4.7) and (4.8) that

$$\omega(\sigma(n)) - \omega(B_1(n)) = \omega(A_1(n)) = O(x_2 x_3 x_4) \qquad (n \in \mathcal{N}_x^{(2)}). \tag{4.9}$$

Now, by the definition of $B_1(n)$, we may write that

$$\sigma(B_1(n)) = \prod_{\substack{q > x_2^2 \\ q \mid B_1(n)}} (q+1) = U(n) \cdot V(n), \tag{4.10}$$

where

$$U(n) = \prod_{\pi < x_2^4} \pi^{f_\pi(\sigma(B_1(n)))}, \qquad V(n) = \prod_{\substack{\pi^{\gamma_\pi} \| \sigma(B_1(n)) \\ \pi \ge x_2^4}} \pi^{\gamma_\pi}.$$

Using Lemma 2, we have

$$\begin{aligned} \sum_{n \in \mathcal{N}_x^{(2)}} \sum_{\pi < x_2^4} f_\pi(\sigma(B_1(n))) \log \pi &\le \sum_{\pi \le x_2^4} (\log \pi) \sum_{\pi \to p_1 \to p_2} f_\pi(p_1) \left\lfloor \frac{x}{p_2} \right\rfloor \le \\ &\le c_1 x \sum_{\substack{\pi \le x_2^4 \\ \pi \to p_1}} (\log \pi) \frac{x_2}{p_1} f_\pi(p_1) \le \\ &\le c_2 x x_2^2 \sum_{\pi \le x_2^4} \frac{\log \pi}{\pi} \le c_3 x x_2^2 x_3 \end{aligned} \tag{4.11}$$

and

$$\begin{aligned} \sum_{n \in \mathcal{N}_x^{(2)}} \sum_{\substack{\pi^2 \mid \sigma(B_1(n)) \\ \pi > x_2^4}} 1 &\le \sum_{\substack{\pi \to p_1 \to p_2 \\ \pi \to Q_1 \to Q_2 \\ p_2 Q_2 \le x}} \frac{x}{p_2 Q_2} \le \\ &\le c x \sum_{\pi > x_2^4} \frac{x_2^4}{\pi^2} \le c \frac{x}{x_3}. \end{aligned} \tag{4.12}$$

It follows from (4.11) and (4.12) that by dropping no more than $o(x)$ integers $n \in \mathcal{N}_x^{(2)}$, say belonging to a set $\mathcal{U}_x^{(3)}$ of size $o(x)$, we may now work with the new set $\mathcal{N}_x^{(3)} := \mathcal{N}_x^{(2)} \setminus \mathcal{U}_x^{(3)}$. In other words, we may now assume that

$$U(n) \le \exp\{x_2^2 x_3 x_5\}, \quad V(n) \text{ is squarefree and } (U(n), V(n)) = 1 \quad (n \in \mathcal{N}_x^{(3)}).$$

Introducing the function

$$V^*(n) := \prod_{\substack{\pi > x_2^4 \\ \pi \mid \sigma(B_1(n)) \\ \pi \nmid \sigma(A_1(n))}} \pi,$$

we can now set

$$A_2(n) = \sigma(A_1(n)) U(n) \prod_{\substack{\pi > x_2^4 \\ \pi \mid (\sigma(B_1(n)), \sigma(A_1(n)))}} \pi, \qquad B_2(n) = V^*(n)$$

and have

$$\sigma_2(n) = A_2(n) B_2(n),$$
$$\text{with } (A_2(n), B_2(n)) = 1, \ B_2(n) \ \text{ squarefree } (n \in \mathcal{N}_x^{(3)}).$$

From this it follows that

$$\begin{aligned} &\omega(\sigma_2(n)) - \omega(B_2(n)) = \\ &= \omega(A_2(n)) \le 2 \log A_1(n) + \log U(n) = O(x_2^2 x_3 x_4). \end{aligned} \tag{4.13}$$

Continuing in this manner, we then write

$$\sigma(B_2(N)) = \prod_{q \le x_2^6} q^{f_q(\sigma(B_2(n)))} \cdot \prod_{q > x_2^6} q^{f_q(\sigma(B_2(n)))} = L(n) T(n),$$

say, with clearly $(L(n), T(n)) = 1$. Hence, proceeding as above, we observe that

$$\sum_{n \le x} \sum_{q \le x_2^6} f_q(\sigma(B_2(n))) \log q \le \sum_{q \le x_2^6} (\log q) \sum_{q \to p_1 \to p_2 \to p_3} f_q(p_1) \left\lfloor \frac{x}{p_3} \right\rfloor \le c x x_2^3 x_3$$

and that $f_q(\sigma(B_2(n)) = 1$ or 0 for every prime $q > x_2^6$ and therefore that $\log L(n) \le x_2^3 x_3 x_4$ with the possible exception of some positive integers n belonging to a set $\mathcal{U}_x^{(3)}$ of size at most $o(x)$. Hence, from here on we only need to consider those integers $n \in \mathcal{N}_x^{(4)} := \mathcal{N}_x^{(3)} \setminus \mathcal{U}_x^{(3)}$. Therefore, for $n \in \mathcal{N}_x^{(4)}$, we let

$$B_3(n) = \prod_{\substack{q > x_2^6 \\ q \mid \sigma(B_2(n)) \\ q \nmid \sigma(A_2(n))}} q,$$
$$A_3(n) = \sigma(A_2(n)) L(n) \prod_{\substack{q > x_2^6 \\ q \mid (\sigma(B_2(n)), \sigma(A_2(n)))}} q.$$

Again, with this set up, we have

$$\sigma_3(n) = A_3(n) B_3(n),$$
$$\text{with } (A_3(n), B_3(n)) = 1, \quad B_3(n) \text{ squarefree} \qquad (n \in \mathcal{N}_x^{(4)})$$

and similarly as before

$$\omega(\sigma_3(n)) - \omega(B_3(n)) = \omega(A_3(n)) \le 3 \log A_2(n) + \log L(n) = O(x_2^3 x_3 x_4). \tag{4.14}$$

Pursuing in this matter, one is able to show that for every positive integer k, we have

$$\sigma_k(n) = A_k(n) B_k(n), \quad \text{with } (A_k(n), B_k(n)) = 1 \qquad (n \in \mathcal{N}_x^{(k+1)})$$

where

$$B_k(n) = \prod_{\substack{\pi > x_2^{2k} \\ \pi \mid \sigma(B_{k-1}(n)) \\ \pi \nmid \sigma(A_{k-1}(n))}} \pi$$

is squarefree and, as with (4.9), (4.13) and (4.14),

$$\omega(\sigma_k(n)) - \omega(B_k(n)) = \omega(A_k(n)) = O(x_2^k x_3 x_4). \tag{4.15}$$

From estimate (4.15), it follows that $\omega(\sigma_k(n))$ will be of the same order as $\omega(B_k(n))$ and therefore that in order to prove Theorem 4, we only need to prove the following.

Theorem 4a. *For every fixed $k \in \mathbb{N}$ and real z,*

$$\lim_{x \to \infty} \frac{1}{x} \# \left\{ n \le x : \frac{\omega(B_k(n)) - a_k x_2^{k+1}}{b_k x_2^{k+1/2}} < z \right\} = \Phi(z).$$

5. Proof of Theorem 4a

We will be using the same arguments as in [2] along with the same lemmas, but by modifying the strongly multiplicative functions $\theta(n)$ and $\tau_k(n)$ introduced in Section 1, namely by defining them on prime numbers p by $\theta(p) = p+1$ and $\tau_0(p) = 1$ and thereafter by $\tau_k(p) = \sum_{q|p+1} \tau_{k-1}(q)$. In the same spirit, we now define a *k-chain* as a $k+1$-tuple of primes $q_0, q_1, \dots, q_k$ which is such that $q_{i-1} \mid q_i + 1$ for $i = 1, 2, \dots, k$. A general k-chain is denoted by Q_k. On the other hand, a k-chain with the property that $q_k \mid n$ is denoted by $Q_k(n)$, while $Q_k(n, q_0)$ denotes those k-chains where q_0 is fixed and $q_k \mid n$.

With these adapted concepts, we can use the same techniques that we developed in [2] to obtain the following.

Proposition 1. *For each $k \in \mathbb{N}$ and every $z \in \mathbb{R}$,*

$$\lim_{x\to\infty} \frac{1}{x} \# \left\{ n \le x : \frac{\tau_k(n) - a_k x_2^{k+1}}{b_k x_2^{k+1/2}} < z \right\} \quad = \quad \Phi(z).$$

Then, repeating the same argument that we used in Section 4, we obtain the following.

Proposition 2. *For each $k \in \mathbb{N}$ and every $z \in \mathbb{R}$,*

$$\lim_{x\to\infty} \frac{1}{x} \# \left\{ n \le x : \frac{\omega(\theta_k(n)) - a_k x_2^{k+1}}{b_k x_2^{k+1/2}} < z \right\} \quad = \quad \Phi(z).$$

Now, letting $y = x_1^2$ and proceeding as in Lemma 5.1 of [1], we have that

$$\begin{aligned} \sum_{n \le x} \sum_{\substack{q_0 \le y \\ q_0 | \theta_k(n)}} |Q_k(n, q_0)| \quad &\le \quad x \sum_{q_0 \le y} \sum_{q_1} \cdots \sum_{q_k} \frac{1}{q_k} \ll \\ &\ll \quad x(Cx_2)^k (\log\log y) \ll x(Cx_2)^k x_3), \end{aligned} \tag{5.1}$$

where we made repetitive use of Lemma 2.

Let $\theta^{(y)}$ be the strongly multiplicative function defined on primes p by

$$\theta^{(y)}(p) = \begin{cases} p+1 & \text{if} \quad p > y, \\ 1 & \text{if} \quad p \le y. \end{cases}$$

As usual the function $\theta_\ell^{(y)}$ stands for the ℓ-fold iterate of the function $\theta^{(y)}$.

It follows from (5.1) that

$$0 \le \omega(\theta_k(n)) - \omega(\theta_k^{(y)}(n)) \le x_2^k x_3 x_4 \tag{5.2}$$

for all but at most $o(x)$ integers $n \le x$.

The following result then follows from (5.2).

Proposition 3. *For each $k \in \mathbb{N}$ and every $z \in \mathbb{R}$, for $y = y(x) = x_1^2$, we have*

$$\lim_{x\to\infty} \frac{1}{x} \# \left\{ n \le x : \frac{\omega(\theta_k^{(y)}(n)) - a_k x_2^{k+1}}{b_k x_2^{k+1/2}} < z \right\} \quad = \quad \Phi(z).$$

Now, it is obvious that if $n \in \mathcal{N}_k(x)$, we have $B_j(n) \mid \theta_j(n)$ for $j = 0, 1, \ldots, k$, from which it follows that

$$0 \le \omega(B_j(n)) \le \omega(\theta_j(n)) \qquad (n \in \mathcal{N}_k(x)).$$

Setting $\kappa^{(k)}(n) := \#\{p \in \wp : p \mid \theta_k^{(y)}(n) \text{ and } p \nmid B_k(n)\}$, it is enough to prove that

$$\frac{1}{x} \sum_{n \in \mathcal{N}_x^{(k)}} \kappa^{(k)}(n) = o\left(x_2^{k+1/2}\right) \qquad (x \to \infty). \tag{5.3}$$

Before moving on, we introduce a new concept. Given a k-chain of primes $(q_0, q_1, \ldots, q_k)$, we shall say that q_0 is a *bad prime* if $q_0 \mid \theta_k^{(y)}(n)$ while $q_0 \nmid B_k(n)$, that q_1 is a *bad prime* if $q_1 \mid \theta_{k-1}^{(y)}(n)$ while $q_1 \nmid B_{k-1}(n)$, and so on for the other primes $q_2, \ldots, q_k$ of the k-chain. Moreover, we will say that $Q_k(n, q_0)$ is a *bad chain* if at least one of the q_i's in $q_0 \to q_1 \to \cdots \to q_k$ is a bad prime.

Now, it is obvious that

$$L := \sum_{n \le x} \kappa^{(k)}(n) \le \sum_{n \le x} \sum_{q_0 \ge y} Q_k^*(n, q_0), \tag{5.4}$$

where $Q_k^*(n, q_0)$ runs over the bad k-chains. We then have

$$L \le \sum_{j=0}^{k} \sum_{\substack{q_0 \ge y \\ q_j \text{ bad}}} Q_k^*(n, q_0) = \sum_{j=0}^{k} T_j, \tag{5.5}$$

say, where in T_j, q_j stands for the smallest prime which is a bad prime.

Observe that, by Lemma 2,

$$T_0 \leq \sum_{\substack{q_0 \to \cdots \to q_k \\ q_0 \text{ bad}}} \frac{x}{q_k} \leq x x_2^k \sum_{q_0 \text{ bad}} \frac{1}{q_0}. \tag{5.6}$$

Since the number of such $q_0 \leq x$ is less than $C x_2^k x_4$, it follows that, if $p_1 < p_2 < \cdots$ stand for the primes in increasing order,

$$\sum_{q_0 \text{ bad}} \frac{1}{q_0} \leq \sum_{j \leq C x_2^k x_4} \frac{1}{p_j} \ll x_4.$$

Using this estimate in (5.6), we obtain that

$$T_0 \ll x x_2^k x_4. \tag{5.7}$$

On the other hand, we have

$$T_1 \leq \sum_{\substack{q_0 \to q_1 \to \cdots \to q_k \\ q_1 \text{ bad}}} \frac{x}{q_k} \leq x x_2^{k-1} \sum_{\substack{q_0 \to q_1 \\ q_1 \text{ bad}}} \frac{1}{q_1} \leq x x_2^{k-1} \leq \sum_{\substack{q_1 \leq x \\ q_1 \text{ bad}}} \frac{\tau_1(q_1+1)}{q_1}. \tag{5.8}$$

Now, it was shown in [1] that

$$\sum_{p \leq x} \frac{\tau_j(p)}{p} = \frac{1}{j+1} x_2^{j+1} + O(x_2^j). \tag{5.9}$$

Using this estimate with $j = 1$, we obtain that

$$\begin{aligned} \sum_{\substack{q_1 \leq x \\ q_1 \text{ bad}}} \frac{\tau_1(q_1+1)}{q_1} &= \sum_{\substack{q_1 \leq x_1 \\ q_1 \text{ bad}}} \frac{\tau_1(q_1+1)}{q_1} + \sum_{\substack{x_1 < q_1 \leq x \\ q_1 \text{ bad}}} \frac{\tau_1(q_1+1)}{q_1} \ll \\ &\ll x_3^2 + \max_{x_1 < q \leq x} \frac{\tau_1(q+1)}{q} \cdot x_2^{k-1} x_4 \ll \\ &\ll x_3^2 + \frac{1}{\sqrt{x_1}} x_2^{k-1} x_4. \end{aligned}$$

More generally, using (5.9), we obtain that

$$\begin{aligned}
T_j &\le \sum_{\substack{q_0\to q_1\to\cdots\to q_k\\ q_j \text{ bad}}} \frac{x}{q_k} \le x x_2^{k-1} \sum_{\substack{q_0\to q_1\to\cdots\to q_k\\ q_j \text{ bad}}} \frac{1}{q_j} \le \\
&\le x x_2^{k-j} \sum_{q_j \text{ bad}} \frac{\tau_j(q_j+1)}{q_j} \le \\
&\le x x_2^{k-j} \sum_{q\le x_1} \frac{\tau_j(q+1)}{q} + x x_2^{k-j} x_2^j x_3 \max_{q>x_1} \frac{\tau_j(q+1)}{q} \le \\
&\le x x_2^{k-j} x_2^{j+1} + x x_2^k x_3 \le \\
&\ll x x_2^{k+1}.
\end{aligned}$$

It follows that (5.3) holds and consequently that

$$\left|\omega(\theta_k^{(y)}(n)) - \omega(\sigma_k(n))\right| \le x_2^k x_3,$$

thus completing the proof of Theorem 4a and thereby of Theorem 4 as well.

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