On the distribution of the difference of some arithmetic functions

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Abstract

Let φ stand for the Euler totient function, $\sigma(n)$ for the sum of the positive divisors of n and $\omega(n)$ for the number of distinct prime factors of n. Then, consider the functions $\ell_1(n) := \omega(\sigma(n)) - \omega(\varphi(n))$, $\ell_2(n) := \omega(\sigma(n+1)) - \omega(\sigma(n))$ and $\ell_3(n) := \omega(\varphi(n+1)) - \omega(\varphi(n))$. Here, we study the distribution of the functions $\ell_1(n)$, $\ell_2(n)$ and $\ell_3(n)$, as well as that of various other related functions.

1 Introduction

In 1940, Paul Erdős and Mark Kac proved [5] what is now known as the Erdős-Kac theorem:

Let $f : \mathbb{N} \to \mathbb{R}$ be a real valued strongly additive function and set

$$A(x) := \sum_{p \le x} \frac{f(p)}{p} \qquad \text{and} \qquad B(x) := \sqrt{\sum_{p \le x} \frac{f^2(p)}{p}}.$$

Then, given any real number z,

$$\lim_{x\to\infty} \frac{1}{x} \#\{n \le x : f(n) \le A(x) + zB(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2}\,dt.$$

Here and in what follows, a *strongly additive function* $f : \mathbb{N} \to \mathbb{R}$ is an additive function such that $f(p^a) = f(p)$ for all primes p and all integers $a \ge 1$. In particular, setting $f(n) = \omega(n)$, we find that

$$\lim_{x\to\infty} \frac{1}{x} \#\{n \le x : \omega(n) \le \log\log x + z\sqrt{\log\log x}\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-t^2/2}\,dt.$$

In a sense, the Erdős-Kac result had shown that the primes behaved like independent random variables.

Generalisations were later obtained, namely the following. Given a real valued strongly additive function f, consider the two expressions

$$A(x) := \sum_{p \le x} \frac{f(p)}{p} \qquad \text{and} \qquad B^2(x) := \sum_{p \le x} \frac{f^2(p)}{p}.$$

We will say that f belongs to the class $\mathcal{H}$ if there exists a function $r(x)$ such that, as $x \to \infty$,

$$\frac{\log r(x)}{\log x} \to 0, \qquad \frac{B(r(x))}{B(x)} \to 1, \qquad B(x) \to \infty.$$

Then, we have the following result, often called the *Kubilius-Shapiro theorem*:

Let $f \in \mathcal{H}$. In order to have

$$\lim_{x\to\infty} \frac{1}{x} \# \left\{ n \le x : \frac{f(n) - A(x)}{B(x)} < z \right\} = \Phi(z),$$

it is necessary and sufficient that, for any fixed $\varepsilon > 0$,

$$\lim_{x\to\infty} \frac{1}{B^2(x)} \sum_{\substack{p \le x \\ |f(p)| > \varepsilon B(x)}} \frac{f^2(p)}{p} = 0.$$

This is Theorem 12.2 in the book of Elliott [3].

Here, we examine the distribution of some arithmetic functions evaluated at consecutive integers and others related to shifted primes.

More precisely, letting φ stand for the Euler totient function, $\sigma(n)$ for the sum of the positive divisors of n and $\omega(n)$ for the number of distinct prime factors of n, consider the functions $\ell_1(n) := \omega(\sigma(n)) - \omega(\varphi(n))$, $\ell_2(n) := \omega(\sigma(n+1)) - \omega(\sigma(n))$ and $\ell_3(n) := \omega(\varphi(n+1)) - \omega(\varphi(n))$. Here, assuming that the famous *Elliott-Halberstam Conjecture* (EHC) holds (see its statement in Section 3 below), we obtain the distribution function of each of the functions $\ell_1(n)$, $\ell_2(n)$ and $\ell_3(n)$, as well as that of the functions $a(n+1) - a(n)$ and $b(n+1) - b(n)$, where $a(n) := \sum_{p|n} \omega(p+1)$ and $b(n) := \sum_{p|n} \omega(p-1)$.

2 Main results

We let $\wp$ stand for the set of all primes and, from here on, the letters p, q and π, with or without subscript, will always represent prime numbers. Also, by $\log_2 x$ (resp. $\log_3 x$) we mean $\max(2, \log\log x)$ (resp. $\max(2, \log\log_2 x)$).

For each $p \in \wp$, letting

$$\Delta(p) := \omega(p+1) - \omega(p-1), \tag{2.1}$$

we introduce the two sums

$$\begin{aligned} E_1(x) &:= \sum_{p \le x} \Delta(p), \\ A(x) &:= \sum_{p \le x} \frac{\Delta(p)}{p}. \end{aligned}$$

We will be using the normal distribution function $\Phi(z) := \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{z} e^{-t^2/2}\,dt$.

Introducing the strongly additive function $\Delta(n)$ defined on primes p by (2.1), we then have the following result.

Theorem 1. *Assuming that the EHC holds, then*

$$\lim_{x\to\infty} \frac{1}{x} \# \left\{ n \le x : \frac{\Delta(n)}{\log_2 x} < z \right\} = \Phi(z).$$

Let $\theta^+(n)$ and $\theta^-(n)$ be the strongly multiplicative functions defined respectively on primes p by

$$\theta^+(p) = p + 1 \qquad \text{and} \qquad \theta^-(p) = p - 1.$$

Now, for each $n \in \mathbb{N}$, set

$$\rho(n) := \omega(\theta^+(n)) - \omega(\theta^-(n)).$$

Our second result is the following.

Theorem 2. *Assuming that the EHC holds, then*

$$\lim_{x\to\infty} \frac{1}{x} \# \left\{ n \le x : \frac{\rho(n)}{\log_2 x} < z \right\} = \Phi(z).$$

We also have the following.

Theorem 3. *Assuming that the EHC holds, then*

$$\lim_{x\to\infty} \frac{1}{x} \# \left\{ n \le x : \frac{\ell_1(n)}{\log_2 x} < z \right\} = \Phi(z).$$

Recalling the definitions of the functions $\ell_2(n)$, $\ell_3(n)$, $a(n)$ and $b(n)$ introduced in Section 1, and introducing the additional functions $u(n) := a(n+1) - a(n)$ and $v(n) := b(n+1) - b(n)$, our fourth result is as follows.

Theorem 4. *Unconditionally, we have*

(i) $$\lim_{x\to\infty} \frac{1}{x} \# \left\{ n \le x : \frac{u(n)}{\sqrt{2/3}(\log_2 x)^{3/2}} < z \right\} = \Phi(z),$$

(ii) $$\lim_{x\to\infty} \frac{1}{x} \# \left\{ n \le x : \frac{v(n)}{\sqrt{2/3}(\log_2 x)^{3/2}} < z \right\} = \Phi(z),$$

(iii) $$\lim_{x\to\infty} \frac{1}{x} \# \left\{ n \le x : \frac{\ell_2(n)}{\sqrt{2/3}(\log_2 x)^{3/2}} < z \right\} = \Phi(z),$$

(iv) $$\lim_{x\to\infty} \frac{1}{x} \# \left\{ n \le x : \frac{\ell_3(n)}{\sqrt{2/3}(\log_2 x)^{3/2}} < z \right\} = \Phi(z).$$

3 Preliminary results

We let $\mathrm{li}(x) := \displaystyle\int_2^x \frac{dt}{\log t}$, $\pi(x) := \sum_{p\le x} 1$ and $\pi(x;k,\ell) := \#\{p \le x : p \equiv \ell \pmod{k}\}$.

In 1968, Elliott and Halberstam [4] stated the following conjecture.

Elliott-Halberstam Conjecture (EHC). *Given arbitrarily small $\varepsilon > 0$ and large $D > 0$, there exists a positive constant C_ε for which*

$$\sum_{k\le x^{1-\varepsilon}} \max_{(\ell,k)=1} \left| \pi(x;k,\ell) - \frac{\mathrm{li}(x)}{\varphi(k)} \right| < C_\varepsilon \frac{x}{\log^D x} \qquad (x \ge 2).$$

Lemma 1. *If the EHC holds, then*

$$A(x) = o(\log_2 x) \qquad (x \to \infty).$$

Proof. Observe that, given an arbitrarily small $\varepsilon > 0$, we have

$$\begin{aligned} E_1(x) &\le \sum_{q\le x^{1-\varepsilon}} |\pi(x;q,-1) - \pi(x;q,1)| \\ &\quad + \sum_{a\le x^\varepsilon} \#\{q \le x/a : aq-1 \text{ is prime}\} + \sum_{a\le x^\varepsilon} \#\{q \le x/a : aq+1 \text{ is prime}\} \\ &= S_1(x) + S_2(x) + S_3(x), \end{aligned} \tag{3.1}$$

say. Assuming the EHC, we have that

$$S_1(x) < C_\varepsilon \frac{x}{\log^2 x} \qquad (x \ge 2). \tag{3.2}$$

On the other hand, using Corollary 5.8.2 in the book of Halberstam and Richert [6], we have that, for some absolute constants $c_1 > 0$ and $c_2 > 0$,

$$\begin{aligned} S_2(x) &\le \sum_{a\le x^\varepsilon} c_1 \frac{x/a}{\log^2(x/a)} \prod_{\pi|a} \frac{\pi-1}{\pi-2} \le \frac{c_1 x}{(1-\varepsilon)^2 \log^2 x} \sum_{a\le x^\varepsilon} \frac{1}{a} \prod_{\pi|a} \frac{\pi-1}{\pi-2} \\ &\le c_2 \varepsilon \frac{x}{\log x}. \end{aligned} \tag{3.3}$$

Similarly we obtain that, for some absolute constant $c_3 > 0$,

$$S_3(x) < c_3 \varepsilon \frac{x}{\log x} \qquad (x \ge 2). \tag{3.4}$$

Gathering estimates (3.2), (3.3) and (3.4) in (3.1), we conclude that

$$E_1(x) = o\left(\frac{x}{\log x}\right) \qquad (x \to \infty). \tag{3.5}$$

On the other hand, it is clear that, in light of (3.5), as $x \to \infty$,

$$\begin{aligned} A(x) &= \int_2^x \frac{1}{u}\, dE_1(u) = \left.\frac{E_1(u)}{u}\right|_2^x + \int_2^x \frac{E_1(u)}{u^2}\, du \\ &= O(1) + \int_2^u \frac{o(1)}{u \log u}\, du = o(\log_2 x), \end{aligned}$$

thereby completing the proof of Lemma 1. □

Let us now introduce the two functions

$$E_2(x) := \sum_{p \le x} \Delta^2(p) \qquad \text{and} \qquad B^2(x) := \sum_{p \le x} \frac{\Delta^2(p)}{p}.$$

Lemma 2. *As* $x \to \infty$,

$$B^2(x) = (\log_2 x)^2 + O(\log_2 x)\,. \tag{3.6}$$

Proof. We start by estimating the size of $E_2(x)$. Let $y = x^{1/5}$ and set

$$\omega_y(n) := \sum_{\substack{q \mid n \\ q < y}} 1, \qquad \Delta_y(p) := \omega_y(p+1) - \omega_y(p-1) \qquad \text{and} \qquad R_y := \sum_{q<y} \frac{1}{q-1}.$$

By the Bombieri-Vinogradov theorem, we have

$$\begin{aligned} T(x) := \sum_{p \le x} \Delta_y^2(p) &= \sum_{p \le x} \omega_y^2(p+1) + \sum_{p \le x} \omega_y^2(p-1) - 2 \sum_{p \le x} \omega_y(p+1)\omega_y(p-1) \\ &= \sum_{\substack{q_1, q_2 \le y \\ q_1 \ne q_2}} (\pi(x; q_1 q_2, -1) + \pi(x; q_1 q_2, 1)) - 2 \sum_{\substack{q_1, q_2 \le y \\ q_1 \ne q_2}} \sum_{\substack{p \le x \\ q_1 \mid p+1,\ q_2 \mid p-1}} 1 \qquad (3.7) \\ &\qquad + \sum_{p \le x} (\omega_y(p+1) + \omega_y(p-1)) + O(\mathrm{li}(x)) \\ &= 2\,\mathrm{li}(x)(R_y^2 + O(1)) - 2\,\mathrm{li}(x)(R_y^2 + O(1)) + 2\,\mathrm{li}(x)\, R_y + O\left(\frac{x}{\log^D x}\right) \\ &= 2\,\mathrm{li}(x)\, R_y + O(\mathrm{li}(x)), \qquad (3.8) \end{aligned}$$

where we used the fact that if q divides both $p-1$ and $p+1$, then $q = 2$, therefore implying that the contribution of these terms to the sum in line (3.7) is $O(\mathrm{li}(x))$.

Since $y = x^{1/5}$, it is clear that $R_y = \log_2 x + O(1)$, in which case we get from (3.8) that

$$T(x) = 2\,\mathrm{li}(x)\, \log_2 x + O(\mathrm{li}(x)). \tag{3.9}$$

Now, observe that

$$|(a+b)^2 - a^2| \le b^2 + 2|ab| \qquad \text{for all real numbers } a, b, \tag{3.10}$$

while, because of our choice of y, we also have that $|\Delta(p)-\Delta_y(p)|\le 4$ for every prime p. Therefore, choosing $a=\Delta_y(p)$ and $b=\Delta(p)-\Delta_y(p)$, it follows from (3.10) that

$$\sum_{p\le x}\left|\Delta^2(p)-\Delta_y^2(p)\right|\le 25\,\pi(x)+10\sum_{p\le x}|\Delta_y(p)|. \tag{3.11}$$

Using the Cauchy-Schwarz inequality and estimate (3.9), we have that, for some absolute constant $c_4>0$,

$$\begin{aligned}\sum_{p\le x}|\Delta_y(p)| &\le \left(\sum_{p\le x}1\right)^{1/2}\left(\sum_{p\le x}\Delta_y^2(p)\right)^{1/2}\\ &\le c_4\ (\mathrm{li}(x))^{1/2}\,(\mathrm{li}(x)\ \log_2 x)^{1/2}\\ &= c_4\,\mathrm{li}(x)\,\sqrt{\log_2 x}. \end{aligned}\tag{3.12}$$

Combining (3.9), (3.11) and (3.12) proves that

$$E_2(x)=2\,\mathrm{li}(x)\ \log_2 x+O(\mathrm{li}(x)). \tag{3.13}$$

Estimate (3.6) then follows immediately from (3.13) by partial summation, and Lemma 2 is proved. □

4 Proofs of Theorems 1 and 2

Theorem 1 is an immediate consequence of Lemmas 1, 2 and of the Kubilius-Shapiro theorem stated in the Introduction, although it remains to establish that the last condition of the Kubilius-Shapiro theorem is satisfied.

For this, we need to prove that, for every $\varepsilon>0$,

$$G(x):=\frac{1}{B^2(x)}\sum_{\substack{p\le x\\ |\Delta(p)|\ge\varepsilon B(x)}}\frac{\Delta^2(p)}{p}\to 0\qquad (x\to\infty). \tag{4.1}$$

First observe that

$$\Delta(p)|=|\omega(p+1)-\omega(p-1)|\le|\omega(p+1)-\log\log p|+|\omega(p-1)-\log\log p|. \tag{4.2}$$

On the other hand, it is well known that

$$\sum_{p\le x}|\omega(p+1)-\log\log p|^4\ll(\log_2 x)^2\pi(x),$$

which implies that

$$\sum_{p\le x}\frac{|\omega(p+1)-\log\log p|^4}{p}\ll(\log_2 x)^3. \tag{4.3}$$

Similarly, we obtain that

$$\sum_{p\le x} \frac{|\omega(p-1)-\log\log p|^4}{p} \ll (\log_2 x)^3. \tag{4.4}$$

Finally, observe that it is obvious that, given any $\varepsilon > 0$,

$$G(x) \le \frac{1}{B^2(x)} \sum_{p\le x} \frac{\Delta^2(p)}{p} \cdot \left(\frac{\Delta(p)}{\varepsilon B(x)}\right)^2. \tag{4.5}$$

Hence, using (4.5), (4.2), (4.3), (4.4) and Lemma 2, we obtain that, for some positive constant c,

$$\begin{aligned} G(x) &\le \frac{1}{\varepsilon^2 \log_2^4 x} \sum_{p\le x} \frac{\Delta^4(p)}{p} \\ &\le \frac{c}{\varepsilon^2 \log_2^4 x} \left(\sum_{p\le x} \frac{|\omega(p+1)-\log\log p|^4}{p} + \sum_{p\le x} \frac{|\omega(p-1)-\log\log p|^4}{p} \right) \\ &\le \frac{2c}{\varepsilon^2 \log_2^4 x} \frac{\log_2^3 x}{\log_2^4 x} \end{aligned}$$

a quantity which tends to 0 as $x \to \infty$. We have thus established (4.1) as requested.

In order to prove Theorem 2, we first observe that

$$\omega(\theta^+(n)) \le \sum_{p|n} \omega(p+1),$$

while

$$\omega(\theta^+(n)) \ge \sum_{\substack{p|n \\ p\ge \log_2 x}} \omega(p+1) - \sum_{q>\log_2 x} \sum_{\substack{p_1p_2|n \\ p_1+1\equiv 0 \pmod q \\ p_2+1\equiv 0 \pmod q \\ p_1\ne p_2}} 1,$$

and that

$$\omega(\theta^-(n)) \le \sum_{p|n} \omega(p-1),$$

while

$$\omega(\theta^-(n)) \ge \sum_{\substack{p|n \\ p\ge \log_2 x}} \omega(p-1) - \sum_{q>\log_2 x} \sum_{\substack{p_1p_2|n \\ p_1-1\equiv 0 \pmod q \\ p_2-1\equiv 0 \pmod q \\ p_1\ne p_2}} 1.$$

It follows from these observations that

$$\frac{1}{x}\sum_{n\le x} |\rho(n)-\Delta(n)| \le \frac{1}{x}\sum_{n\le x} \sum_{\substack{p|n \\ p\le \log_2 x}} |\Delta(p)| + \sum_{q>\log_2 x} \sum_{\substack{q|p_1+1 \\ q|p_2+1}} \frac{1}{p_1p_2} + \sum_{q>\log_2 x} \sum_{\substack{q|p_1-1 \\ q|p_2-1}} \frac{1}{p_1p_2}$$

$$(4.6) \qquad = H_1(x) + H_2(x) + H_3(x),$$

say. On the one hand, we have that for some absolute constant $c_5 > 0$,

$$\begin{aligned} H_1(x) &\le \sum_{p \le \log\log x} \frac{|\Delta(p)|}{p} \\ &\le \sum_{p \le \log\log x} \frac{1}{p} \left(|\omega(p+1) - \log\log p| + |\omega(p-1) - \log\log p| \right) \\ &\le c_5 (\log_4 x)^{3/2}, \qquad (4.7) \end{aligned}$$

where for this last inequality, we used the known estimate

$$\sum_{p \le U} |\omega(p+1) - \log\log p| \ll \frac{U}{\log U} \sqrt{\log_2 U} \quad (U \to \infty).$$

According to Lemma 6 in Kátai [7], there exists an absolute constant $c > 0$ such that

$$\sum_{\substack{k \le p \le x \\ p \equiv \ell \pmod k}} \frac{1}{p} < c \frac{\log_2 x}{\varphi(k)}.$$

We will use this result in the special cases $\ell = -1$ and $\ell = 1$. In particular, we easily obtain that, for some absolute constants $c_6 > 0$ and $c_7 > 0$,

$$\begin{aligned} H_2(x) &\le \sum_{q > \log_2 x} \sum_{\substack{p_1, p_2 \le x \\ q \mid p_1 + 1 \\ q \mid p_2 + 1}} \frac{1}{p_1 p_2} \le c_6 (\log_2 x)^2 \sum_{q > \log_2 x} \frac{1}{q^2} \\ &\le c_7 (\log_2 x)^2 \frac{1}{\log_2 x \log_3 x} = c_7 \frac{\log_2 x}{\log_3 x}. \qquad (4.8) \end{aligned}$$

Similarly, we obtain

$$(4.9) \qquad H_3(x) \le c_8 \frac{\log_2 x}{\log_3 x}.$$

Combining (4.6), (4.7), (4.8) and (4.9), we obtain that

$$\rho(n) = \Delta(n) + A_n,$$

where

$$\frac{1}{x} \sum_{n \le x} |A_n| \ll \frac{\log_2 x}{\log_3 x}.$$

From this it follows that Theorem 2 is an immediate consequence of Theorem 1.

5 Proof of Theorem 3

Any positive integer n can be written as $n = Km$, where $K = K(n)$ is squarefull and $m = m(n)$ is squarefree, with $(K, m) = 1$. Doing so, we clearly have

$$0 \le \omega(\varphi(n)) - \omega(\varphi(m)) \le \omega(\varphi(K)), \tag{5.1}$$
$$0 \le \omega(\sigma(n)) - \omega(\sigma(m)) \le \omega(\sigma(K)) \tag{5.2}$$

as well as

$$\omega(\varphi(m)) = \omega(\theta^-(m)), \qquad \omega(\sigma(m)) = \omega(\theta^+(m)). \tag{5.3}$$

On the other hand, given any function $s(x)$ which tends to ∞ arbitrarily slowly as $x \to \infty$, we have

$$\frac{1}{x}\#\{n \le x : K(n) > s(x)\} \to 0 \qquad \text{as } x \to \infty.$$

This is immediate because, letting μ stand for the Mëbius function, we have

$$\frac{1}{x} \sum_{\substack{Km \le x \\ (m,K)=1,\ K>s(x)}} \mu^2(m) = \frac{1}{x} \sum_{\substack{K \le x \\ K > s(x)}} \sum_{\substack{m \le \frac{x}{K} \\ (m,K)=1}} \mu^2(m) \le \sum_{\substack{K \le x \\ K > s(x)}} \frac{1}{K} \ll \frac{1}{\sqrt{s(x)}}.$$

This is why we may assume that $K(n) \le s(x)$, in which case, in light of (5.1), (5.2) and (5.3), we have that $\rho(n) = \rho(m) + O(s(x))$ and $\ell_1(n) = \ell_1(m) + O(s(x))$. Therefore, with $n = Km$ and $K \le s(x)$,

$$\begin{aligned} \rho(n) &= \rho(m) + O(s(x)) = \omega(\theta^+(m)) - \omega(\theta^-(m)) + O(s(x)) \\ &= \omega(\sigma(m)) - \omega(\varphi(m)) + O(s(x)) = \ell_1(m) + O(s(x)) \\ &= \ell_1(n) + O(s(x)). \end{aligned}$$

From this and by choosing $s(x) = \log_3 x$, we obtain that

$$\ell_1(n) = \rho(n) + O(\log_3 x).$$

From this observation, Theorem 3 follows immediately from Theorem 2.

6 Proof of Theorem 4

We begin by proving part (i). We will use the "method of moments" provided in the book of Billingsley [1].

For $n \le x$, we have $a(n+1) - a(n) = \sum_{p \le x} X_p(n)\omega(p+1)$, where $X_p(n) = \mathbf{1}_{p|n} - \mathbf{1}_{p|n+1}$. Now, observe that

$$\sum_{n \le x} \left| \sum_{y < p \le x} X_p(n)\omega(p+1) \right| \ll (\log\log x)\log(\log x/\log y).$$

Choosing $y = x^{1/\log\log x}$ in the above implies that for almost all n, the sum $\displaystyle\sum_{y<p\le x} X_p(n)\omega(p+1)$ is smaller than the variance, so that it suffices to work only with the primes $p \le y$.

Given p, the random variable X_p takes the values 1, -1 and 0 with approximate probabilities $1/p$, $1/p$ and $1-2/p$, respectively. Moreover, for different p_1 and p_2, the random variables X_{p_1} and X_{p_2} are approximately independent. Now, define truly independent random variables Y_p, for p prime, such that $Y_p = 1$ with probability $1/p$, $Y_p = -1$ with probability $1/p$, and $Y_p = 0$ with probability $1-2/p$. Clearly, $\mathbb{E}[Y_p] = 0$ and $\mathrm{Var}[Y_p] = 2/p$. Hence,

$$\mathrm{Var}\left[\sum_{p\le x} Y_p\omega(p+1)\right] \sim \frac{2}{3}(\log\log x)^3 \qquad (x\to\infty)$$

and it is easy to see that $\displaystyle\sum_{p\le x} Y_p\omega(p+1)/\sqrt{2/3}\,(\log\log x)^{3/2}$ tends to the standard normal distribution by the Central Limit Theorem. Therefore, to complete the proof of Theorem 4(i), it suffices to show that

(6.1)
$$\mathbb{E}_{n\le x}\left[\left(\sum_{p\le y} X_p(n)\omega(p+1)\right)^k\right] - \mathbb{E}\left[\left(\sum_{p\le y} Y_p\omega(p+1)\right)^k\right] = o((\log\log x)^{3k/2}) \quad (x\to\infty).$$

Expanding the k-th powers, we obtain that

$$\begin{aligned}\mathbb{E}_{n\le x}&\left[\left(\sum_{p\le y} X_p(n)\omega(p+1)\right)^k\right] - \mathbb{E}\left[\left(\sum_{p\le y} Y_p\omega(p+1)\right)^k\right]\\ &= \sum_{p_1,\dots,p_k\le y} \omega(p_1+1)\cdots\omega(p_k+1)\left(\mathbb{E}_{n\le x}[X_{p_1}(n)\cdots X_{p_k}(n)] - \mathbb{E}[Y_{p_1}\cdots Y_{p_k}]\right).\end{aligned}$$

Since the above innermost difference is $O_k(1/x)$ by the Chinese Remainder Theorem, we may conclude that (6.1) follows, thereby concluding the proof of Theorem 4(i).

It remains to prove statements (ii), (iii) and (iv) of Theorem 4. First of all, the proof of (ii) can be treated similarly as that of (i), and we shall therefore omit it. On the other hand, one can deduce (iv) from (ii) and (iii) from (i) by simply observing that, given any $\varepsilon > 0$,

$$\frac{1}{x}\#\left\{n\le x : \left|\frac{\omega(\sigma(n)) - a(n)}{x^{3/2}}\right| > \varepsilon\right\} \to 0 \qquad (x\to\infty)$$

and

$$\frac{1}{x}\#\left\{n\le x : \left|\frac{\omega(\varphi(n)) - b(n)}{x^{3/2}}\right| > \varepsilon\right\} \to 0 \qquad (x\to\infty).$$

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