On properties of sharp normal numbers and of non-Liouville numbers

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Abstract

We show that some sequences of real numbers involving sharp normal numbers or non-Liouville numbers are uniformly distributed modulo 1. In particular, we prove that if $\tau(n)$ stands for the number of divisors of n and α is a binary sharp normal number, then the sequence $(\alpha\tau(n))_{n\geq 1}$ is uniformly distributed modulo 1 and that if $g(x)$ is a polynomial of positive degree with real coefficients and whose leading coefficient is a non-Liouville number, then the sequence $(g(\tau(\tau(n))))_{n\geq 1}$ is also uniformly distributed modulo 1.

Résumé

Nous montrons que certaines suites de nombres réels impliquant des nombres normaux robustes et des nombres non-Liouville sont uniformément réparties modulo 1. En particulier, nous démontrons que si $\tau(n)$ représente le nombre de diviseurs de n, alors, étant donné un nombre normal binaire robuste α, la suite correspondante $(\alpha\tau(n))_{n\geq 1}$ est uniformément répartie modulo 1 et nous démontrons également que si $g(x)$ est un polynôme à coefficients réels de degré positif et dont le coefficient principal est un nombre non-Liouville, alors la suite $(g(\tau(\tau(n))))_{n\geq 1}$ est uniformément répartie modulo 1.

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1 Introduction and notation

Let us first recall the concept of *sharp normality*, recently introduced by De Koninck, Kátai and Phong [3].

The *discrepancy* of a set of N real numbers $x_1, \ldots, x_N$ is the quantity

$$D(x_1, \ldots, x_N) := \sup_{[a,b)\subseteq[0,1)} \left| \frac{1}{N} \sum_{\substack{n=1 \\ \{x_n\}\in[a,b)}}^{N} 1 - (b-a) \right|. \tag{1.1}$$

Here and in what follows, $\{y\}$ stands for the fractional part of the real number y.

Recall that a sequence $(x_n)_{n\in\mathbb{N}}$ of real numbers is said to be *uniformly distributed modulo 1* if for each subinterval $[a,b)$ of $[0,1)$,

$$\lim_{N\to\infty} \frac{1}{N} \#\{n \le N : \{x_n\} \in [a,b)\} = b - a.$$

Recall also that, given a fixed integer $q \ge 2$, an irrational number is said to be a *q-normal number* if, in the base q expansion of this number, any preassigned block of k digits appears at the expected frequency, namely $1/q^k$. Equivalently, given a positive irrational number $\eta < 1$ whose base q expansion is

$$\eta = 0.a_1a_2a_3\ldots = \sum_{j=1}^{\infty} \frac{a_j}{q^j}, \text{ where each } a_j \in \{0,1,\ldots,q-1\},$$

we say that η is a *q-normal number* if the sequence $(\{q^m\eta\})_{m\ge1}$ is uniformly distributed in the interval $[0,1)$.

This paves the way for the introduction of the notions of "sharp distribution modulo 1" and of a "sharp normal number".

For each positive integer N, let

$$M = M_N = \lfloor \delta_N \sqrt{N} \rfloor, \qquad \text{where } \delta_N \to 0 \text{ and } \delta_N \log N \to \infty \text{ as } N \to \infty. \tag{1.2}$$

We shall say that a sequence of real numbers $(x_n)_{n\ge1}$ is *sharply uniformly distributed* modulo 1 if

$$D(x_{N+1},\ldots,x_{N+M}) \to 0 \qquad \text{as} \qquad N \to \infty$$

for every choice of δ_N satisfying (1.2). Given a fixed integer $q \ge 2$, we then say that an irrational number α is a *sharp normal number* in base q (or a sharp q-normal number) if the sequence $(\alpha q^n)_{n\ge1}$ is sharply uniformly distributed modulo 1. In [3], it is shown that the Lebesgue measure of the set of all those real numbers $\alpha \in [0,1]$ which are not sharp q-normal is equal to 0.

Before we move on, we make two remarks.

Remark 1. *Our original paper on sharp normality appeared in Uniform Distribution Theory under the title "On strong normality". After its publication, we became aware that the term "strong normal number" had been used by other authors with a different meaning. For instance, Adrian Belshaw and Peter Borwein [1] call α a strong normal number in base b if every string of digits in the base b expansion of α appears with the frequency expected for random digits and the discrepancy fluctuates as is expected by the law of the iterated logarithm. With this concept of "strong normality", they then showed that almost all numbers are strong normal numbers (as we do in the present document, but for different reasons). This being said, in order to avoid confusion, in this paper and in other papers in which we will further expand on properties regarding this new concept, we shall always use the term "sharp normal numbers".*

Remark 2. *Instead of choosing* $M_N = \lfloor \delta_N \sqrt{N} \rfloor$ *in (1.2), we could have chosen* $M_N = \lfloor \delta_N N^\gamma \rfloor$ *for some fixed number* $\gamma \in (0,1)$*, thereby introducing the notion of* γ*-sharp distribution modulo 1 and the corresponding notion of* γ*-sharp normal number. With such definitions, it can be shown that, given* $0 < \gamma_1 < \gamma_2 < 1$*, any* γ_1*-sharp normal number is also a* γ_2*-sharp normal number. One can then show that, given* $\gamma \in (0,1)$*, almost all real numbers are* γ*-sharp normal numbers. Various alternatives for the choice of* $M = M_N$ *in (1.2) are discussed in De Koninck, Kátai and Phong [3].*

We shall also need the concept of discrepancy of a set of N t-tuples $\underline{y}_1, \underline{y}_2, \ldots, \underline{y}_N$, where $\underline{y}_n = (x_1^{(n)}, \ldots, x_t^{(n)})$ for $n = 1, 2, \ldots, N$, with each $x_i^{(n)} \in \mathbb{R}$. The discrepancy of a set of N such vectors $\underline{y}_1, \ldots, \underline{y}_N$ is defined as the quantity

$$D(\underline{y}_1, \ldots, \underline{y}_N) := \sup_{I \subseteq [0,1)^t} \left| \frac{1}{N} \sum_{\substack{n=1 \\ \{\underline{y}_n\} \in I}}^{N} 1 - \prod_{i=1}^{t} (\beta_i - \alpha_i) \right|,$$

where $\{\underline{y}_n\}$ stands for $(\{x_1\}, \ldots, \{x_n\})$ and where the above supremum runs over all possible subsets $I = [\alpha_1, \beta_1) \times \cdots \times [\alpha_t, \beta_t)$ of the t-dimensional unit interval $[0,1)^t$.

Recall also that an irrational number β is said to be a *Liouville number* if for each integer $m \geq 1$, there exist two integers t and $s > 1$ such that

$$0 < \left| \beta - \frac{t}{s} \right| < \frac{1}{s^m}.$$

In a sense, one might say that a Liouville number is an irrational number which can be well approximated by a sequence of rational numbers.

Here, we show that some sequences of real numbers involving sharp normal numbers or non-Liouville numbers are uniformly distributed modulo 1. We also study the discrepancy of a sequence of t-tuples of real numbers involving sharp normal numbers.

Throughout this paper, $\wp$ stands for the set of all primes. Given an integer $n \geq 2$, we let $\gamma(n)$ (resp. $\omega(n)$) stand for the product (resp. number) of distinct prime factors of n, with $\gamma(1) = 1$ and $\omega(1) = 0$. Moreover, given a set $\mathcal{B} \subseteq \wp$, we let

$$\omega_{\mathcal{B}}(n) = \sum_{\substack{p|n \\ p \in \mathcal{B}}} 1.$$

We also let τ stand for the number of divisors function. More generally, given an integer $\ell \geq 2$, we let $\tau_\ell(n)$ stand for the number of ways of writing n as the product of ℓ positive integers. Also, we let φ stand for the Euler function and write $e(y)$ for $e^{2\pi i y}$. Finally, by $\log_2 x$ (resp. $\log_3 x$) we mean $\max(2, \log\log x)$ (resp. $\max(2, \log\log_2 x)$).

2 Main results

If α is an irrational number, it is well known that the sequence $(\alpha n)_{n\geq 1}$ is uniformly distributed modulo 1, while there is no guarantee that the sequence $(\alpha\tau(n))_{n\geq 1}$ will itself be uniformly distributed modulo 1. However, if α is a sharp normal number, the situation is different, as is shown in our first result.

Theorem 1. *Let $q \geq 2$ be a fixed integer. If α is a sharp q-normal number, then the sequence $(\alpha\tau_q(n))_{n\geq 1}$ is uniformly distributed modulo 1.*

In an earlier paper [2], we showed that if $g(x) = \alpha x^k + \alpha_{k-1}x^{k-1} + \cdots + \alpha_1 x + \alpha_0 \in \mathbb{R}[x]$ is a polynomial of positive degree, where α is a non-Liouville number, and if h belongs to a particular set of arithmetic functions, then the sequence $(g(h(n))_{n\geq 1}$ is uniformly distributed modulo 1. Our next result goes along the same lines.

Theorem 2. *Let $g(x) = \alpha x^k + \alpha_{k-1}x^{k-1} + \cdots + \alpha_1 x + \alpha_0 \in \mathbb{R}[x]$ be a polynomial of positive degree, where α is a non-Liouville number. Then, the sequence $(g(\tau(\tau(n))))_{n\geq 1}$ is uniformly distributed modulo 1.*

Now, consider the following (plausible) conjecture.

Conjecture 1. *Let ε_x be some function which tends to 0 as $x \to \infty$. Then, if $|k-\ell| \leq \varepsilon_x\sqrt{\log_2 x}$, we have, uniformly for $|k-\log_2 x| \leq \frac{1}{\varepsilon_x}\sqrt{\log_2 x}$ and $|\ell-\log_2 x| \leq \frac{1}{\varepsilon_x}\sqrt{\log_2 x}$, as $x \to \infty$,*

$$\frac{1}{x}\#\{n \leq x : \omega(n) = k \text{ and } \omega(n+1) = \ell\}$$
$$= (1+o(1))\frac{1}{x}\#\{n \leq x : \omega(n) = k\} \cdot \frac{1}{x}\#\{n \leq x : \omega(n+1) = \ell\}$$

and more generally, if $|\ell_i - \ell_j| \leq \varepsilon_x\sqrt{\log_2 x}$ for all $i \neq j$, then, uniformly for $|\ell_j - \log_2 x| \leq \frac{1}{\varepsilon_x}\sqrt{\log_2 x}$, for each $j = 0, 1, \ldots, t-1$, as $x \to \infty$,

$$\frac{1}{x}\#\{n \leq x : \omega(n+j) = \ell_j, \text{ with } j = 0, 1, \ldots, t-1\}$$
$$= (1+o(1))\prod_{j=0}^{t-1}\frac{1}{x}\#\{n \leq x : \omega(n+j) = \ell_j\}.$$

It is interesting to observe that, using the ideas mentioned at the beginning of Theorem 3, the following result would follow immediately from Conjecture 1.

> Let $q_0, q_1, \ldots, q_{t-1}$ be integers larger than 1 and, for each $j = 0, 1, \ldots, t-1$, let α_j be a sharp q_j-normal number. Consider the sequence of t-tuples $(\underline{x}_n)_{n\geq 1}$ defined by
>
> $$\underline{x}_n := \left(\{\alpha_0 q_0^{\omega(n)}\}, \{\alpha_1 q_1^{\omega(n+1)}\}, \ldots, \{\alpha_{t-1} q_{t-1}^{\omega(n+t-1)}\}\right) \in [0,1)^t.$$
>
> Then, the sequence $(\underline{x}_n)_{n\geq 1}$ is uniformly distributed modulo $[0,1)^t$.

This observation explains the importance of the following result.

Theorem 3. *Let w_x and Y_x be two increasing functions both tending to ∞ as $x \to \infty$ and satisfying the conditions*

$$\frac{\log Y_x}{\log x} \to 0, \qquad \frac{Y_x}{\log x} \to \infty, \qquad w_x \ll \log_2 x \qquad (x \to \infty).$$

Set $\mathcal{B} = \mathcal{B}_x = \{p \in \wp : w_x < p < Y_x\}$ and let $q_0, q_1 \dots, q_{t-1}$ be t integers larger than 1 and for each $i = 0, 1, \dots, t-1$, let α_i be a sharp normal number in base q_i. Consider the sequence of t-tuples $(\underline{y}_n)_{n\geq 1}$ defined by

$$\underline{y}_n := \left(\{\alpha_0 q_0^{\omega_{\mathcal{B}}(n)}\}, \{\alpha_1 q_1^{\omega_{\mathcal{B}}(n+1)}\}, \dots, \{\alpha_{t-1} q_{t-1}^{\omega_{\mathcal{B}}(n+t-1)}\}\right) \in [0,1)^t.$$

If $D_{\lfloor x \rfloor}$ stands for the discrepancy of the set $\{\underline{y}_1, \dots, \underline{y}_{\lfloor x \rfloor}\}$, then $D_{\lfloor x \rfloor} \to 0$ as $x \to \infty$.

Finally, the following result is essentially the case $t = 1$ of the previous theorem.

Corollary 1. *Given an integer $q \geq 2$, let α be a sharp q-normal number. Let w_x, Y_x and $\mathcal{B} = \mathcal{B}_x$ be as in Theorem 3 and consider the sequence $(y_n)_{n\geq 1}$ defined by $y_n = \{\alpha q^{\omega_{\mathcal{B}}(n)}\}$. Then, the discrepancy $D(y_1, y_2, \dots, y_{\lfloor x \rfloor})$ tends to 0 as $x \to \infty$.*

3 Preliminary results

Lemma 1. *If α is a sharp q-normal number and m a positive integer, then $m\alpha$ is also a sharp q-normal number.*

Proof. Let $x_n \in [0,1)$ for $n = 1, 2, \dots, N$ and consider the corresponding numbers $y_n = \{m x_n\}$ for $n = 1, 2, \dots, N$. If we can prove the inequality

$$D(y_1, y_2, \dots, y_N) \leq m D(x_1, x_2, \dots, x_N), \tag{3.1}$$

the proof of Lemma 1 will be complete. In order to prove (3.1), first observe that, for each integer $n \in \{1, 2, \dots, N\}$, we have that $y_n \in [a, b) \subseteq [0,1)$ if and only if $m x_n \in \bigcup_{\ell=0}^{m-1} [\ell + a, \ell + b)$, which is equivalent to

$$x_n \in \bigcup_{\ell=0}^{m-1} \left[\frac{\ell}{m} + \frac{a}{m}, \frac{\ell}{m} + \frac{b}{m}\right) =: \bigcup_{\ell=0}^{m-1} J_\ell.$$

Since

$$\left| \frac{1}{N} \sum_{\substack{n=1 \\ x_n \in J_\ell}}^{N} 1 - \frac{b-a}{m} \right| \leq D(x_1, x_2, \dots, x_N),$$

it follows that

$$\left| \frac{1}{N} \sum_{\substack{n=1 \\ y_n \in [a,b)}}^{N} 1 - (b-a) \right| \le \sum_{\ell=0}^{m-1} \left| \frac{1}{N} \sum_{\substack{n=1 \\ x_n \in J_\ell}}^{N} 1 - \frac{b-a}{m} \right| \le mD(x_1, x_2, \ldots, x_N).$$

Taking the supremum of the first two of the above quantities over all possible subintervals $[a,b)$ of $[0,1)$, inequality (3.1) follows immediately. □

The following result is Lemma 3 in Spiro [5].

Lemma 2. *Let B_1, B_2 and B_3 be three fixed positive numbers. Assume that $x \ge 3$ and that both y and ℓ are positive integers satisfying $y \le B_1 \log_2 x$, $\ell \le \exp\{\log^{B_2} x\}$ and $\gamma(\ell) \le \log^{B_3} x$. Then, uniformly for y and ℓ,*

$$\begin{aligned} \pi_\ell(x,y) &:= \#\{n \le x : \omega(n) = y,\ \mu^2(n) = 1,\ (n,\ell) = 1\} \\ &= \frac{x(\log_2 x)^{y-1}}{(y-1)!\,\log x} \left\{ F\left(\frac{y-1}{\log_2 x}\right) F_\ell\left(\frac{y-1}{\log_2 x}\right) + O_{B_1,B_3}\left(y \frac{(\log_3(16\ell))^3}{(\log_2 x)^2} \right) \right\}, \end{aligned}$$

where

$$F(z) = \frac{1}{\Gamma(z+1)} \prod_p \left(1 + \frac{z}{p}\right) \left(1 - \frac{1}{p}\right)^z, \qquad F_\ell(z) = \prod_{p|\ell} \left(1 + \frac{z}{p}\right)^{-1}.$$

The following result is Lemma 2.1 in the book of Elliott [4].

Lemma 3. *Let $f(n)$ be a real valued non negative arithmetic function. Let a_n, $n = 1, \ldots, N$, be a sequence of integers. Let r be a positive real number, and let $p_1 < p_2 < \cdots < p_s \le r$ be prime numbers. Set $Q = p_1 \cdots p_s$. If $d|Q$, then let*

$$\sum_{\substack{n=1 \\ a_n \equiv 0 \pmod d}}^{N} f(n) = \rho(d) X + R(N,d), \tag{3.2}$$

where X and $R(N,d)$ are real numbers, $X \ge 0$, and $\rho(d_1 d_2) = \rho(d_1)\rho(d_2)$ whenever d_1 and d_2 are co-prime divisors of Q.

Assume that for each prime p, $0 \le \rho(p) < 1$. Setting

$$I(N,Q) := \sum_{\substack{n=1 \\ (a_n,Q)=1}}^{N} f(n), \qquad S = S(Q) := \sum_{p|Q} \frac{\rho(p)}{1-\rho(p)} \log p.$$

then the estimate

$$I(N,Q) = \{1 + 2\theta_1 H\} X \prod_{p|Q} (1 - \rho(p)) + 2\theta_2 \sum_{\substack{d|Q \\ d \le z^3}} 3^{\omega(d)} |R(N,d)|$$

holds uniformly for $r \ge 2$, $\max(\log r, S) \le \frac{1}{8} \log z$, where $|\theta_1| \le 1$, $|\theta_2| \le 1$, and

$$H = \exp\left(-\frac{\log z}{\log r} \left\{ \log\left(\frac{\log z}{S}\right) - \log\log\left(\frac{\log z}{S}\right) - \frac{2S}{\log z} \right\} \right).$$

Lemma 4. *Let w_x, Y_x and $\mathcal{B} = \mathcal{B}_x$ be as in Theorem 3 and let $\mathcal{N}(\mathcal{B})$ be the semigroup generated by $\mathcal{B}$. Further let r_x be a function which tends to ∞ as $x \to \infty$, while satisfying the two conditions*

$$r_x \ll \log_3 x \qquad \textit{and} \qquad \lim_{x\to\infty} \frac{r_x \log Y_x}{\log x} = 0. \tag{3.3}$$

Moreover, let $D_j \in \mathcal{N}(\mathcal{B})$, $j = 0, 1, \ldots, t-1$, with $(D_i, D_j) = 1$ for $i \neq j$, and let

$$\mathcal{N}_{D_0,D_1,\ldots,D_{t-1}}(x) := \#\left\{n \le x : D_j \mid n+j, j = 0,1,\ldots,t-1,\ \left(\frac{n+j}{D_j}, \mathcal{B}\right) = 1\right\}. \tag{3.4}$$

Then, as $x \to \infty$,

$$\frac{1}{x}\#\{n \le x : D_j \mid n+j, j = 0,1,\ldots,t-1 \textit{ and } \max(D_0, D_1, \ldots, D_{t-1}) > Y_x^{r_x}\} \to 0 \tag{3.5}$$

and, uniformly for $D_j \le Y_x^{r_x}$, $j = 0, 1, \ldots, t-1$,

$$\mathcal{N}_{D_0,D_1,\ldots,D_{t-1}}(x) = (1 + o(1))x\, \kappa(D_0)\kappa(D_1)\cdots\kappa(D_{t-1})L_x^t$$

as $x \to \infty$, where κ is the multiplicative function defined on primes p by

$$\kappa(p) = \frac{1}{p} \cdot \frac{p-t+1}{p-t}$$

and $L_x := \dfrac{\log w_x}{\log Y_x}$.

Proof. First observe that (3.5) is easily proved. We may therefore assume that $D_j \le Y_x^{r_x}$ for $j = 0, 1, \ldots, t-1$. In order to use the same notation as in Lemma 3, we set

$$\mathcal{B} = \{p_1, \ldots, p_s\}, \quad Q = p_1 \cdots p_s, \quad E = D_0 D_1 \cdots D_{t-1}, \quad D_j \mid Q \text{ for } j = 0, 1, \ldots, t-1.$$

Observe that the condition $D_j \mid n + j$ for $(j = 0, 1, \ldots, t-1)$ in the definition of $\mathcal{N}_{D_0,D_1,\ldots,D_{t-1}}(x)$ (see (3.4)) holds for exactly one residue class $n \pmod E$. Letting this residue class be $\ell \pmod E$, we then have

$$\mathcal{N}_{D_0,D_1,\ldots,D_{t-1}}(x) = \#\left\{m \le \left\lfloor \frac{x}{E} \right\rfloor : \left(\frac{\ell + mE + j}{D_j}, Q\right) = 1,\ j = 0,1,\ldots,t-1\right\} + O(1).$$

Choose $N = \left\lfloor \dfrac{x}{E} \right\rfloor$ and $f(m) = 1$, while further setting $a_m := \displaystyle\prod_{j=0}^{t-1} \frac{\ell + mE + j}{D_j}$.

Using Lemma 3 with $X = N$, we then get that if $d \mid Q$, relation (3.2) can be written as

$$\sum_{\substack{m=1 \\ a_m \equiv 0 \pmod d}}^{N} 1 = \rho(d)N + R(N, d).$$

Here, $\rho(d)$ is multiplicative and defined by

$$\rho(p) = \begin{cases} t/p & \text{if } p \mid Q/E, \\ (t-1)/p & \text{if } p \mid E. \end{cases}$$

On the other hand, $|R(N,d)| \le \tau_t(d) = (t+1)^{\omega(d)}$ (since d is squarefree), which implies that

$$\sum_{\substack{d|Q \\ d\le z^3}} 3^{\omega(d)}|R(N,d)| \le \sum_{\substack{d|Q \\ d\le z^3}} 3^{\omega(d)}\tau_t(d) \le \sum_{d\le z^3} (3(t+1))^{\omega(d)} \le Cz^3 \log^A z,$$

where A and C are suitable constants depending only on t. Again, with the notation used in Lemma 3, we have

$$\begin{aligned} S &= \sum_{p|Q} \frac{\rho(p)}{1-\rho(p)} \log p = \sum_{p|Q/E} \frac{t\log p}{p(1-t/p)} + \sum_{p|E} \frac{(t-1)\log p}{p(1-(t-1)/p)} \\ &= t\sum_{p|Q/E} \frac{\log p}{p} + (t-1)\sum_{p|E} \frac{\log p}{p} + O(1) \\ &= t\sum_{p|Q} \frac{\log p}{p} - \sum_{p|E} \frac{\log p}{p} + O(1). \qquad (3.6) \end{aligned}$$

Observing that $\sum_{p|E} \frac{\log p}{p} \le t\frac{r_x \log Y_x}{w_x} \to 0$ as $x \to \infty$ (because of (3.3)), it follows from (3.6) that

$$S = t\log(Y_x/w_x) + O\left(\frac{r_x \log Y_x}{w_x}\right).$$

Choosing $r = p_s$ and since

$$s = \pi(Y_x) - \pi(w_x) = \pi(Y_x)\left(1 - \frac{\pi(w_x)}{\pi(Y_x)}\right),$$

it follows, since $\log r = \log s + \log\log s + O(1)$, that

$$\log r = \log Y_x + O(\log\log Y_x).$$

Finally, choose $z = Y_x^{8t\nu_x}$, where $\nu_x \to \infty$ very slowly as $x \to \infty$. One can then easily check that the conditions of Lemma 3 are satisfied, thus allowing us to conclude that

$$H = \exp\left(-8t\nu_x\left(\log(8\nu_x) - \log\log(8\nu_x) + O(1)\right)\right),$$

thereby implying, since $\nu_x \to \infty$ as $x \to \infty$, that

$$H = H_{x,\nu_x} = o(1) \qquad (x \to \infty). \qquad (3.7)$$

Now, writing

$$\prod_{p|Q}(1-\rho(p)) = \prod_{p|Q}\left(1-\frac{t}{p}\right)\cdot\prod_{p|E}\frac{1-\frac{t-1}{p}}{1-t/p} =: \lambda(E),$$

we may conclude from (3.7) that

$$\mathcal{N}_{D_0,D_1,\ldots,D_{t-1}}(x) = (1+o(1))\frac{x}{E}\lambda(E) + O(z^3\log^A z). \tag{3.8}$$

It remains to check that the above error term is not too large compared to the main term $\frac{x}{E}\lambda(E)$. Indeed, if ν_x tends to ∞ slowly enough, this will guarantee that $z^4 \le \sqrt{x}$, say, while on the other hand, in light of conditions (3.3), we have that, for any small $\varepsilon > 0$,

$$\frac{x}{E} \ge \frac{x}{Y_x^{tr_x}} = \frac{x}{e^{tr_x\log Y_x}} \ge \frac{x}{e^{t\varepsilon\log x}} = \frac{x}{x^{t\,\varepsilon}} > x^{3/4},$$

say. Finally, since $\lambda(E) \ge C/\log Y_x$ for some constant $C > 0$, we may conclude that indeed the error term in (3.8) is of smaller order than the main term of (3.8). Consequently, uniformly for $D_j \le Y_x^{r_x}$, $j = 0, 1, \ldots, t-1$, we find that

$$\begin{aligned}\mathcal{N}_{D_0,D_1,\ldots,D_{t-1}}(x) &= (1+o(1))\frac{x}{D_0D_1\cdots D_{t-1}}\prod_{\substack{p\nmid D_0D_1\cdots D_{t-1}\\ p\in\mathcal{B}}}\left(1-\frac{t}{p}\right)\cdot\prod_{p|D_0D_1\cdots D_{t-1}}\left(1-\frac{t-1}{p}\right)\\ &= (1+o(1))\frac{x}{D_0D_1\cdots D_{t-1}}\prod_{p|D_0D_1\cdots D_{t-1}}\frac{1-\frac{t-1}{p}}{1-\frac{t}{p}}\cdot\prod_{p\in\mathcal{B}}\left(1-\frac{t}{p}\right).\end{aligned}$$

Since

$$\prod_{p\in\mathcal{B}}\left(1-\frac{t}{p}\right) = (1+o(1))L_x^t \qquad (x\to\infty),$$

the proof of Lemma 4 is complete. □

The following result is Lemma 1 in our paper [2].

Lemma 5. *Let $g(x) = \alpha x^k + \alpha_{k-1}x^{k-1} + \cdots + \alpha_1 x + \alpha_0 \in \mathbb{R}[x]$ be a polynomial of positive degree, where α be a non-Liouville number. Then,*

$$\sup_{U\ge1}\frac{1}{N}\left|\sum_{n=U+1}^{U+N} e(g(n))\right| \to 0 \qquad \text{as } N\to\infty.$$

Lemma 6. *Assume that the set of natural integers $\mathbb{N}$ is written as a disjoint union of sets N_K, where K runs through the elements of a particular set $\mathcal{P}$ of positive* *integers, that is,* $\mathbb{N} = \bigcup_{K\in\mathcal{P}} N_K$*. Assume that, for each* $K \in \mathcal{P}$*, the counting function* $N_K(x) := \#\{n \le x : n \in N_K\}$ *satisfies*

$$\lim_{x\to\infty} \frac{N_K(x)}{x} = c_K,$$

where the c_K *are positive real numbers such that* $\sum_{K\in\mathcal{P}} c_K = 1$*. Moreover, let* $(x_n)_{n\ge1}$ *be a sequence of real numbers which is such that, for each* $K \in \mathcal{P}$*, the corresponding sequence* $(x_n)_{n\in N_K}$ *is uniformly distributed modulo 1, that is, for each integer* $h \ge 1$*,*

$$S_K^{(h)}(x) := \sum_{\substack{n\le x\\ n\in N_K}} e(hx_n) = o(N_K(x)) \qquad \text{as } x\to\infty. \tag{3.9}$$

Then, the sequence $(x_n)_{n\ge1}$ *is uniformly distributed modulo 1.*

Proof. According to an old and very important result of Weyl [6], a sequence $(x_n)_{n\ge1}$ is uniformly distributed modulo 1 if for every non negative integer h,

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e(hx_n) = 0.$$

Therefore, in light of Weyl's criteria, we only need to prove that, for each positive integer h,

$$S^{(h)}(x) := \sum_{K\in\mathcal{P}} S_K^{(h)}(x) \to 0 \text{ as } x\to\infty. \tag{3.10}$$

Given any $z > 0$ and writing

$$S^{(h)}(x) = \sum_{\substack{K\in\mathcal{P}\\ K<z}} S_K^{(h)}(x) + \sum_{\substack{K\in\mathcal{P}\\ K\ge z}} S_K^{(h)}(x),$$

it follows that

$$\left|\frac{S^{(h)}(x)}{x}\right| \le \sum_{K<z,K\in\mathcal{P}} \frac{N_K(x)}{x}\cdot\frac{1}{N_K(x)}|S_K^{(h)}(x)| + \frac{1}{x}\#\left\{n\le x : n\in \bigcup_{K\in\mathcal{P},K\ge z} N_K\right\}. \tag{3.11}$$

Since, in light of (3.9), we have that $\frac{1}{N_K(x)}|S_K^{(h)}(x)| = o(1)$ as $x\to\infty$, it follows from (3.11) that, for some $C > 0$,

$$\limsup_{x\to\infty}\left|\frac{S^{(h)}(x)}{x}\right| \le C\cdot(\sum_{K<z,K\in\mathcal{P}} c_K)\cdot o(1) + \sum_{K\ge z,K\in\mathcal{P}} c_K,$$

which is as small as we want provided z is chosen large enough, thus proving (3.10). □

4 Proof of Theorem 1

An integer n is called *squarefull* if $p \mid n$ implies that $p^2 \mid n$. Let $\mathcal{P}$ be the set of all squarefull numbers. For convenience, we let $1 \in \mathcal{P}$. To each squarefull number K, we associate the set $N_K := \{n = Km : (m, K) = 1,\, \mu^2(m) = 1\}$, where μ stands for the Möbius function. Since each positive integer n belongs to one and only one such set N_K, we have that

$$\mathbb{N} = \bigcup_{K \in \mathcal{P}} N_K.$$

For any $n \in N_K$, we have $\tau_q(n) = \tau_q(Km) = \tau_q(K) q^{\omega(m)}$.

Now, in light of Lemma 6, the theorem will follow if we can prove that for each fixed $K \in \mathcal{P}$,

(4.1) the sequence $(\{\alpha\tau_q(n)\})_{n \in N_K}$ is uniformly distributed modulo 1 over N_K.

To prove this last statement, we use Lemma 2. First, observe that for $\ell = K$ fixed, we have that $\gamma(\ell) = \gamma(K)$ is bounded and that we can also assume that, given any function δ_x which tends to 0 sufficiently slowly as $x \to \infty$, say with $1/\delta_x < \log_3 x$,

$$|y - \log_2 x| \leq \frac{1}{\delta_x}\sqrt{\log_2 x}, \tag{4.2}$$

so that each of the two quantities $F\left(\dfrac{y-1}{\log_2 x}\right)$ and $F_\ell\left(\dfrac{y-1}{\log_2 x}\right)$ is equal to $1 + o(1)$ as $x \to \infty$ for y in the range (4.2). From there and the fact that α is a sharp normal number, it is clear that (4.1) follows.

5 Proof of Theorem 2

Given a squarefull number K, let N_K and $\mathcal{P}$ be as in the proof of Theorem 1. Any integer $n \in N_K$ can be written as $n = Km$, where $(K, m) = 1$ and $\mu^2(m) = 1$. Moreover, write $\tau(K) = k_1 \cdot 2^{\rho_K}$ for some odd positive integer k_1 and some non negative integer ρ_K. From this set up, it follows that $\tau(n) = \tau(Km) = k_1 \cdot 2^{\rho_K + \omega(m)}$, from which it follows that

$$\tau(\tau(n)) = \tau(k_1)\left(\omega(m) + \rho_K + 1\right). \tag{5.1}$$

Now, for $n \in N_K$ with $\omega(m) = t$, we have, using (5.1),

$$g(\tau(\tau(n))) = \alpha\tau(k_1)^k (t + \rho_K + 1)^k + \cdots = \alpha\tau(k_1)^k t^k + P_{k-1}(t), \tag{5.2}$$

where $P_{k-1}(t)$ stands for some polynomial of degree no larger than $k-1$.

We shall now use Weyl's criteria, already stated in the proof of Lemma 6. So, let h be an arbitrary positive integer. For each $K \in \mathcal{P}$, set

$$S_K(x) := \sum_{\substack{n \leq x \\ n \in N_K}} e(hg(\tau(\tau(n)))).$$

In light of (5.2), we have, writing t for $\omega(m)$,

$$S_K(x) = \sum_{t\geq 1} e(h\alpha\tau(k_1)^k t^k + P_{k-1}(t)) \cdot \pi_K(x,t),$$

were $\pi_k(x,t)$ was defined in Lemma 2. Setting $R(t) := \alpha\tau(k_1)^k t^k + P_{k-1}(t)$, we may write the above as

$$S_K(x) = \sum_{t\geq 1} e(hR(t)) \cdot \pi_K(x,t).$$

Our goal will be to establish that, given any $K \in \mathcal{P}$,

$$S_K(x) = o(x) \qquad (x \to \infty). \tag{5.3}$$

If we can accomplish this, then, in light of Lemma 6, the proof of Theorem 2 will be complete.

To prove (5.3), we first observe that

$$\sum_{\substack{t\geq 1 \\ |t-\log_2 x|>\sqrt{\log_2 x}/\varepsilon_x}} \pi_K(x,t) = o(x) \qquad (x \to \infty) \tag{5.4}$$

and furthermore that

$$\max_{\substack{t_1 \\ |t_1-\log_2 x|\leq\sqrt{\log_2 x}/\varepsilon_x}} \max_{\substack{t_2 \\ |t_2-t_1|\leq\varepsilon_x\sqrt{\log_2 x}}} \left|\frac{\pi_K(x,t_1)}{\pi_K(x,t_2)} - 1\right| \to 0 \text{ as } x \to \infty. \tag{5.5}$$

Now, consider the sequence of real numbers $(z_n)_{n\geq 0}$ defined by

$$z_0 = \log_2 x - \frac{\sqrt{\log_2 x}}{\varepsilon_x} \text{ and for each } m \geq 1 \text{ by } z_m = z_{m-1} + \varepsilon_x\sqrt{\log_2 x},$$

and, setting $M = \left\lfloor \frac{(2/\varepsilon_x)\sqrt{\log_2 x}}{\varepsilon_x\sqrt{\log_2 x}} \right\rfloor = \left\lfloor \frac{2}{\varepsilon_x^2} \right\rfloor$, further consider the intervals

$$I_j := [\lfloor z_j \rfloor, z_{j+1}) \qquad (j = 0, 1, \ldots, M).$$

Now, observe that, uniformly for $j \in \{0, 1, \ldots, M\}$, as $x \to \infty$,

$$\left|\sum_{t\in I_j} e(hR(t))\pi_K(x,t) - \pi_K(x, \lfloor z_j \rfloor) \sum_{t\in I_j} e(hR(t))\right| \leq o(1) \sum_{t\in I_j} \pi_K(x,t). \tag{5.6}$$

Using the fact that the above intervals I_j are all of the same length, say $\mathcal{L} = \mathcal{L}_x$, it follows from Lemma 5 that, uniformly for $j \in \{0, 1, \ldots, M\}$,

$$\frac{1}{\mathcal{L}} \sum_{t\in I_j} e(hR(t)) \to 0 \qquad (x \to \infty). \tag{5.7}$$

Combining (5.6) and (5.7) allows us to conclude that

$$\left|\sum_{j=0}^{M}\sum_{t\in I_j} e(hR(t))\pi_K(x,t)\right| = o(x).$$

Using this last estimate and recalling estimates (5.4) and (5.5), it follows that estimate (5.3) holds, thus completing the proof of Theorem 2.

6 Proof of Theorem 3

Given a large number x, let $T = T_x := \sum_{w_x\le p\le Y_x} \frac{1}{p}$, and observe that

$$T = \log\left(\frac{\log Y_x}{\log w_x}\right) + o(1) = \log L_x^{-1} + o(1) \qquad (x\to\infty). \tag{6.1}$$

Further let δ_x be a function which tends to 0 as $x\to\infty$, but not too fast in the sense that $\frac{1}{\delta_x} = O(\log_2 T)$.

We will be using the fact that, as a consequence of Lemma 4, as $x\to\infty$,

$$\frac{1}{x}\#\{n\le x : \omega_{\mathcal{B}}(n+j) = k_j,\ j=0,1,\dots,t-1\} = (1+o(1))\prod_{j=0}^{t-1}\frac{1}{x}\#\{n\le x : \omega_{\mathcal{B}}(n) = k_j\}$$

uniformly for positive integers $k_0, k_1, \dots, k_{t-1}$ satisfying $|k_j - T| \le \frac{1}{\delta_x}\sqrt{T}$ and also that

$$\frac{1}{x}\#\left\{n\le x : \frac{|\omega_{\mathcal{B}}(n) - T|}{\sqrt{T}} > \frac{1}{\delta_x}\right\} \to 0 \quad \text{as} \quad x\to\infty.$$

We begin by obtaining an upper bound for the sum

$$S := \sum_{\substack{D_0,D_1,\dots,D_{t-1}\\ D_\nu\in\mathcal{N}(\mathcal{B}),\ D_\nu\le Y_x^{r_x}\\ (D_i,D_j)>1 \text{ for some } i\neq j}} \kappa(D_0)\kappa(D_1)\cdots\kappa(D_{t-1})L_x^t,$$

where r_x is as in Lemma 4, keeping in mind that we allow the above sum to run only over those $D_\nu \le Y_x^{r_x}$, because, as was shown in (3.5), the total contribution of those terms for which at least one of the D_ν exceeds $Y_x^{r_x}$ is negligible. So, let us fix i,j and consider the sum

$$S_{i,j} := \sum_{\substack{D_i,D_j\in\mathcal{N}(\mathcal{B})\\ (D_i,D_j)>1\\ D_i,D_j\le Y_x^{r_x}}} \kappa(D_i)\kappa(D_j)L_x^2.$$

Writing $D_i = UD_i'$ and $D_j = VD_j'$, where U and V have the same prime divisors, $(D_i', D_j') = (U, D_i') = (V, D_j') = 1$, we then have

$$\kappa(D_i)\kappa(D_j) = \kappa(D_i')\kappa(D_j')\kappa(U)\kappa(V).$$

Observe also that, for some positive constant c_1, we have

$$\kappa(U)\kappa(V) < c_1 \left(\prod_{p|U} p^2 \right)^{-1}.$$

From these observations, it follows that, for some positive constant c_2,

$$\begin{aligned} S_{i,j} &< c_2 \sum_{\substack{m=2 \\ m\in\mathcal{N}(\mathcal{B})}}^{\infty} \frac{1}{m^2} \cdot \left(L_x \sum_{D\in\mathcal{N}(\mathcal{B})} \kappa(D) \right)^2 \\ &= c_2 \sum_{\substack{m=2 \\ m\in\mathcal{N}(\mathcal{B})}}^{\infty} \frac{1}{m^2} \cdot \prod_{p\in\mathcal{B}} (1+\kappa(p))^2 \cdot L_x^2. \end{aligned} \tag{6.2}$$

On the other hand, using (6.1),

$$\begin{aligned} \prod_{p\in\mathcal{B}} (1+\kappa(p)) &= \exp\left(\sum_{p\in\mathcal{B}} \log(1+\kappa(p)) \right) \\ &= \exp\left(\sum_{p\in\mathcal{B}} \frac{1}{p} + O(1) \right) = \exp(T + O(1)) \\ &= \exp(-\log L_x + O(1)). \end{aligned}$$

Using this last estimate and the fact that

$$\sum_{\substack{m=2 \\ m\in\mathcal{N}(\mathcal{B})}}^{\infty} \frac{1}{m^2} < \sum_{m>w_x} \frac{1}{m^2} < \frac{2}{w_x},$$

say, it follows from (6.2) that, for some positive constant c_3,

$$S_{i,j} \le \frac{c_3}{w_x} \cdot \frac{1}{L_x^2} \cdot L_x^2 = \frac{c_3}{w_x}.$$

Moreover, in light of the fact that

$$L_x \sum_{\substack{D_\nu\in\mathcal{N}(\mathcal{B}) \\ D_\nu \le Y_x^{rx} \\ \text{for every } \nu=0,1,\ldots,t-1}} \kappa(D_\nu) \le c_4$$

for some absolute constant $c_4 > 0$, we obtain after gathering our estimates that

$$S = O\left(\frac{1}{w_x}\right). \tag{6.3}$$

Now, given arbitrary subsets $E_0, E_1, \ldots, E_{t-1}$ of $\{D : D \in \mathcal{N}(\mathcal{B}),\ D \le Y_x^{r_x}\}$, we have, as $x \to \infty$, in light of (6.3),

$$\sum_{\substack{D_0 \in E_0, \ldots, D_{t-1} \in E_{t-1} \\ (D_i, D_j)=1 \text{ for } i \ne j}} \kappa(D_0)\kappa(D_1)\cdots\kappa(D_{t-1})L_x^t = \prod_{j=0}^{t-1}\left(L_x \sum_{D \in E_j} \kappa(D)\right) + o(x). \tag{6.4}$$

Observe that to the discrepancy $D_N := D(x_1, \ldots, x_N)$ of the real numbers $x_1, \ldots, x_N$ (as defined by (1.1)), one can associate the so-called *star discrepancy*

$$D_N^* = D^*(x_1, \ldots, x_N) := \sup_{0 \le \beta < 1} \left| \frac{1}{N} \sum_{\substack{i=1 \\ \{x_i\} < \beta}}^{N} 1 - \beta \right|$$

and establish that $D_N^* \le D_N \le 2D_N^*$. In light of this observation, defining the function $H_u : [0,1) \to \{0,1\}$ by

$$H_u(y) := \begin{cases} 1 & \text{if } 0 \le y < u, \\ 0 & \text{if } u \le y < 1, \end{cases} \tag{6.5}$$

one can easily establish that

$$D_N^* = \max_{u \in [0,1)} \left(\frac{1}{N}\sum_{n=1}^{N} H_u(x_n) - u\right),$$

implying that if we can show that this last expression tends to 0 as $N \to \infty$, it will allow us to conclude that $D_N = D_{\lfloor x \rfloor} \to 0$ as $N \to \infty$.

To do so, given real numbers $u_0, u_1, \ldots, u_{t-1} \in [0,1)$, choose

$$E_j := \{D \in \mathcal{N}(\mathcal{B}) : |\omega(D) - T| \le \sqrt{T}/\delta_x,\ D \le Y_x^{r_x},\ H_{u_j}(\{\alpha_j q_j^{\omega(D)}\}) = 1\}$$

and apply estimate (6.4).

It follows from this that, if we can prove that

$$\left(\{\alpha_j q_j^{\omega_{\mathcal{B}}(n+j)}\}\right)_{n \ge 1} \quad \text{is uniformly distributed modulo 1} \tag{6.6}$$

for each $j = 0, 1, \ldots, t-1$, it will imply that, as $x \to \infty$,

$$\sum_{\substack{D_j \in \mathcal{N}(\mathcal{B}) \\ D_j \le Y_x^{r_x}}} H_{u_j}(\{\alpha_j q_j^{\omega(D_j)}\})\kappa(D_j)L_x \to u_j \qquad (j = 0, 1, \ldots, t-1),$$

thus allowing us to conclude that

$$\prod_{j=0}^{t-1}\left(\sum_{\substack{D_j\in\mathcal{N}(\mathcal{B})\\ D_j\le Y_x^{r_x}}} H_{u_j}(\{\alpha_j q_j^{\omega(D_j)}\})\kappa(D_j)L_x\right) = u_0u_1\cdots u_{t-1}+o(1) \qquad (x\to\infty),$$

thereby establishing that the sequence $(\underline{y}_n)_{n\ge1}$ is uniformly distributed mod $[0,1)^t$.

Thus, it remains to prove (6.6). To do so, it is enough to prove Corollary 1.

7 Proof of Corollary 1

Let

$$A(n) := \prod_{\substack{p^a\|n\\ p\in\mathcal{B}}} p^a \quad\text{and}\quad M_x := \prod_{p\in\mathcal{B}}\left(1-\frac{1}{p}\right).$$

For every $D\in\mathcal{N}(\mathcal{B})$ with $D\le Y_x^{r_x}$, we have

$$\#\{n\le x : A(n)=D\} = \left(1+O\left(\frac{1}{\log w_x}\right)\right)\frac{x}{D}M_x \qquad (x\to\infty),$$

from which it follows that, as $x\to\infty$,

$$\begin{aligned} B_k(x) &:= \frac{1}{x}\#\{n\le x:\omega_{\mathcal{B}}(n)=k\}\\ &= (1+o(1))M_x\sum_{\substack{D\in\mathcal{N}(\mathcal{B})\\ \omega(D)=k}}\frac{1}{D}+O(U_k(x)), \end{aligned} \tag{7.1}$$

where

$$U_k(x) = M_x\sum_{\substack{D\in\mathcal{N}(\mathcal{B})\\ \omega(D)=k\\ D>Y_x^{r_x}}}\frac{1}{D}+\frac{1}{x}\#\{n\le x : A(n)>Y_x^{r_x},\ \omega(A(n))=k\},$$

thereby implying that

$$\sum_{k\ge1}U_k(x)\to 0 \qquad \text{as } x\to\infty. \tag{7.2}$$

For each positive integer k, let $z_k=\{\alpha q^k\}$. Further, let $H_u(y)$ be the function defined in the proof of Theorem 3 (see (6.5)).

In light of estimate (7.1), we have, as $x\to\infty$,

$$R_x := \frac{1}{x}\sum_{n\le x}H_u(y_n) = \sum_{k\ge1}H_u(z_k)B_k(x)$$

$$(7.3) \qquad = (1+o(1))\sum_{k\geq 1} H_u(z_k) M_x \sum_{\substack{D\in\mathcal{N}(\mathcal{B})\\ \omega(D)=k}} \frac{1}{D} + O\left(\sum_{k\geq 1} U_k(x)\right).$$

Observing that

$$\sum_{a\geq 1, p\in\mathcal{B}} \frac{1}{ap^a} = \sum_{p\in\mathcal{B}} \frac{1}{p} + O\left(\frac{1}{w_x}\right)$$

allows us to write that

$$(7.4) \qquad M_x = \exp\left\{-\sum_{p\in\mathcal{B}} \frac{1}{p} + O\left(\frac{1}{w_x}\right)\right\} = \exp\left\{-T + O\left(\frac{1}{w_x}\right)\right\},$$

say. Hence, it follows from (7.2), (7.3) and (7.4) that

$$(7.5) \qquad R_x = (1+o(1))\sum_{k\geq 1} H_u(z_k)\exp\{-T\}\cdot\frac{T^k}{k!} + o(1) \qquad (x\to\infty).$$

Now, since, for any function δ_x which tends to 0 as $x\to\infty$,

$$\sum_{\frac{|k-T|}{\sqrt{T}} > \frac{1}{\delta_x}} \exp\{-T\}\cdot\frac{T^k}{k!} \to 0 \qquad \text{as } x\to\infty,$$

we obtain that (7.5) can be replaced by

$$(7.6) \qquad R_x = (1+o(1))\sum_{\frac{|k-T|}{\sqrt{T}} \leq \frac{1}{\delta_x}} H_u(z_k) K_k + o(1) \qquad (x\to\infty),$$

where $K_k := \exp\{-T\}\cdot\dfrac{T^k}{k!}$.

On the other hand, observe that for any function ε_x which tends to 0 as $x\to\infty$, we have

$$(7.7) \qquad \max_{\substack{k_1\\ \left|\frac{k_1-T}{\sqrt{T}}\right|\leq\frac{1}{\delta_x}}} \max_{\substack{k_2\\ |k_2-k_1|<\varepsilon_x\sqrt{T}}} \left|\frac{K_{k_2}}{K_{k_1}} - 1\right| \to 0 \qquad \text{as } x\to\infty.$$

Let us now subdivide the interval $[T-\sqrt{T}/\delta_x, T+\sqrt{T}/\delta_x]$ into intervals $I_1, I_2, \ldots, I_s$, where $s = \lfloor 2/(\delta_x\varepsilon_x)\rfloor$, each of length $\varepsilon_x\sqrt{T}$. Since, in light of (7.7), we have

$$(7.8) \qquad \max_{j=1,\ldots,s}\ \max_{k_1,k_2\in I_j} \left|\frac{K_{k_2}}{K_{k_1}} - 1\right| \to 0 \qquad \text{as } x\to\infty$$

and since α is a sharp q-normal number, it follows that, for each $j\in\{1,\ldots,s\}$,

$$\sum_{k\in I_j} H_u(z_k) = (1+o(1))\sum_{k\in I_j} 1 \qquad (x\to\infty).$$

Using this last statement in (7.6), recalling (7.8), and writing $|I_j|$ for the length of the interval I_j, we obtain that, as $x \to \infty$,

$$\begin{aligned}
R_x &= (1+o(1)) \sum_{j=1}^{s} \sum_{k\in I_j} H_u(z_k) K_k \\
&= (1+o(1)) \sum_{j=1}^{s} \sum_{k\in I_j} H_u(z_k) \left(\frac{1}{|I_j|} \sum_{k_1\in I_j} K_{k_1} \right) \\
&= (1+o(1)) \sum_{j=1}^{s} \left(\frac{1}{|I_j|} \sum_{k_1\in I_j} K_{k_1} \right) \sum_{k\in I_j} H_u(z_k) \\
&= (1+o(1)) \sum_{j=1}^{s} \left(\frac{1}{|I_j|} \sum_{k_1\in I_j} K_{k_1} \right) (1+o(1)) u |I_j| \\
&= (1+o(1)) u \sum_{j=1}^{s} \sum_{k\in I_j} K_k \\
&= (1+o(1)) u \sum_{\substack{k \\ |k-T|\le\sqrt{T}/\delta_x}} K_k \\
&= (1+o(1)) u.
\end{aligned}$$

Since this last estimate holds for every real $u \in [0,1)$, it follows that $R_x = o(1)$ as $x \to \infty$ and the proof of Corollary 1 is complete.

8 Final remarks

Using the same techniques as above, one could prove the following result regarding the discrepancy of a t-tuples sequence.

> Let $f_1, f_2, \ldots, f_t \in \mathbb{R}[x]$ be polynomials of positive degree such that the coefficient of the leading term of each f_j is some non-Liouville number α_j. Moreover, let $a_1, a_2, \ldots, a_t$ be distinct integers and let $\mathcal{B}$ be as in Theorem 3. Set
>
> $$\underline{y}_n := \left(f_1(\omega_{\mathcal{B}}(n+a_1)), f_2(\omega_{\mathcal{B}}(n+a_2)), \ldots, f_t(\omega_{\mathcal{B}}(n+a_t)) \right).$$
>
> Then,
>
> $$D(\underline{y}_1, \underline{y}_2, \ldots, \underline{y}_{\lfloor x \rfloor}) \to 0 \qquad \text{as } x \to \infty$$
>
> and similarly, if p_i and $\pi(x)$ stand respectively for the i-th prime and the number of primes not exceeding x,
>
> $$D(\underline{y}_2, \underline{y}_3, \underline{y}_5, \ldots, \underline{y}_{p_{\pi(x)}}) \to 0 \qquad \text{as } x \to \infty.$$

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