Normal numbers in generalized number systems in Euclidean spaces

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Abstract

We introduce the notion of normal numbers for generalized number systems in Euclidean spaces and then explore the relevance of certain conjectures to normality.

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1 Generalized number systems in Euclidean spaces

Given a positive integer k, let $\mathbb{R}_k$ and $\mathbb{Z}_k$ stand respectively for the k-dimensional real Euclidean space and the ring of k-dimensional vectors with integer entries. Fix k and let M be a $k \times k$ matrix with integer elements. Assume that M has k distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ such that $|\lambda_1| > |\lambda_2| > \cdots > |\lambda_k| > 1$. Let $\mathcal{L} := M\,\mathbb{Z}_k$. Then, $\mathcal{L}$ is a subgroup of $\mathbb{Z}_k$ and let t stand for the order of $\mathbb{Z}_k/\mathcal{L}$, so that $t = |\det M|$. Further let $A_0, A_1, \ldots, A_{t-1}$ stand for the residue classes mod $\mathcal{L}$ and let $A_0 = \mathcal{L}$. For each $j \in \{0, 1, \ldots, t-1\}$, choose an arbitrary element $\underline{a}_j \in A_j$ such that the vector $\underline{a}_0$ is the zero vector $\underline{0} = (0, 0, \ldots, 0)$, and then write

$$\mathcal{A} := \{\underline{a}_0, \underline{a}_1, \ldots, \underline{a}_{t-1}\}.$$

If the norm $\|\underline{n}\|$ of $\underline{n} = (n_1, \ldots, n_k)$ is $\|\underline{n}\| = \max_{1\le i\le k} |n_i|$ or $\|\underline{n}\| = \sum_{1\le i\le k} |n_i|$, then the operator norm $\|\cdot\|$ of M^{-1} is $1/|\lambda_k|$ while that of M is $|\lambda_1|$.

Let us now introduce the function $J : \mathbb{Z}_k \to \mathbb{Z}_k$ as follows. Since for each $\underline{n} \in \mathbb{Z}_k$, there exist a unique $\underline{b}_0 \in \mathcal{A}$ for which $\underline{n} - \underline{b}_0 \in \mathcal{L}$ and a unique $\underline{n}_1 \in \mathbb{Z}_k$ for which $\underline{n} = \underline{b}_0 + M\,\underline{n}_1$, that is, $\underline{n}_1 = M^{-1}(\underline{n} - \underline{b}_0)$, we define $J : \mathbb{Z}_k \to \mathbb{Z}_k$ by $J(\underline{n}) = \underline{n}_1$.

We further define the real numbers K, ξ and L by

$$K = \max_{b\in\mathcal{A}} \|b\|, \qquad \xi = \frac{1}{\min_{1\le j\le k} |\lambda_j|} = \left|M^{-1}\right|, \qquad L = \frac{K\xi}{1-\xi}.$$

In [3], the following result was proved.

Lemma 1. *(a) If $\|\underline{n}\| > L$, then $\|J(\underline{n})\| < \|\underline{n}\|$.*

(b) If $\|\underline{n}\| \le L$, then $\|J(\underline{n})\| \le L$.

Since the disks contain only a finite number of elements of $\mathbb{Z}_k$, it follows that the path

$$\underline{n}, \quad J(\underline{n}), \quad J^2(\underline{n}), \ \ldots$$

is ultimately periodic.

Now, let $\mathcal{P}$ stand for the set of periodic elements. Then, $\underline{n} \in \mathcal{P}$ if there is an integer $j \geq 1$ such that $J^j(\underline{n}) = \underline{n}$. The directed graph (over $\mathcal{P}$) is defined by $\underline{n} \to J(\underline{n})$ ($\underline{n} \in \mathcal{P}$). It is clear that $\underline{n} \in \mathcal{P}$ implies that $J(\underline{n}) \in \mathcal{P}$ and that the directed graph $J\mathcal{P} \to \mathcal{P}$, which we denote by $G(\mathcal{P})$, is the union of disjoint directed circles (allowing for loops). Moreover, $\underline{0}\,(\to \underline{0}) \in \mathcal{P}$, and if $\pi \in \mathcal{P}$, then $\|\pi\| \leq L$.

Now, for each $\underline{n} \in \mathbb{Z}_k$ and integer $h \geq 1$, we have

$$\begin{aligned} \underline{n} &= \underline{b}_0 + M\,\underline{b}_1 + \cdots + M^{h-1}\underline{b}_{h-1} + M^h \underline{n}_h, \\ \underline{n}_h &= J^h(\underline{n}_0), \qquad \underline{b}_\nu \in \mathcal{A}. \end{aligned}$$

Further define

$$\ell(\underline{n}) := \begin{cases} 0 & \text{if} \quad \underline{n} \in \mathcal{P}, \\ h & \text{if} \quad \underline{n} \notin \mathcal{P}, \end{cases}$$

where h is the smallest integer for which $\underline{n}_h \in \mathcal{P}$. For this reason, we will say and write that the standard expansion of $\underline{n}$ is $(\underline{b}_0, \underline{b}_1, \ldots, \underline{b}_{h-1} | \pi)$, where $\pi = \underline{n}_h$. In the special case where $\underline{n} = \pi \in \mathcal{P}$, the expansion is written as $(*|\pi)$.

We say that $(\mathcal{A}, M)$ is a *number system* (written for short as NS) in $\mathbb{Z}_k$ if each $\underline{n} \in \mathbb{Z}_k$ can be written as

$$\underline{n} = \underline{b}_0 + M\,\underline{b}_1 + \cdots + M^{h-1}\underline{b}_{h-1}.$$

In other words, $(\mathcal{A}, M)$ is a number system in $\mathbb{Z}_k$ if and only if $\mathcal{P} = \{\underline{0}\}$.

Let H be the set of those $\underline{z} \in \mathbb{R}_k$ which can be expanded as

$$\underline{z} = \sum_{\nu=1}^{\infty} M^{-\nu} \underline{b}_\nu, \qquad \underline{b}_\nu \in \mathcal{A}.$$

The set H is called the *fundamental region* with respect to $(\mathcal{A}, M)$.

For each integer $h \geq 0$, let

$$\Gamma_h := \left\{ \underline{n} : \underline{n} = \sum_{j=0}^{h} M^j \underline{b}_j, \quad \underline{b}_j \in \mathcal{A} \right\},$$

so that in particular $\Gamma_h \subseteq \Gamma_{h+1}$. Letting $\Gamma = \bigcup_{h=0}^{\infty} \Gamma_h$, we have that $\Gamma \subseteq \mathbb{Z}_k$ and one can easily see that $\Gamma = \mathbb{Z}_k$ if and only $(\mathcal{A}, M)$ is a number system.

Since we can write the fundamental region H as

$$H = \bigcup_{\underline{a} \in \mathcal{A}} \left(M^{-1}\underline{a} + M^{-1} H \right),$$

it is easily seen that H is a compact set.

The following result was proved in [3].

Theorem A. *Let λ stand for the Lebesgue measure in $\mathbb{R}_k$.*

(a) We have $\bigcup_{\underline{n}\in\mathbb{Z}_k}(H+\underline{n})=\mathbb{R}_k$.

(b) If $\underline{n}_1,\underline{n}_2\in\Gamma$, $\underline{n}_1\neq\underline{n}_2$, then

$$\lambda(H+\underline{n}_1\cap H+\underline{n}_2)=0.$$

(c) If $\Gamma=\mathbb{Z}_k$, that is if $(\mathcal{A},M)$ is a number system, then

$$\lambda(H+\underline{n}_1\cap H+\underline{n}_2)=0$$

for every $\underline{n}_1,\underline{n}_2\in\mathbb{Z}_k$ with $\underline{n}_1\neq\underline{n}_2$.

2 Just touching covering system

We now introduce the concept of just touching covering system. We say that $(\mathcal{A},M)$ is a *just touching covering system* (for short JTCS) if $\lambda(H+\underline{n}_1\cap H+\underline{n}_2)=0$ for every $\underline{n}_1,\underline{n}_2\in\mathbb{Z}_k$ with $\underline{n}_1\neq\underline{n}_2$.

Interestingly, if $(\mathcal{A},M)$ is a JTCS, then

$$\lambda(M^{-h}\underline{n}_1+M^{-h}H\cap M^{-h}\underline{n}_2+M^{-h}H)=0$$

for every $\underline{n}_1,\underline{n}_2\in\mathbb{Z}_k$ with $\underline{n}_1\neq\underline{n}_2$.

The next two results reveal interesting properties regarding JTCS.

Theorem B. ([4]) *The number system $(\mathcal{A},M)$ is a JTCS if $\Gamma-\Gamma=\mathbb{Z}_k$, that is if every $\underline{n}\in\mathbb{Z}_k$ can be written as $\underline{n}_1-\underline{n}_2$, where $\underline{n}_1,\underline{n}_2\in\mathbb{Z}_k$.*

Theorem C. ([6]) *Given $D\in\mathbb{Z}\setminus\{0\}$, let $A=\{a_0,a_1,\ldots,a_{|D|-1}\}$ (where $a_0=0$) be a complete residue system mod D. Then, $(\mathcal{A},D)$ is a JTCS if and only if $gcd(a_1,\ldots,a_{|D|-1})=1$.*

Let $(\mathcal{A},M)$ be a JTCS and let

$$\xi=\sum_{\ell=-r}^{\infty}M^{-\ell}\underline{c}_\ell \qquad (\underline{c}_\ell\in\mathcal{A}).$$

We write the "integer part" and "fractional part" of ξ as follows:

$$\lfloor\xi\rfloor \;=\; \sum_{\ell=-r}^{0}M^{-\ell}\underline{c}_\ell \qquad (\in\mathbb{Z}_k),$$

$$\{\xi\} \;=\; \sum_{\ell=1}^{\infty} M^{-\ell}\underline{c}_\ell \qquad (\in H).$$

Observe that it is clear that

$$\{M^u\xi\} = \sum_{\ell=1}^{\infty} M^{-\ell}\underline{c}_{u+\ell} \qquad (\in H).$$

Moreover, letting $\beta = \underline{b}_1\underline{b}_2\ldots\underline{b}_k$, let us define

$$H_\beta := \left\{\eta : \eta = \sum_{\ell=1}^{\infty} M^{-\ell}\underline{c}_\ell : \underline{c}_\ell = \underline{b}_\ell \text{ for } \ell = 1,2,\ldots,k\right\}.$$

It is clear that, for a fixed k, any two H_{β_1} and H_{β_2} will be isomorphic since

$$H = \sum_{\ell=1}^{k} M^{-\ell}\underline{b}_\ell + M^{-k}H$$

and

(i) $H = \bigcup_{\beta\in A^k} H_\beta$,

(ii) $\lambda(H_{\beta_1}\cap H_{\beta_2}) = 0$,

(iii) $\lambda(H_{\beta_1}) = \lambda(H_{\beta_2})$,

(iv) $\lambda(H_\beta)t^k = \lambda(H)$.

3 Normal sequences and normal numbers in $\mathbb{R}$

Let $A = \{a_1,\ldots,a_N\}$ be a finite set of letters. Let A^* be the set of finite words over A. Given a word $\alpha\in A^*$, we write $\lambda(\alpha)$ to denote its length (that is, the number of letters in the word α). We let Λ stand for the empty word and write $\lambda(\Lambda) = 0$. The operation $(\alpha,\beta)\to\alpha\beta$ is called *concatenation*. The expression $A^{\mathbb{N}}$ stands for the set of infinite sequences over A, that is, $\beta\in A^{\mathbb{N}}$ if it can be written as $\beta = b_1b_2b_3\ldots$, where each $b_i\in A$. Moreover, given $\beta\in A^{\mathbb{N}}$ and a positive integer T, we set $\beta^T := b_1b_2\ldots b_T$. Given $\gamma,\delta\in A^*$, we let $S(\delta|\gamma)$ stand for $\#\{\epsilon_1,\epsilon_2\in A^* : \gamma = \epsilon_1\delta\epsilon_2\}$, that is, the number of occurrences of δ as a subword in γ.

Definition. Let $\beta\in A^{\mathbb{N}}$. We say that β is a *normal sequence* (over A) if

$$\lim_{T\to\infty}\frac{S(\alpha|\beta^T)}{T} = \frac{1}{N^{\lambda(\alpha)}}$$

for every $\alpha\in A^*$.

4 Normal sequences and normal numbers in $\mathbb{R}_k$

Definition. Let $(\mathcal{A}, M)$ be a number system and let $\eta = \sum_{\ell=1}^{\infty} M^{-\ell} \underline{b}_\ell$, with each $\underline{b}_\ell \in \mathcal{A}$. We say that η is a *normal number in* $\mathbb{R}_k$ with respect to $(\mathcal{A}, M)$ if, for every $\beta \in A^*$,

$$\lim_{N\to\infty} \frac{1}{N} \#\{n \le N : \{M^n \eta\} \in H_\beta\} = \frac{1}{t^{\lambda(\beta)}},$$

where $t = |\det M|$.

The following two assertions are obvious.

- (I) η is a normal number in $\mathbb{R}_k$ with respect to $(\mathcal{A}, M)$ if and only if $\beta = \underline{b}_1 \underline{b}_2 \dots$ is a normal sequence over A.
- (II) Let $E = \{e_1, \dots, e_k\}$, $D = \{d_1, \dots, d_k\}$, $\varphi : E \to D$ defined by $\varphi(e_j) = d_j$, $\beta = b_1 b_2 \dots \in E^{\mathbb{N}}$, $\varphi(\beta) = \varphi(b_1)\varphi(b_2) \dots$ $(\in D^{\mathbb{N}})$. Then, β is a normal sequence in $E^{\mathbb{N}}$ if and only if $\varphi(\beta)$ is a normal sequence in $D^{\mathbb{N}}$.

In light of these assertions, one can easily prove the following theorem.

Theorem 1. *Let $(\mathcal{A}, M)$ be a JTCS with $\mathcal{A} = \{\underline{a}_0 = \underline{0}, \underline{a}_1, \dots, \underline{a}_{t-1}\}$, where $t = |det M|$. Moreover, let $E = \{0, 1, \dots, t-1\}$ and let $\eta = 0.\epsilon_1 \epsilon_2 \dots$ be an arbitrary t-ary normal number. Then, $\psi = \sum_{\ell=1}^{\infty} M^{-\ell} \underline{a}_{\epsilon_\ell}$ is a normal number in $\mathbb{R}_k$ with respect to $(\mathcal{A}, M)$.*

5 Construction of base Q normal numbers

Fix an integer $Q \ge 2$. Let $\mathcal{A}_Q = \{0, 1, \dots, Q-1\}$ and let $\mathcal{A}_Q^*$ stand for the set of words over $\mathcal{A}_Q$. For each integer $N \ge 1$, let $J_N = [Q^{N-1}, Q^N - 1]$. Given an integer $n \in J_N$, write it as $n = \sum_{\nu=0}^{N-1} \epsilon_\nu(n) Q^\nu$ and define $\overline{n} := \epsilon_1(n)\epsilon_1(n) \dots \epsilon_{N-1}(n) \in \mathcal{A}_Q^*$. Finally, we let $\lambda(\overline{n}) = N$ stand for the length of $\overline{n}$.

For each integer $N \ge 3$, consider a subset S_N of $\{1, 2, \dots, N-1\}$, writing it as $S_N = \{\ell_1^{(N)}, \dots, \ell_{r_N}^{(N)}\}$, where the $\ell_i^{(N)}$'s are in increasing order. Assume that $r_N \ge 1$ and that $(r_1 + \dots + r_{N-1})/r_N \to \infty$ as $N \to \infty$.

To each prime $p \in J_N$, let us associate the number

$$\kappa(p) = \epsilon_{\ell_1^{(N)}}(p) \dots \epsilon_{\ell_{r_N}^{(N)}}(p).$$

Let $p_1 < \dots < p_{\pi(J_N)}$ be all the primes included in J_N. Moreover, let σ_N be an arbitrary permutation of $\{1, \dots, \pi(J_N)\}$. Further define

$$\eta_N := \kappa(p_{\sigma_N(1)}) \dots \kappa(p_{\sigma_N(\pi(J_N))}).$$

Finally, consider the number

$$\alpha = 0.\eta_1 \eta_2 \dots$$

Theorem 2. *The number α is a normal number in base Q.*

Proof. This is an easy consequence of an earlier result obtained by Harman and Kátai [8] and according to which, given integer r integers $(1 \leq)\ j_1 < \cdots < j_r\ (\leq N-1)$, setting

$$\Pi\left(J_N \,\middle|\, \begin{matrix} j_1, \ldots, j_r \\ b_1, \ldots, b_r \end{matrix}\right) := \#\{p \in J_N : a_{j_\ell}(p) = b_{j_\ell} \text{ for } \ell = 1, \ldots, r\},$$

we have

$$\max_{\substack{1 \leq j_1 < \cdots < j_r \leq N-1 \\ b_1, \ldots, b_r}} \left| \frac{Q^r \Pi\left(J_N \,\middle|\, \begin{matrix} j_1, \ldots, j_r \\ b_1, \ldots, b_r \end{matrix}\right)}{\pi(J_N)} - 1 \right| \to 0 \qquad (N \to \infty)$$

for every fixed integer $r \geq 1$. □

Theorem 3. *If $S_N = \{1, \ldots, N-1\}$, then Theorem 2 holds without the condition $(r_1 + \cdots + r_{N-1})/r_N \to \infty$ as $N \to \infty$.*

Theorem 4. *Let $\wp_N$ be the set of primes in J_N. Given a prime $p \in J_N$, write its Q-ary expansion as*

$$\overline{p} = \varepsilon_0(p)\epsilon_1(p)\ldots\epsilon_{N-1}(p).$$

Then, set

$$\gamma_N = Concat(\overline{p} : p \in \wp_N).$$

Fix an integer $D \in \mathbb{N}$ and consider the real number

$$\alpha = 0.\gamma_D\gamma_{2D}\ldots = 0.a_1a_2\ldots,$$

say. Further consider the number

$$\alpha^{(\ell)} = 0.Concat(a_m : m \equiv \ell \pmod{D}) = 0.a_\ell a_{D+\ell}a_{D+2\ell}\ldots,$$

say. Let $\ell_1, \ldots, \ell_h$ be a set of distinct residues mod D and consider the real number

$$\delta = 0.Concat(a_m : m \equiv \ell \pmod{D} \text{ for some } \ell \in \{\ell_1, \ldots, \ell_h\}).$$

Then the numbers α, $\alpha^{(\ell)}$ for each $\ell = 0, 1, \ldots, D-1$, δ for each $\ell \in \{\ell_1, \ldots, \ell_h\}$, are all Q-normal numbers.

Proof. The proof can be obtained along the same lines as that of Theorem 2. □

6 The relevance of certain conjectures to normality

6.1 On the conjecture of Chowla and its generalisations

Let $\Omega(1) = 0$ and, for each integer $n \geq 2$, let $\Omega(n) := \sum_{p^a \| n} a$. Then, the Liouville function λ is defined on positive integers n by $\lambda(n) = (-1)^{\Omega(n)}$. An old conjecture of Chowla states that, for any given positive integers $a_1 < a_2 < \cdots < a_k$,

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n \leq x} \lambda(n)\lambda(n+a_1)\cdots\lambda(n+a_k) = 0. \tag{6.1}$$

If the Chowla conjecture were true, then, given any predetermined vector $(\delta_0, \delta_1, \ldots, \delta_k)$, where each $\delta_j \in \{-1, 1\}$, it would follow that

$$\lim_{x\to\infty} \frac{1}{x} \#\{n \leq x : \lambda(n+j) = \delta_j \text{ for } j = 0, 1, \ldots, k\} = \frac{1}{2^{k+1}},$$

in which case, by setting $\epsilon_n = (\lambda(n)+1)/2$, it would also follow that the number

$$\alpha = 0.\epsilon_1\epsilon_2\ldots \tag{6.2}$$

is a binary normal number.

Recently, Terence Tao [9] obtained an important result in this direction, namely by proving that, given any fixed positive integer a,

$$\lim_{x\to\infty} \frac{1}{\log x} \sum_{n \leq x} \frac{\lambda(n)\lambda(n+a)}{n} = 0. \tag{6.3}$$

From this, setting $b_n = (\lambda(n)+1)/2$ and

$$\gamma = 0.b_1b_2\ldots, \tag{6.4}$$

it follows that

$$\lim_{x\to\infty} \frac{1}{\log x} \sum_{\substack{n \leq x \\ b_n = \epsilon_1,\ b_{n+1} = \epsilon_2}} \frac{1}{n} = \frac{1}{4}$$

for every choice of $(\epsilon_1, \epsilon_2) \in \{0, 1\}^2$.

If the Chowla conjecture is true (in the form given by (6.1)), one can prove that

$$\lim_{x\to\infty} \frac{1}{\log x} \sum_{n \leq x} \frac{\lambda(n)\lambda(n+a_1)\cdots\lambda(n+a_k)}{n} = 0. \tag{6.5}$$

Perhaps (6.5) is easier to prove that the original conjecture (6.1).

In any event, from conjecture (6.5), it would follow that the real number γ (in (6.4)) is a binary normal number with "weight $1/n$", meaning that if for each positive integer n, we set $\gamma_n := 0.b_{n+1}b_{n+2}\dots$ and, for any given interval $E = [a, b) \subseteq [0, 1)$, we consider the characteristic function $\chi_E(x) = \begin{cases} 1 & \text{if} \quad x \in E, \\ 0 & \text{if} \quad x \notin E \end{cases}$ along with the corresponding function $S_N(E) = \sum_{n=1}^{N} \frac{1}{n}\chi_E(\gamma_n)$, then

$$\lim_{N\to\infty} \frac{S_N(E)}{\log N} = b - a,$$

namely the length of the interval E.

6.2 A conjecture of Elliott

The following conjecture was stated by Elliott [7] in 1994.

Conjecture 1. (Elliott) *Let $g_1, \dots, g_k$ be multiplicative functions such that $|g_j(n)| \le 1$ for all integers $n \ge 1$, for each $j \in \{1, 2, \dots, k\}$. Moreover, for each $j = 1, 2, \dots, k$, let $a_j \in \mathbb{N}$ and $b_j \in \mathbb{Z}$ be such that $a_r b_t - a_t b_r \ne 0$ when ever $1 \le r < t \le k$. Then, there exist constants $A, \alpha \in \mathbb{R}$ and a slowly oscillating function $L(u)$ such that $|L(u)| = 1$ for all $u \in \mathbb{R}$, such that, as $x \to \infty$,*

$$s(x) := \frac{1}{x}\sum_{n\le x} g_1(a_1 n + b_1)\cdots g_k(a_k n + b_k) = A\, x^{i\alpha} L(\log x) + o(1).$$

If $\limsup_{x\to\infty} |s(x)| = |A| > 0$, then there are Dirichlet characters χ_j and real numbers τ_j for which the series

$$\Re\left(\sum_p \frac{1 - g_j(p)\chi_j(p)p^{-i\tau_j}}{p}\right)$$

converges.

It is clear that the Chowla conjecture would follow from the Elliott conjecture. Another interesting consequence of Conjecture 1 is the following yet unproven result.

Conjecture 2. Let g be a multiplicative function such that $|g(n)| = 1$ for all $n \in \mathbb{N}$ and assume that, for every $\tau \in \mathbb{R}$ and Dirichlet character χ,

$$\sum_p \frac{\Re(1 - g(p)\chi(p)p^{i\tau})}{p} = \infty.$$

Then, given arbitrary positive integers $a_1 < a_2 < \cdots < a_k$,

$$\lim_{x\to\infty} \frac{1}{x}\sum_{n\le x} g(n)g(n + a_1)\cdots g(n + a_k) = 0. \tag{6.6}$$

As a special case of Conjecture 2, one has the following. Fix an integer $Q \geq 2$ and assume that $g(n)^Q = 1$ for all integers $n \geq 1$. Hence the range of $g(\mathbb{N})$ is $\{\xi^\ell : \ell = 0, 1, \ldots, Q-1\}$ for some root of unity ξ, namely $\xi = e^{2\pi i/Q}$. We can therefore write $g(n)$ as $g(n) = \xi^{\epsilon_n}$, where each $\epsilon_n \in \mathcal{A}_Q$. With this set up, let us introduce the real number

$$\alpha = 0.\epsilon_1\epsilon_2\ldots \tag{6.7}$$

If (6.6) were true, then this would imply that α is a normal number in base Q.

Observe that the multiplicative function g could have been chosen differently. Here are some appropriate choices for Q and g:

(I) $Q = 2$ and $g(n) = (-1)^{\Omega(n)}$.

(II) $Q = 2$ and $g(n) = (-1)^{\omega(n)}$.

(III) $Q \geq 2$, $\xi = e^{2\pi i\ell/Q}$ with $(\ell, Q) = 1$ and then choose $g(p) = \xi$ for each prime p and, for each $k \geq 2$, choose choose $g(p^k)$ in an arbitrary way as long as $|g(p^k)| = 1$.

(IV) $Q \geq 2$, $\xi = e^{2\pi i/Q}$ and then, if $p \equiv \ell \pmod{K}$ for any given ℓ and K with $(\ell, K) = 1$ and $(e_\ell, Q) = 1$, choose $g(p) = \xi^{e_\ell}$ for each prime p, and $g(p) = 1$ if $p \mid K$, while choosing $g(p^k)$ in an arbitrary way for each $k \geq 2$ as long as $|g(p^k)| = 1$.

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