ON THE PROXIMITY OF MULTIPLICATIVE FUNCTIONS TO THE NUMBER OF DISTINCT PRIME FACTORS FUNCTION

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Abstract. Given an additive function f and a multiplicative function g, let $E(f,g;x) = \#\{n \le x : f(n) = g(n)\}$. We study the size of $E(\omega,g;x)$ and $E(\Omega,g;x)$, where $\omega(n)$ stands for the number of distinct prime factors of n and $\Omega(n)$ stands for the number of prime factors of n counting multiplicity. In particular, we show that $E(\omega,g;x)$ and $E(\Omega,g;x)$ are $O\left(\frac{x}{\sqrt{\log\log x}}\right)$ for any integer valued multiplicative function g. This improves an earlier result of De Koninck, Doyon and Letendre.

1. Introduction

Given an additive function f and a multiplicative function g, let $E(f,g;x) = \#\{n \le x : f(n) = g(n)\}$. De Koninck, Doyon and Letendre [3] proved that if f is an integer valued additive function such that the corresponding sums

$$A_f(x) := \sum_{p^\alpha \le x} \frac{f(p^\alpha)}{p^\alpha}\left(1-\frac{1}{p}\right) \quad \text{and} \quad B_f(x)^2 := \sum_{p^\alpha \le x} \frac{|f(p^\alpha)|^2}{p^\alpha}$$

satisfy the conditions

(i) $\varphi(x) = \varphi_f(x) := \dfrac{B_f(x)}{|A_f(x)|} \to 0 \quad$ as $x \to \infty$,

(ii) $\max\limits_{z \in \mathbb{C}} \#\{n \le x : f(n) = z\} = O\left(\dfrac{x}{H(x)}\right)$,
where $H(x) = H_f(x) \to \infty$ as $x \to \infty$,

then, given any multiplicative function g, we have $E(f,g;x) = o(x)$ as $x \to \infty$. They also observed that in the case $f = \omega$, we have $A_\omega(x) = (1+o(1))\log\log x$ and $B_\omega(x) = (1+o(1))\sqrt{\log\log x}$ as $x \to \infty$, so that $\varphi(x) = (1+o(1))/\sqrt{\log\log x}$ as $x \to \infty$, while $H(x)$ can be taken as $\sqrt{\log\log x}$ by a result of Balazard [1]. Hence, they showed in particular that $E(\omega,g;x) = o(x)$ as $x \to \infty$. Moreover, observe that this result also applies to $\Omega(n)$.

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Here, we improve the result of De Koninck, Doyon and Letendre in the case where g takes only integer values.

2. Main results

Theorem 1. *Let $g : \mathbb{N} \to \mathbb{Z}$ be an arbitrary multiplicative function. Then,*

$$\#\{n \le x : \omega(n) = g(n)\} \ll \frac{x}{\sqrt{\log\log x}}.$$

Theorem 2. *Let $g : \mathbb{N} \to \mathbb{Z}$ be an arbitrary multiplicative function. Then,*

$$\#\{n \le x : \Omega(n) = g(n)\} \ll \frac{x}{\sqrt{\log\log x}}.$$

Observe that according to Theorem 2 in [3], given any $\epsilon > 0$, there exists a multiplicative function g and an infinite sequence of integers x_n such that

$$E(\omega, g; x_n) \gg \frac{x_n}{(\log\log x_n)^{1/2+\epsilon}}$$

as $n \to \infty$. In their proof, the authors construct a function g which takes only integers values. This means that in some sense, our Theorem 1 is very close to being optimal.

3. Notation and the idea of the proof

We shall write $\mathcal{P}$ for the set of all prime numbers, while the letter p will always stand for a prime number. Let also $\pi(x)$ stand for the number of primes not exceeding x. Also, by $\log_2 x$, we mean $\max(1, \log\log x)$.

Let

$$\begin{aligned} P_1 &:= \{p : p \equiv 1 \mod 7 \,\text{or}\, p \equiv 2 \mod 7\} \cup \{7\}, \\ P_2 &:= \{p : p \equiv 3 \mod 7 \,\text{or}\, p \equiv 4 \mod 7\}, \\ P_3 &:= \{p : p \equiv 5 \mod 7 \,\text{or}\, p \equiv 6 \mod 7\}, \\ A &:= \{n : p|n \Rightarrow p \in P_1 \cup P_2\}, \\ B &:= \{n : p|n \Rightarrow p \in P_3\} \end{aligned}$$

and, for an integer valued multiplicative fonction g, let

$$g_k(n) := \prod_{\substack{p^r \| n \\ p \in P_k}} g(p^r) \quad \text{for} \quad k \in \{1, 2, 3\}.$$

Also, let $A(x) := A \cap [1, x]$, $B(x) := B \cap [1, x]$,

$$a(n) := \prod_{\substack{p^r \| n \\ p \in P_1 \cup P_2}} p^r$$

and

$$b(n) := \prod_{\substack{p^r \| n \\ p \in P_3}} p^r.$$

A key tool for our demonstration is the next two lemmas, which essentially follow from the Turán-Kubilius inequality.

Lemma 1. *Uniformly for* $0 \le \xi(x) \le \sqrt{\log_2 x}$,

$$\#\left\{n \le x : |\omega(n) - \log_2 x| > \xi(x)\sqrt{\log_2 x}\right\} \ll xe^{-\xi(x)^2/3}.$$

Proof. This result follows immediately from Tenenbaum [7, Theorem 3.7] with $t = x$. □

In particular, choosing $\xi(x) = (\log_2 x)^{1/12}$, we have that for $n \le x$,

$$\log_2 x - (\log_2 x)^{7/12} \le \omega(n) \le \log_2 x + (\log_2 x)^{7/12} \tag{3.1}$$

with at most $O(xe^{-(\log_2 x)^{1/6}/3})$ exceptions. Since $xe^{-(\log_2 x)^{1/6}/3} \ll \dfrac{x}{\log_2 x}$, we can assume for the purpose of the demonstration that (3.1) holds for all $n \le x$.

Lemma 2. *Let* $\xi(x) \to \infty$. *We have*

$$\#\left\{n \le x : \left|\omega(a(n)) - \frac{2}{3}\log_2 x\right| > \frac{2\xi(x)}{3}\sqrt{\log_2 x}\right\} \ll \frac{x}{\xi(x)^2}.$$

Proof. This result follows immediately from Tenenbaum [7, Theorem 3.4], with $A(x) = \frac{2}{3}\log_2 x + O(1)$, $B(x)^2 = \frac{2}{3}\log_2 x + O(1)$ and $\epsilon(x) = \frac{\xi(x)}{\sqrt{\log_2 x}}$. □

In particular, choosing $\xi(x) = (\log_2 x)^{1/4}$, we have that for $n \le x$,

$$\frac{2}{3}(\log_2 x - (\log_2 x)^{3/4}) \le \omega(a(n)) \le \frac{2}{3}(\log_2 x + (\log_2 x)^{3/4}) \tag{3.2}$$

with at most $O\left(\dfrac{x}{\sqrt{\log_2 x}}\right)$ exceptions. Therefore, we can also assume that (3.2) holds for all $n \le x$.

Observe that the above two lemmas are also valid if we replace $\omega(n)$ by $\Omega(n)$. In fact, the inequalities (3.1) and (3.2) with the $\omega(n)$ function replaced by the $\Omega(n)$ function will allow us to use Lemma 6 in order to prove Theorem 2.

If $g(n) = \omega(n)$, we have by (3.1) that

$$|g_1(n)g_2(n)g_3(n)| = \omega(n) > \frac{1}{2}\log_2 x. \tag{3.3}$$

It follows that at least one of the three inequalities

$$|g_1(n)g_2(n)| \ge \left(\frac{1}{2}\log_2 x\right)^{2/3},$$

$$|g_1(n)g_3(n)| \geq \left(\frac{1}{2}\log_2 x\right)^{2/3}$$

and

$$|g_2(n)g_3(n)| \geq \left(\frac{1}{2}\log_2 x\right)^{2/3}$$

holds. Indeed, if it was not the case, then we would have $g(n)^2 < (\frac{1}{2}\log_2 x)^2$, thus contradicting (3.3).

In order to prove our results, without any loss in generality, we shall assume that when $g(n) = \omega(n)$,

$$|g_1(n)g_2(n)| \geq \left(\frac{1}{2}\log_2 x\right)^{2/3}.$$

Since $|g(a(n))| \geq \left(\frac{1}{2}\log_2 x\right)^{2/3}$, there exists for x large enough at most one multiple of $|g(a(n))|$ in the interval $[\log_2 x - (\log_2 x)^{7/12}, \log_2 x + (\log_2 x)^{7/12}]$. Hence, given any x, if there exists a unique multiple of an integer m in this interval, we write it as $\kappa(m)$; else, we simply write $\kappa(m) = 0$.

Now, observe that

$$\begin{aligned}
\#\{n \leq x : \omega(n) = g(n)\} &\leq \#\{n \leq x : \omega(n) = \kappa(a(n))\} \\
&= \#\{n \leq x : \omega(b(n)) = \kappa(a(n)) - \omega(a(n))\} \\
&= \sum_{a \in A(x)} \#\left\{b \in B\left(\frac{x}{a}\right) : \omega(b) = \kappa(a) - \omega(a)\right\} \\
&\leq \sum_{a \in A\left(x^{1-\frac{1}{\sqrt{\log x}}}\right)} \#\left\{b \in B\left(\frac{x}{a}\right) : \omega(b) = \kappa(a) - \omega(a)\right\} \\
&\qquad + \#\left\{n \leq x : a(n) > x^{1-\frac{1}{\sqrt{\log x}}}\right\} \\
&= \Sigma_1 + \Sigma_2, \qquad (3.4)
\end{aligned}$$

say.

Furthermore, since Lemma 1 and Lemma 2 also hold for the $\Omega(n)$ function, the above argument also applies if we replace $\omega(n)$ by $\Omega(n)$.

In the next sections, we evaluate both Σ_1 and Σ_2, the latter being a simple consequence of a result of Spearman and Williams [6].

4. Key lemmas

For each $m \geq 1$, let

$$(\mathbb{Z}/m\mathbb{Z})^* = \{h \in \mathbb{N} : 1 \leq h \leq m, (h, m) = 1\}$$

be the set of invertible classes modulo m. Given a subset J of $(\mathbb{Z}/m\mathbb{Z})^*$, for convenience, we write $|J|$ for $\#J$. Hence, we clearly have $|J| \le \phi(m)$, where ϕ stands for the Euler totient function. For each $j \in J$, we set

$$P_j = P_{j,m} = \{p : p \equiv j \mod m\}$$

and

$$P_J = \bigcup_{j \in J} P_j.$$

For each $m \in \mathbb{N}$ and $j \in J$, let

$$P(m,j) = \prod_p \left(1 - \frac{1}{p}\right)^{\theta(p)},$$

where $\theta(p) = \phi(m) - 1$ if $p \equiv j \mod m$ and $\theta(p) = -1$ otherwise. This product is convergent by the prime number theorem for arithmetic progressions.

We shall be using the following result of Ben Saïd and Nicolas [2].

Lemma 3. *Fix $z \in \mathbb{C}$ and let $J \subset (\mathbb{Z}/m\mathbb{Z})^*$ and $a_J(n,z) \in \mathbb{C}$, $b_J(n,z) \in \mathbb{R}^+ \cup \{0\}$ be two multiplicative functions in n such that, for all integers $n \ge 1$, $|a_J(n,z)| \le b_J(n,z)$ and for which the corresponding Dirichlet series*

$$F_J(s,z) = \sum_{n=1}^{\infty} \frac{a_J(n,z)}{n^s}$$

and

$$F_J^+(s,z) = \sum_{n=1}^{\infty} \frac{b_J(n,z)}{n^s}$$

are holomorphic in the half-plane $\Re s > 1$. Moreover, assume that there exist real numbers $B > 0$, $0 < c < 1/2$ and $0 \le \delta < 1$ such that in the half-plane $\Re s > 1$, the series $F_J(s,z)$ has an Euler product representation of the form

$$F_J(s,z) = H_J(s,z) \prod_{j \in J} \prod_{p \equiv j \mod m} \left(1 - \frac{1}{p^s}\right)^{-z},$$

where $H_J(s,z)$ is holomorphic in

$$D_c := \left\{ s : \Re s \ge 1 - \frac{c}{\log(2 + |\Im s|)} \right\}$$

and satisfies

$$|H_J(s,z)| \le B(3 + |\Im s|)^{\delta} \qquad (s \in D_c). \tag{4.1}$$

Moreover, assume that in the half-plane $\Re s > 1$, the series $F_J^+(s,z)$ has a representation of the form

$$F_J^+(s,z) = H_J^+(s,z)\zeta(s)^z,$$

where $H_J^+(s,z)$ is holomorphic in D_c and satisfies (4.1).

Letting

$$A_J(x,z) = \sum_{n\le x} a_J(n,z),$$

we then have

$$A_J(x,z) = \frac{x}{(\log x)^{1-|J|z/\phi(m)}}\left(\frac{H_J(1,z)C_{J,m}^z}{\Gamma(|J|z/\phi(m))} + O\left(\frac{\log_2 x}{\log x}\right)\right),$$

where the constant in the O term depends on m, B, c and δ, with the convention that $1/\Gamma(0) = 0$, and

$$C_{J,m} = \prod_{j\in J} P(m,j)^{-1/\phi(m)}.$$

Now, let $J \subset (\mathbb{Z}/m\mathbb{Z})^*$ and consider the multiplicative function $\varrho_{J,m}$ defined on prime powers p^α by

$$\varrho_{J,m}(p^\alpha) = \begin{cases} 1 & \text{if there exists } j \in J \text{ such that } p \equiv j \mod m, \\ 0 & \text{otherwise.} \end{cases}$$

The next two results are direct applications of Lemma 3 with $a_J(n,z) = \varrho_{J,m}(n)z^{\omega(n)}$(respectively $a_J(n,z) = \varrho_{J,m}(n)z^{\Omega(n)}$) and $b_J(n,z) = \varrho_{J,m}(n)|z|^{\omega(n)}$ (respectively $b_J(n,z) = \varrho_{J,m}(n)|z|^{\Omega(n)}$). We will now obtain asymptotic formulas for

$$U(x,z) := \sum_{n\le x} \varrho_{J,m}(n)z^{\omega(n)} \quad \text{and} \quad V(x,z) := \sum_{n\le x} \varrho_{J,m}(n)z^{\Omega(n)}. \tag{4.2}$$

Lemma 4. *For any real $R > 0$, there exists a real constant $D_1 > 0$ such that*

$$U(x,z) = \frac{x}{(\log x)^{1-|J|z/\phi(m)}}\left(D_1^z \frac{\prod_{\substack{j\in J \\ p\equiv j \ (m)}} \left(1+\frac{z}{(p-1)}\right)\left(1-\frac{1}{p}\right)^z}{\Gamma(|J|z/\phi(m))} + O\left(\frac{\log_2 x}{\log x}\right)\right) \tag{4.3}$$

uniformly for $|z| < R$. Moreover, for any real $\delta > 0$, there exists a real constant $D_2 > 0$ such that

$$V(x,z) = \frac{x}{(\log x)^{1-|J|z/\phi(m)}}\left(D_2^z \frac{\prod_{\substack{j\in J \\ p\equiv j \ (m)}} \left(1-\frac{z}{p}\right)^{-1}\left(1-\frac{1}{p}\right)^z}{\Gamma(|J|z/\phi(m))} + O\left(\frac{\log_2 x}{\log x}\right)\right) \tag{4.4}$$

uniformly for $|z| < 2 - \delta$.

Proof. The proof follows essentially along the same lines as the discussion in Tenenbaum [7, pp. 204–205]. Here are the details.

For $\Re s > 1$,

$$F(s,z) = \sum_{n=1}^{\infty} \frac{\varrho_{J,m}(n) z^{\omega(n)}}{n^s} = \prod_{\substack{j \in J \\ p \equiv j \mod m}} \left(1 + \frac{z}{p^s - 1}\right)$$
$$= H_1(s,z) \prod_{\substack{j \in J \\ p \equiv j \mod m}} \left(1 - \frac{1}{p^s}\right)^{-z} \tag{4.5}$$

with

$$H_1(s,z) = \prod_{\substack{j \in J \\ p \equiv j \mod m}} \left(1 + \frac{z}{p^s - 1}\right)\left(1 - \frac{1}{p^s}\right)^{z} = \sum_{n=1}^{\infty} \frac{c(n,z)}{n^s},$$

say, where $c(n,z)$ is clearly a multiplicative function of n. If p is a prime such that $p \not\equiv j \mod m$ for all $j \in J$, then we easily see that $c(p^\nu, z) = 0$ for every integer $\nu \geq 1$. Also, for a prime p such that $p \equiv j \mod m$ for some $j \in J$, the identity

$$1 + \sum_{\nu=1}^{\infty} c(p^\nu, z)\xi^\nu = \left(1 + \frac{\xi z}{1-\xi}\right)(1-\xi)^z \qquad (|\xi| < 1) \tag{4.6}$$

holds. Indeed, for such a prime p, observe that

$$\left(1 + \frac{p^{-s}z}{1-p^{-s}}\right)(1-p^{-s})^z = \sum_{\nu=0}^{\infty} \frac{c(p^\nu,z)}{p^{\nu s}} = 1 + \sum_{\nu=1}^{\infty} \frac{c(p^\nu,z)}{p^{\nu s}},$$

so that (4.6) follows using the uniqueness of representation of Dirichlet series and the substitution $\xi = p^{-s}$ for all primes p. It follows in particular that $c(p,z) = 0$. The Cauchy formula now gives, for $|z| < R$,

$$|c(p^\nu, z)| \leq M(R) 2^{\nu/2},$$

where

$$M(R) := \sup_{|z| \leq R, |\xi| \leq 1/\sqrt{2}} \left| \left(1 + \frac{\xi z}{1-\xi}\right)(1-\xi)^z \right|.$$

Hence, for any $\sigma > \frac{1}{2}$, we have

$$\sum_p \sum_{\nu=1}^{\infty} |c(p^\nu, z)| p^{-\nu\sigma} \leq 2M(R) \sum_p \frac{1}{p^\sigma(p^\sigma - \sqrt{2})} < \infty.$$

We can thus conclude that for every $\epsilon > 0$, the associated Dirichlet series $H_1(s,z)$ is convergent and bounded for $\sigma \geq \frac{1}{2} + \epsilon$.

Similarly, letting

$$H_1^+(s,z) = \prod_{\substack{j\in J \\ p\equiv j \mod m}} \left(1+\frac{|z|}{p^s-1}\right)^{-1}\left(1-\frac{1}{p^s}\right)^{|z|},$$

we can also conclude that $H_1^+(s,z)$ is bounded for $\sigma \geq \frac{1}{2}+\epsilon$. Hence, the conditions of Lemma 3 are fulfilled with $a_J(n,z) = \varrho_{J,m}(n)z^{\omega(n)}$ and $b_J(n,z) = |z|^{\omega(n)}$, and the proof of (4.3) is complete.

Proceeding as above, for $\Re s > 1$,

$$\begin{aligned} G(s,z) = \sum_{n=1}^{\infty} \frac{\varrho_{J,m}(n)z^{\Omega(n)}}{n^s} &= \prod_{\substack{j\in J \\ p\equiv j \mod m}} \left(1-\frac{z}{p^s}\right)^{-1} \\ &= H_2(s,z) \prod_{\substack{j\in J \\ p\equiv j \mod m}} \left(1-\frac{1}{p^s}\right)^{-z} \end{aligned}$$

with

$$H_2(s,z) = \prod_{\substack{j\in J \\ p\equiv j \mod m}} \left(1-\frac{z}{p^s}\right)^{-1}\left(1-\frac{1}{p^s}\right)^{z} = \sum_{n=1}^{\infty}\frac{d(n,z)}{n^s},$$

say, where $d(n,z)$ is also a multiplicative function of n. If p is a prime such that $p \not\equiv j \mod m$ for all $j \in J$, then again $d(p^\nu) = 0$ for every integer $\nu \geq 1$. Also, for a prime p such that $p \equiv j \mod m$ for some $j \in J$, we obtain an identity similar to (4.6), namely

$$1 + \sum_{\nu=1}^{\infty} d(p^\nu, z)\xi^\nu = (1-\xi z)^{-1}(1-\xi)^z. \tag{4.7}$$

It follows in particular that $d(p,z) = 0$ for all primes p. Since the right hand side of (4.7) is a holomorphic function in the disk $|\xi| < \min(1, |z|^{-1})$, the Cauchy formula gives, for all $0 < \delta < 1$ and $|z| < 2 - 2\delta$,

$$|d(p^\nu, z)| \leq N(\delta)(2-\delta)^\nu,$$

where

$$N(\delta) := \sup_{\substack{|\xi|\leq 1/(2-\delta) \\ |z|\leq 2-2\delta}} |(1-\xi z)^{-1}(1-\xi)^z|.$$

Hence, for any given $\sigma > \frac{1}{2}$, we have

$$\sum_p \sum_{\nu=1}^{\infty} |d(p^\nu, z)|p^{-\nu\sigma} \leq N(\delta)(2-\delta)^2 \sum_p \frac{1}{p^\sigma(p^\sigma - (2-\delta))} < \infty.$$

As in the proof of (4.3), we obtain that for every $\epsilon > 0$, the associated Dirichlet series is convergent and bounded for $\sigma \geq \frac{1}{2}+\epsilon$.

Then, setting

$$H_2^+(s,z) := \prod_p \left(1 - \frac{|z|}{p^s}\right)^{-1} \left(1 - \frac{1}{p^s}\right)^{-|z|},$$

we conclude that $H_2^+(s,z)$ is also bounded for $\sigma > \frac{1}{2} + \epsilon$. Again, the conditions of Lemma 3 are fulfilled with $a_J(n,z) = \varrho_{J,m}(n) z^{\Omega(n)}$ and $b_J(n,z) = |z|^{\Omega(n)}$, which proves (4.4). □

With the help of Lemma 4, we are now able to obtain asymptotic formulas for

$$\pi_{J,m,k}(x) := \#\{n \le x : p|n \Rightarrow \exists j \in J \text{ such that } p \equiv j \mod m, \omega(n) = k\}$$

and

$$\sigma_{J,m,k}(x) := \#\{n \le x : p|n \Rightarrow \exists j \in J \text{ such that } p \equiv j \mod m, \Omega(n) = k\}.$$

With the same notation as in Lemma 4, we set $g_1(z) := \frac{D_1^z H_1(1,z)}{\Gamma(1+|J|z/\phi(m))}$ and $g_2(z) := \frac{D_2^z H_2(1,z)}{\Gamma(1+|J|z/\phi(m))}$.

Lemma 5. *Let $R > 0$ and $\delta > 0$ be real numbers. Under the assumptions of Lemma 3, and setting $\varrho_{J,m} z^{\omega(n)} = \sum_{k=0}^{\infty} c_k(n) z^k$, we have uniformly for $1 \le k \le R|J|\log_2 x/\phi(m)$,*

$$\begin{aligned}\pi_{J,m,k}(x) &= \frac{x}{\log x} \frac{\left(\frac{|J|\log_2 x}{\phi(m)}\right)^{k-1}}{(k-1)!} \left(g_1\left(\frac{\phi(m)(k-1)}{|J|\log_2 x}\right)\right. \\ &\quad \left. + O\left(\left(\frac{|J|\log_2 x}{\phi(m)}\right)^{-2}(k-1) + \frac{(\log_2 x)^2}{k \log x}\frac{|J|}{\phi(m)}\right)\right).\end{aligned}$$

Also, setting $\varrho_{J,m} z^{\Omega(n)} = \sum_{k=0}^{\infty} d_k(n) z^k$, we have uniformly for $k \le (2-\delta)|J|\log_2 x/\phi(m)$,

$$\begin{aligned}\sigma_{J,m,k}(x) &= \frac{x}{\log x} \frac{\left(\frac{|J|\log_2 x}{\phi(m)}\right)^{k-1}}{(k-1)!} \left(g_2\left(\frac{\phi(m)(k-1)}{|J|\log_2 x}\right)\right. \\ &\quad \left. + O\left(\left(\frac{|J|\log_2 x}{\phi(m)}\right)^{-2}(k-1) + \frac{(\log_2 x)^2}{k \log x}\frac{|J|}{\phi(m)}\right)\right).\end{aligned}$$

Proof. We will only prove the first statement, the proof of the second statement being similar. Since $\varrho_{J,m} z^{\omega(n)} = \sum_{k=0}^{\infty} c_k(n) z^k$, we clearly have

$$c_k(n) = \frac{1}{2\pi i} \int_{|z|=r} \frac{\varrho_{J,m} z^{\omega(n)}}{z^{k+1}} dz,$$

for any $r \leq R$. Observing that $\pi_{J,m,k}(x) = \sum_{n\leq x} c_k(n)$, and letting $U(x,z)$ be as in (4.2), we have that

$$\pi_{J,m,k}(x) = \sum_{n\leq x} c_k(n) = \frac{1}{2\pi i}\sum_{n\leq x}\int_{|z|=r}\frac{\varrho_{J,m} z^{\omega(n)}}{z^{k+1}}\,dz = \frac{1}{2\pi i}\int_{|z|=r}\frac{U(x,z)}{z^{k+1}}\,dz.$$

Hence, writing $f(z)$ for $H_1(1,z)$ and for any $r, r_1, r_2 \leq R$, we have

$$\begin{aligned}
\pi_{J,m,k}(x) &= \frac{1}{2\pi i}\int_{|z|=r}\frac{x}{z^{k+1}(\log x)^{1-|J|z/\phi(m)}}\left(\frac{D_1^z f(z)}{\Gamma(|J|z/\phi(m))}\right.\\
&\qquad\qquad\left. + O\left(\frac{\log_2 x}{\log x}\right)\right)dz\\
&= \frac{1}{2\pi i}\int_{|z|=r_1}\frac{\frac{D_1^z f(z)}{\Gamma(|J|z/\phi(m))}x(\log x)^{|J|z/\phi(m)-1}}{z^{k+1}}\,dz\\
&\quad + O\left(\int_{|z|=r_2}\frac{x\log_2 x(\log x)^{|J|\Re z/\phi(m)-2}}{|z|^{k+1}}\,|dz|\right)\\
&= \frac{1}{2\pi i}\int_{|z|=r_1}\frac{\frac{D_1^z f(z)}{\Gamma(|J|z/\phi(m))}x(\log x)^{|J|z/\phi(m)-1}}{z^{k+1}}\,dz\\
&\quad + O\left(\frac{x\log_2 x}{{r_2}^k(\log x)^2}\int_0^{2\pi}(\log x)^{|J|r_2\cos\theta/\phi(m)}d\theta\right)\\
&= \frac{x}{\log x}\frac{1}{2\pi i}\int_{|z|=r_1}\frac{D_1^z f(z)}{z^k\Gamma(|J|z/\phi(m)+1)}(\log x)^{|J|z/\phi(m)}\,dz\\
&\quad + O\left(x\frac{\log_2 x}{(\log x)^2}\left(\frac{|J|\log_2 x}{k\phi(m)}\right)^k\int_0^{2\pi}e^{k\cos\theta}\,d\theta\right), \qquad (4.8)
\end{aligned}$$

where we chose $r_2 := \dfrac{k\phi(m)}{|J|\log_2 x} \leq R$.

Also, we have that

$$\begin{aligned}
\int_0^{2\pi} e^{k\cos\theta}\,d\theta &\leq 2\int_0^{\pi/2} e^{k\cos\theta}\,d\theta + \pi = 2\int_0^1 e^{kt}\frac{dt}{\sqrt{1-t^2}} + \pi\\
&\leq 2\int_0^1 \frac{e^{kt}}{\sqrt{1-t}}\,dt + \pi. \qquad (4.9)
\end{aligned}$$

Using the relation

$$\Gamma(s)n^{-s} = \int_0^\infty u^{s-1}e^{-nu}\,du \qquad (n\in\mathbb{N},\ \Re(s)>0) \qquad (4.10)$$

with $n = k$ and $s = 1/2$, we have that

$$\begin{aligned}\int_0^1 \frac{e^{kt}}{\sqrt{1-t}}\,dt &\leq \int_{-\infty}^1 \frac{e^{kt}}{\sqrt{1-t}}\,dt = \int_0^\infty \frac{e^{k(1-u)}}{\sqrt{u}}\,du \\ &= e^k \int_0^\infty \frac{e^{-ku}}{\sqrt{u}}\,du = e^k\Gamma(1/2)k^{-1/2} \\ &\leq e\Gamma(1/2)\frac{k^k}{k!}, \qquad (4.11)\end{aligned}$$

where the last inequality follows from Stirling's formula.

Using (4.11) in (4.9) yields

$$\int_0^{2\pi} e^{k\cos\theta}\,d\theta \leq 2e\Gamma(1/2)\frac{k^k}{k!} + \pi \leq c\frac{k^k}{k!} \qquad (4.12)$$

for some absolute constant $c > 0$. Hence, inserting (4.12) in (4.8), we get that

$$\begin{aligned}\pi_{J,m,k}(x) &= \frac{x}{\log x}\left(\frac{1}{2\pi i}\int_{|z|=r_1} \frac{D_1^z f(z)}{z^k\Gamma(|J|z/\phi(m)+1)}(\log x)^{|J|z/\phi(m)}\,dz\right. \\ &\qquad\left. +O\left(\frac{\log_2 x}{k!\log x}\left(\frac{|J|\log_2 x}{\phi(m)}\right)^k\right)\right).\end{aligned}$$

We will now evaluate the integral

$$\begin{aligned}I_k &= I_{J,m,k} := \frac{1}{2\pi i}\int_{|z|=r_1} \frac{D_1^z f(z)}{z^k\Gamma(|J|z/\phi(m)+1)}(\log x)^{|J|z/\phi(m)}\,dz \\ &= \frac{1}{2\pi i}\int_{|z|=r_1} \frac{g_1(z)}{z^k}(\log x)^{|J|z/\phi(m)}\,dz. \qquad (4.13)\end{aligned}$$

Here, it is somewhat simpler to take $r_1 := \frac{(k-1)\phi(m)}{|J|\log_2 x}$. This clearly yields the same result since the only singularity of the integrand in I_k is at $z = 0$.

Observing that, with this new radius r_1, we get

$$\begin{aligned}\frac{1}{2\pi i}\int_{|z|=r_1} (z-r_1)\frac{(\log x)^{|J|z/\phi(m)}}{z^k}\,dz &= \mathrm{Res}_{z=0}(z-r_1)\frac{(\log x)^{|J|z/\phi(m)}}{z^k} \\ &= \frac{(|J|\log_2 x/\phi(m))^{k-2}}{(k-2)!} \\ &\qquad -r_1\frac{(|J|\log_2 x/\phi(m))^{k-1}}{(k-1)!} \\ &= 0,\end{aligned}$$

it follows from (4.13), using the residue theorem, that

$$I_k = \frac{g_1(r_1)}{2\pi i}\int_{|z|=r_1} \frac{(\log x)^{|J|z/\phi(m)}}{z^k}\,dz$$

$$
\begin{aligned}
&+\ \frac{1}{2\pi i}\int_{|z|=r_1}\left(g_1(z)-g_1(r_1)-g_1'(r_1)(z-r_1)\right)\frac{(\log x)^{|J|z/\phi(m)}}{z^k}\,dz\\
&=\frac{g_1(r_1)}{(k-1)!}\left(\frac{|J|\log_2 x}{\phi(m)}\right)^{k-1}\\
&\quad+\ O\left(\int_{|z|=r_1}\left(g_1(z)-g_1(r_1)-g_1'(r_1)(z-r_1)\right)\frac{(\log x)^{|J|z/\phi(m)}}{z^k}\,dz\right)\\
&=\frac{g_1(r_1)}{(k-1)!}\left(\frac{|J|\log_2 x}{\phi(m)}\right)^{k-1}+O\left(L(k)\right), \qquad (4.14)
\end{aligned}
$$

say. On the other hand, integration by part yields

$$
g_1(z)-g_1(r_1)-g_1'(r_1)(z-r_1)=(z-r_1)^2\int_0^1(1-t)g_1''(r_1+t(z-r_1))\,dt. \qquad (4.15)
$$

Moreover, observe that for each $t\in[0,1]$,

$$
|r_1+t(z-r_1)|=|r_1(1-t)+tz|\le r_1(1-t)+tr_1=r_1. \qquad (4.16)
$$

Since the closed disk centered at 0 of radius R is compact, the function $g_1''(z)$ reaches its maximum on the boundary of the disk. Letting c_0 be this maximum value, it follows from (4.15) and (4.16) that, using the change of variable $z=r_1e^{i\theta}$, we have

$$
\begin{aligned}
|L(k)| &\le \int_{C'}\left|(z-r_1)^2\int_0^1(1-t)g_1''(r_1+t(z-r_1))\,dt\right|\left|\frac{(\log x)^{|J|z/\phi(m)}}{z^k}\right||dz|\\
&\le c_0\int_{|z|=r_1}|z-r_1|^2\left|\frac{(\log x)^{|J|z/\phi(m)}}{z^k}\right||dz|\\
&= c_0{r_1}^{3-k}\int_0^{2\pi}|e^{i\theta}-1|^2e^{r_1\cos\theta(|J|\log_2 x)/\phi(m)}\,d\theta\\
&= 2c_0{r_1}^{3-k}\int_0^{2\pi}(1-\cos\theta)e^{(k-1)\cos\theta}\,d\theta. \qquad (4.17)
\end{aligned}
$$

Now observe that

$$
\begin{aligned}
\int_0^{2\pi}(1-\cos\theta)e^{(k-1)\cos\theta}\,d\theta &= 2\int_0^{\pi/2}(1-\cos\theta)e^{(k-1)\cos\theta}\,d\theta\\
&\quad+\int_{\pi/2}^{3\pi/2}(1-\cos\theta)e^{(k-1)\cos\theta}\,d\theta \qquad (4.18)\\
&\le 2\int_0^{\pi/2}(1-\cos\theta)e^{(k-1)\cos\theta}\,d\theta+\int_{\pi/2}^{3\pi/2}2\,d\theta\\
&= 2\int_0^{\pi/2}(1-\cos\theta)e^{(k-1)\cos\theta}\,d\theta+2\pi
\end{aligned}
$$

$$
\begin{aligned}
&= 2\int_0^1 \frac{(1-t)e^{(k-1)t}}{\sqrt{1-t^2}}\,dt + 2\pi \\
&\le 2\int_0^1 e^{(k-1)t}\sqrt{1-t}\,dt + 2\pi \\
&\le 2\int_{-\infty}^1 e^{(k-1)t}\sqrt{1-t}\,dt + 2\pi \\
&= 2\int_0^\infty \sqrt{u}e^{(k-1)(1-u)}\,du + 2\pi \\
&= 2e^{k-1}\int_0^\infty \sqrt{u}e^{-(k-1)u}\,du + 2\pi. \qquad (4.19)
\end{aligned}
$$

Using once more relation (4.10) but this time with $n = k-1$ and $s = 3/2$, we can replace (4.18) by

$$
\begin{aligned}
\int_0^{2\pi} (1-\cos\theta)e^{(k-1)\cos\theta}\,d\theta &\le 2e^{k-1}\Gamma(3/2)(k-1)^{-3/2} + 2\pi \\
&\le 2e\Gamma(3/2)\frac{(k-1)^{k-2}}{(k-1)!} + 2\pi, \qquad (4.20)
\end{aligned}
$$

where, as in (4.11), we used Stirling's formula.

Substituting (4.20) in (4.17), we obtain that, for some positive constant C,

$$
|L(k)| \le Cr_1^{3-k}\frac{(k-1)^{k-2}}{(k-1)!}. \qquad (4.21)
$$

Hence, combining (4.8), (4.13), (4.14) and (4.21), we obtain

$$
\begin{aligned}
\pi_{J,m,k}(x) &= \frac{x}{\log x}\Bigg(g_1\left(\frac{(k-1)\phi(m)}{|J|\log_2 x}\right)\frac{\left(\frac{|J|\log_2 x}{\phi(m)}\right)^{k-1}}{(k-1)!} \\
&\quad + O\Bigg(\frac{(k-1)}{(k-1)!}\left(\frac{|J|\log_2 x}{\phi(m)}\right)^{k-3} + \frac{\log_2 x}{k!\log x}\left(\frac{|J|\log_2 x}{\phi(m)}\right)^{k}\Bigg)\Bigg) \\
&= \frac{x}{\log x}\frac{\left(\frac{|J|\log_2 x}{\phi(m)}\right)^{k-1}}{(k-1)!}\Bigg(g_1\left(\frac{(k-1)\phi(m)}{|J|\log_2 x}\right) \\
&\quad + O\left((k-1)\left(\frac{|J|\log_2 x}{\phi(m)}\right)^{-2} + \frac{(\log_2 x)^2}{k\log x}\frac{|J|}{\phi(m)}\right)\Bigg)
\end{aligned}
$$

uniformly for $k \le R|J|\log_2 x/\phi(m)$, thereby completing the proof of Lemma 5. □

Using Lemma 5 with $J = \{5, 6\}$ and $m = 7$, we have in particular

$$\pi_{B,k}(x) := \pi_{J,7,k}(x) \ll \frac{x\left(\frac{\log_2 x}{3}\right)^{k-1}}{(k-1)!\log x} \tag{4.22}$$

uniformly for $k \leq (R/3)\log_2 x$ and

$$\sigma_{B,k}(x) := \sigma_{J,7,k}(x) \ll \frac{x\left(\frac{\log_2 x}{3}\right)^{k-1}}{(k-1)!\log x} \tag{4.23}$$

uniformly for $k \leq ((2-\delta)/3)\log_2 x$.

Lemma 6. *The estimate*

$$\pi_{B,k}(y) \ll \frac{y}{(\log y)^{2/3}\sqrt{\log_2 y}}$$

holds uniformly for all integers $1 \leq k \leq \frac{R}{3}\log_2 x$, *while*

$$\sigma_{B,k}(y) \ll \frac{y}{(\log y)^{2/3}\sqrt{\log_2 y}}$$

holds uniformly for all integers $1 \leq k \leq \frac{2-\delta}{3}\log_2 x$.

Proof. We use (4.22) and Stirling's formula to get

$$\pi_{B,k}(y) \ll \frac{y}{\log y}\frac{\left(\frac{\log_2 y}{3}\right)^{k-1}}{(k-1)!} \ll \frac{y}{\sqrt{k-1}\log y}\left(\frac{\frac{e}{3}\log_2 y}{k-1}\right)^{k-1} \tag{4.24}$$

uniformly for all integers $1 \leq k \leq \frac{A}{3}\log_2 x$. Note that the second bound in this Lemma can be established in the same manner but this time by using (4.23) instead of (4.22). Now, fix $M > e^2$ and, for $t \geq 1$, set $\varphi(t) := \frac{(M/t)^t}{\sqrt{t}}$ and $\psi(t) := (M/t)^t$. Taking logarithms of $\varphi(t)$ and then taking derivatives, we obtain

$$\frac{d}{dt}\left(t\log(M/t) - \frac{\log t}{2}\right) = \log M - \log t - 1 - \frac{1}{2t}.$$

Since this last expression is positive if $t \leq M/e^2$, it follows that $\sup_{t\geq 1}\varphi(t) \leq \frac{e}{\sqrt{M}}\sup_{t\geq M/e^2}\psi(t)$. Similarly, we have that $\psi(t)$ attains its maximum when $t = M/e$ in which case $\psi(M/e) = e^{M/e}$. Thus, it follows that $\sup_{t\geq 1}\varphi(t) \leq \frac{e^{M/e+1}}{\sqrt{M}}$. Choosing $M = \frac{e}{3}\log_2 y$, we obtain

$$\frac{1}{\sqrt{k-1}}\left(\frac{\frac{e}{3}\log_2 y}{k-1}\right)^{k-1} \ll \frac{e^{\frac{\log_2 y}{3}}}{\sqrt{\log_2 y}} \ll \frac{(\log y)^{1/3}}{\sqrt{\log_2 y}}. \tag{4.25}$$

Inserting (4.25) in (4.24), the result follows. □

Lemma 7. *There exist constants $C > 0$ and $D > 0$ such that*

$$A(y) = C(1 + o(1))\frac{y}{(\log y)^{1/3}}$$

and

$$B(y) = D(1 + o(1))\frac{y}{(\log y)^{2/3}}.$$

Proof. This follows immediately from Theorem 1.1 in the paper of Spearman and Williams [6]. □

5. Proof of Theorems 1 and 2

Since by (3.1) and (3.2), we have

$$\kappa(a(n)) - \omega(a(n)) \le \log_2 x + (\log_2 x)^{7/12} - \frac{2}{3}(\log_2 x - (\log_2 x)^{3/4}) \le \frac{1}{2}\log_2 x,$$

we can use Lemma 6 with $y = x/a$ and Lemma 7 to obtain

$$\begin{aligned}
\Sigma_1 &\ll \sum_{a \in A\left(x^{1-\frac{1}{\sqrt{\log x}}}\right)} \frac{x}{a(\log(x/a))^{2/3}\sqrt{\log_2 x/a}} \\
&\ll \frac{1}{\sqrt{\log_2 x}} \sum_{a \in A\left(x^{1-\frac{1}{\sqrt{\log x}}}\right)} \frac{x}{a(\log(x/a))^{2/3}} \\
&\ll \frac{1}{\sqrt{\log_2 x}} \sum_{a \in A(x)} \#B\left(x/a\right) \\
&= \frac{\lfloor x \rfloor}{\sqrt{\log_2 x}} \le \frac{x}{\sqrt{\log_2 x}}. \qquad (5.1)
\end{aligned}$$

It remains to show that

$$\Sigma_2 = \#\left\{n \le x : a(n) > x^{1-\frac{1}{\sqrt{\log x}}}\right\} \ll \frac{x}{\sqrt{\log_2 x}}. \qquad (5.2)$$

We can in fact obtain a much better upper bound. Indeed,

$$\begin{aligned}
\Sigma_2 &= \sum_{b \in B(\exp(\sqrt{\log x}))} A(x/b) \\
&\asymp \sum_{b \in B(\exp(\sqrt{\log x}))} \frac{x}{b(\log(x/b))^{1/3}} \\
&= \frac{B(\exp(\sqrt{\log x}))x^{1-\frac{1}{\sqrt{\log x}}}}{\left(\log x - \sqrt{\log x}\right)^{1/3}} + \int_1^{\exp(\sqrt{\log x})} \frac{xB(t)(3\log(x/t) - 1)}{3t^2\log^{4/3}(x/t)}\,dt
\end{aligned}$$

$$
\begin{aligned}
&\asymp \frac{\exp(\sqrt{\log x})x^{1-\frac{1}{\sqrt{\log x}}}}{(\log x)^{1/3}\left(\log x-\sqrt{\log x}\right)^{1/3}}+x\int_1^{\exp(\sqrt{\log x})}\frac{B(t)}{t^2(\log(x/t))^{1/3}}\,dt\\
&\asymp \frac{x}{(\log x)^{2/3}}+x\int_e^{\exp(\sqrt{\log x})}\frac{dt}{t(\log t)^{2/3}(\log(x/t))^{1/3}}\\
&= \frac{x}{(\log x)^{2/3}}+x\int_1^{\sqrt{\log x}}\frac{du}{u^{2/3}(\log x-u)^{1/3}},
\end{aligned}
$$

where we set $u = \log t$ in the last integral.

Hence,

$$
\begin{aligned}
\Sigma_2 &\ll \frac{x}{(\log x)^{2/3}}+\frac{x}{(\log x-\sqrt{\log x})^{1/3}}\int_1^{\sqrt{\log x}}\frac{du}{u^{2/3}}\\
&\ll \frac{x}{(\log x)^{2/3}}+\frac{x(\log x)^{1/6}}{(\log x)^{1/3}}\ll \frac{x}{(\log x)^{1/6}},
\end{aligned}
$$

which proves (5.2). Using (5.1) and (5.2) in (3.4), the proof of Theorem 1 is complete.

Since Lemma 1 and Lemma 2 both apply to $\Omega(n)$, we also have

$$
\kappa(a(n))-\Omega(a(n))\ll \frac{1}{2}\log_2 x
$$

and the proof of Theorem 2 follows along the same lines as that of the proof of Theorem 1.

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