On the k-fold iterates of the Euler totient function at shifted primes

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Dedicated to the memory of Marejke Wisjmuller

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Abstract

Let $\gamma(n)$ stand for the product of the prime factors of n. The index of composition $\lambda(n)$ of an integer $n \geq 2$ is defined as $\lambda(n) = \log n / \log \gamma(n)$ with $\lambda(1) = 1$. Given an arbitrary integer $k \geq 0$ and letting $\phi_k(n)$ stand for the k-fold iterate of the Euler totient function, we show that, given any real number $\varepsilon > 0$, $\lambda(\phi_k(p-1)) < 1 + \varepsilon$ for almost all prime numbers p.

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1 Introduction and notation

Let $\gamma(n)$ stand for the product of all the prime factors of the positive integer n. The index of composition of an integer, defined by $\lambda(1) = 1$ and for $n \geq 2$ by $\lambda(n) := \log n / \log \gamma(n)$ was studied by De Koninck and Doyon [2] and thereafter by many more (see [3], [6], [9]). In 2007, De Koninck and Luca [7] showed that the normal order of $\lambda(\sigma(n))$, where $\sigma(n)$ stands for the sum of the divisors function, is equal to 1. Let $\sigma_k(n)$ stand for the k-fold iterate of the $\sigma(n)$ function, that is, let $\sigma_0(n) = n$, $\sigma_1(n) = \sigma(n)$, $\sigma_2(n) = \sigma(\sigma(n))$, and so on. Recently, the authors [4] proved that, for every $\varepsilon > 0$,

$$\frac{1}{x}\#\{n \leq x : \lambda(\sigma_k(n)) \geq 1 + \varepsilon\} \to 0 \qquad (x \to \infty). \tag{1.1}$$

They also showed that (1.1) holds if $\sigma_k(n)$ is replaced by $\phi_k(n)$, the k-fold iterate of the Euler ϕ function.

Here, we prove an analogous result for the shifted primes, namely the following.

Theorem 1. *Given any $\varepsilon > 0$ and letting $\pi(x)$ stand for the number of primes not exceeding x, then*

$$\frac{1}{\pi(x)}\#\{p \leq x : \lambda(\phi_k(p-1)) \geq 1 + \varepsilon\} \to 0 \qquad (x \to \infty). \tag{1.2}$$

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In the following, we denote by $p(n)$ and $P(n)$ the smallest and largest prime factors of n, respectively. We let $\mu(n)$ stand for the Moebius function. For each integer $n \geq 2$, we let $\omega(n)$ stand for the number of distinct prime factors of n and $\Omega(n)$ for the total number of prime factors of n counting multiplicity and we set $\omega(1) = \Omega(1) = 0$. The letters p, q, π, ρ and Q, with or without subscript, will stand exclusively for primes. On the other hand, the letters c and C, with or without subscript, will stand for absolute constants but not necessarily the same at each occurrence. Moreover, we shall use the abbreviations $x_1 = \log x$, $x_2 = \log\log x$, and so on. We denote the logarithmic integral $\displaystyle\int_2^x \frac{dt}{\log t}$ by $\mathrm{li}(x)$. Finally, we shall write $\pi(x; k, \ell)$ for $\#\{p \leq x : p \equiv \ell \pmod{k}\}$.

2 Preliminary results

Lemma 1. *Given an arbitrary positive number $\delta < 1/20$, then,*

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \#\{p \leq x : P(p-1) > x^{1-\delta}\} < C_1\delta \tag{2.1}$$

for some absolute constant $C_1 > 0$.

Proof. For a proof see Theorem 4.2 in the book of Halberstam and Richert [8]. □

Let us now set

$$\mathcal{N}_x^{(1)} := \{p \leq x \text{ and } P(p-1) \leq x^{1-\delta}\}.$$

Also, for each positive $\delta < 1/20$, let us introduce the functions

$$\omega_\delta(n) := \sum_{\substack{p|n \\ x^\delta < p < x^{1/5}}} 1 \qquad \text{and} \qquad A_\delta(x) := \sum_{x^\delta < p < x^{1/5}} \frac{1}{p}.$$

It is easy to show that

$$A_\delta(x) = \log\frac{1}{5\delta} + o(1) \qquad (x \to \infty).$$

The following Turán-Kubilius type inequality can be deduced using the Bombieri-Vinogradov inequality.

Lemma 2. *Given $\delta \in (0, 1/20)$, there exists an absolute constant $C_2 > 0$ such that*

$$\frac{1}{\pi(x)} \sum_{p \leq x} (\omega_\delta(p-1) - A_\delta(x))^2 \leq C_2 A_\delta(x).$$

Letting $\mathcal{A}_x^{(1)} := \{p \leq x : \omega_\delta(p-1) \leq 4\}$, then the following result is easily established.

Lemma 3. *Given $\delta \in (0, 1/20)$, there exist real numbers C_3 and $x_0 = x_0(\delta)$ such that, for all $x \geq x_0$, we have*

$$\frac{1}{\pi(x)} \# \mathcal{A}_x^{(1)} \leq C_3 \delta.$$

Given positive integers k and D, set $U_k(x; D) := \#\{n \leq x : D \mid \phi_k(n)\}$. The following result was established by Bassily, Kátai and Wijsmuller [1].

Lemma 4. *Given positive integers k and D, there exists a constant $C_4 = C_4(k, \Omega(D))$ such that*

$$U_k(x; D) \leq C_4 \frac{x\ x_2^{k\Omega(D)}}{D}.$$

Letting $\ell_k(x) = x_5$ if $k = 0$ and $x_1 x_2^{2k}$ if $k \geq 1$. Then, for each integer $k \geq 0$, setting

$$\mathcal{B}_x^{(k)} = \{p \leq x : \text{ there exists } q > \ell_k(x) \text{ such that } q^2 \mid \phi_k(p-1)\},$$

the following result follows from Lemma 4.

Lemma 5. *There exists an absolute constant $C_5 > 0$ such that*

$$\frac{1}{\pi(x)} \# \mathcal{B}_x^{(k)} \leq \frac{C_5}{x_2} \qquad (k = 0, 1, \ldots).$$

For each integer $k \geq 0$, let $a_k = 1/(k+1)!$ and, given a real number $\kappa > 0$, set

$$\mathcal{D}_x^{(k)} := \{p \leq x : \omega(\phi_k(p-1)) > (1+\kappa) a_k x_2^{k+1}\}.$$

In Bassily, Kátai and Wijsmuller [1], it was proved that, for each integer $k \geq 0$ and for every real number z,

$$\lim_{x \to \infty} \frac{1}{\pi(x)} \# \left\{ p \leq x : \frac{\omega(\phi_k(p-1)) - a_k x_2^{k+1}}{b_k x_2^{k+1/2}} < z \right\} = \Phi(z),$$

where $b_k = 1/(k! \sqrt{2k+1})$ and where

$$\Phi(z) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-u^2/2}\, du$$

stands for the standard Gaussian law.

It then follows from this result that the following is true.

Lemma 6. *For each integer $k \geq 0$,*

$$\frac{1}{\pi(x)} \# \mathcal{D}_x^{(k)} \to 0 \qquad (x \to \infty).$$

We will also need the following, which is a particular case of Lemma 2.5 in Bassily, Kátai and Wisjmuller [1].

Lemma 7. *Letting* $\delta(x,k) := \sum_{\substack{p\le x\\ p\equiv 1 \pmod k}} \frac{1}{p}$, *there exists an absolute constant* $C_6 > 0$ *such that*

$$\delta(x,k) \le \frac{C_6 x_2}{\phi(k)},$$

provided $k \le x$ *and* $x \ge 3$.

We say that a $k+1$-tuple of primes $(q_0, q_1, \ldots, q_k)$ is a *k-chain* if $q_{i-1} \mid q_i + 1$ for $i = 1, 2, \ldots, k$, in which case we write $q_0 \to q_1 \to \cdots \to q_k$. We then have the following result, whose proof can be deduced from Lemma 2 established in our earlier paper [5].

Lemma 8. *For any fixed prime* q_0 *and integer* $k \ge 1$, *there exist absolute constants* $c_1, c_2, \ldots, c_k$ *such that*

$$\sum_{\substack{q_0\to q_1\\ q_1\le x}} \frac{1}{q_1} \le \frac{c_1 x_2}{q_0}, \qquad \sum_{\substack{q_0\to q_1\to q_2\\ q_2\le x}} \frac{1}{q_2} \le \frac{c_2 x_2^2}{q_0}, \qquad \ldots \quad , \quad \sum_{\substack{q_0\to q_1\to\cdots\to q_k\\ q_k\le x}} \frac{1}{q_k} \le \frac{c_k x_2^k}{q_0}.$$

Moreover, summing over those $k+1$ *chains for which* $q_0 \equiv 1 \pmod D$, *then there exists a constant* $C_7 > 0$ *such that*

$$\sum_{\substack{q_0\to q_1\to\cdots\to q_k\\ q_k\le x}} \frac{1}{q_k} \le \frac{C_7 x_2^{k+1}}{\phi(D)}.$$

Now, let

$$\mathcal{N}_x^{(2)} = \mathcal{N}_x^{(1)} \setminus \left(\left(\bigcup_{j=0}^{k} \mathcal{D}_x^{(j)} \right) \bigcup \left(\bigcup_{j=0}^{k} \mathcal{B}_x^{(j)} \right) \right).$$

Defining $L_k(x) = x_5$ if $k = 0$ and x_2^{2k} if $k \ge 1$, let us introduce the function

$$S_k(n) = \prod_{\substack{q^\alpha \| \phi_k(n)\\ q > L_k(x)}} q^\alpha, \tag{2.2}$$

We then have the following result.

Lemma 9. *For each integer* $j = 0, 1, \ldots, k$,

$$\frac{1}{\pi(x)} \#\{p \in \mathcal{N}_x^{(2)} : \mu(S_j(p-1)) = 0\} \to 0 \qquad (x \to \infty).$$

Proof. The result is almost obvious if $k = 0$. Indeed, first observe that

$$\#\{p \le x : q^2 \mid p-1 \text{ for some prime } q > L_0(x)\} \le \sum_{q > L_0(x)} \pi(x; q^2, 1). \tag{2.3}$$

Recall that according to the Brun-Titchmarsh theorem, given $\delta \in (0,1)$, there exists a constant $c_1 = c_1(\delta) > 0$ such that

$$\pi(x; k, \ell) < c_1 \frac{\mathrm{li}(x)}{\phi(k)} \qquad \text{provided } k < x^{1-\delta}. \tag{2.4}$$

Thus, using (2.4), we may write that, for some absolute constant $C_8 > 0$,

$$\sum_{q > L_0(x)} \pi(x; q^2, 1) \le C_8 \mathrm{li}(x) \sum_{L_0(x) < q < x^{1/5}} \frac{1}{\phi(q^2)} + \sum_{q \ge x^{1/5}} \frac{x}{q^2} = o(\mathrm{li}(x)), \tag{2.5}$$

so that the result follows by combining (2.3) and (2.5).

So, let us assume that $k \ge 1$. Let us first count the number of primes $p \in \mathcal{N}_x^{(2)}$ such that $S_j(p-1)$ is square-free for $j = 0, 1, \ldots, k-1$ and for which there exists some prime $q > L_k(x)$ such that $q^2 \mid \phi_k(p-1)$. Since $p \notin \mathcal{B}_x^{(k)}$, it follows that $q \le \ell_k(x)$. On the other hand, since $q^2 \mid \phi_k(p-1)$, then

- either there exist two primes $\pi_1 \ne \rho_1$ such that $q \to \pi_1$ and $q \to \rho_1$ (meaning that $\pi_1 \equiv 1 \pmod q$ and $\rho_1 \equiv 1 \pmod q$), with $\pi_1 \rho_1 \mid \phi_{k-1}(p-1)$,
- or there exists a prime π such that $\pi \equiv 1 \pmod{q^2}$ and $\pi \mid \phi_{k-1}(p-1)$.

In other words, one of the following two situations (1) and (2) will occur.

(1) There exist two $k+1$-chains

$$\begin{aligned} &q \to \pi_1 \to \cdots \to \pi_k \quad (\to p), \\ &q \to \rho_1 \to \cdots \to \rho_k \quad (\to p), \end{aligned}$$

where $\pi_\nu, \rho_\nu \ \ (\nu = 1, \ldots, k)$ are distinct primes and $\pi_k \rho_k \mid p-1$.

(2) There exists a positive integer h such that

$$\begin{aligned} &\pi_\nu \rho_\nu \mid \phi_{k-\nu}(p-1) \text{ for } \nu = 0, \ldots, h, \\ &Q_{h+1} \mid \phi_{k-h-1}(p-1), \quad Q_{h+1} \equiv 1 \pmod{\pi_h \rho_h}, \\ &Q_{h+1} \to Q_{h+2} \to \cdots \to Q_k \quad (\to p). \end{aligned}$$

It follows from the above that if we set

$$M_q := \#\{p \in \mathcal{N}_x^{(2)} : q^2 \mid \phi_k(p-1)\},$$

then

$$M_q \leq \sum_{\substack{q\to\pi_1\to\cdots\to\pi_k\\ q\to\rho_1\to\cdots\to\rho_k}} \pi(x;\pi_k\rho_k,1) + \sum_{h=0}^{k-1} \sum_{\substack{q\to\pi_1\to\cdots\to\pi_h\to Q_{h+1}\to\cdots\to Q_k\\ q\to\rho_1\to\cdots\to\rho_h\to Q_{h+1}\to\cdots\to Q_k}} \pi(x;Q_k,1). \tag{2.6}$$

But since $p \in \mathcal{N}_x^{(2)}$ implies that $\omega_\delta(p-1) > 4$, we obtain that $\pi_k\rho_k < x^{1-\delta}$ and $Q_k < x^{1-\delta}$. Hence, in light of Lemmas 1, 2 and 3, we may use (2.4) in (2.6) and obtain that, for some constant $C_9 > 0$,

$$M_q \leq C_9\mathrm{li}(x) \sum_{\substack{q\to\pi_1\to\cdots\to\pi_k\\ q\to\rho_1\to\cdots\to\rho_k}} \frac{1}{\pi_k\rho_k} + C_9\mathrm{li}(x) \sum_{\substack{q\to\pi_1\to\cdots\to\pi_h\to Q_{h+1}\to\cdots\to Q_k\\ q\to\rho_1\to\cdots\to\rho_h\to Q_{h+1}\to\cdots\to Q_k}} \frac{1}{Q_k}. \tag{2.7}$$

using Lemma 8, inequality (2.7) yields

$$M_q \leq C_{10}\mathrm{li}(x)\frac{x_2^{2k}}{q^2} \tag{2.8}$$

for some positive constant C_{10}. Since, for some $C_{11} > 0$,

$$\sum_{q > L_k(x)} \frac{1}{q^2} \leq \frac{C_{11}}{L_k(x)\log L_k(x)},$$

it follows from (2.8) that

$$\sum_{q > L_k(x)} M_q \leq C_{10}\mathrm{li}(x)x_2^{2k}\frac{C_{11}}{x_2^{2k}2kx_3} \ll \frac{x}{x_3},$$

thus completing the proof of Lemma 9. □

Recalling the definition of $S_k(n)$ provided in (2.2), we now introduce the function

$$T_k(n) = \frac{\phi_k(n)}{S_k(n)} \qquad (k = 0, 1, \ldots) \tag{2.9}$$

and prove the following result.

Lemma 10. *For each $j = 0, 1, \ldots, k$, we have*

$$\frac{1}{\pi(x)}\#\left\{p \in \mathcal{N}_x^{(2)} : \frac{\log T_j(p-1)}{\log x} \geq \frac{1}{x_2}\right\} \to 0 \qquad (x \to \infty). \tag{2.10}$$

Proof. Consider the set

$$\mathcal{N}_x^{(3)} := \{p \in \mathcal{N}_x^{(2)} : \mu^2(S_j(p-1)) = 1 \text{ for } j = 0, 1, \ldots, k\}.$$

Since $\#(\mathcal{N}_x^{(2)} \setminus \mathcal{N}_x^{(3)}) = o(\pi(x))$ as $x \to \infty$, in order to prove Lemma 9, we need to find an adequate upper bound for the number of primes $p \in \mathcal{N}_x^{(3)}$.

First of all, it is clear that (2.10) is true for $j = 0$. Indeed, by definition (2.9) for $k = 0$, we have

$$p - 1 = T_0(p-1)S_0(p-1),$$

where $S_0(p-1)$ is square-free, $p(S_0(p-1)) > x_5$ and $p(T_0(p-1)) \le x_5$. Hence, $(T_0(p-1), S_0(p-1)) = 1$, and therefore

$$\phi(p-1) = \phi(T_0(p-1)) \cdot \phi(S_0(p-1)),$$

with

$$\phi(S_0(p-1)) = \prod_{\substack{\pi^\alpha \| \phi(S_0(p-1)) \\ \pi \le L_1(x)}} \pi^\alpha \cdot \prod_{\substack{\pi | \phi(S_0(p-1)) \\ \pi > L_1(x)}} \pi,$$

since in $\mathcal{N}_x^{(3)}$, $\pi^2 \nmid \phi(S_0(p-1))$ if $\pi > L_1(x)$.

It follows from this that

$$T_1(p-1) = \phi(T_0(p-1)) \cdot \prod_{\substack{\pi^\alpha \| \phi(S_0(p-1)) \\ \pi \le L_1(x)}} \pi^\alpha$$

and

$$\phi(p-1) = T_1(p-1) \cdot S_1(p-1),$$

where $P(T_1(p-1)) \le L_1(x)$ and $p(S_1(p-1)) > L_1(x)$, thus implying in particular that $(T_1(p-1), S_1(p-1)) = 1$, so that

$$\phi_2(p-1) = \phi(T_1(p-1)) \cdot \phi(S_1(p-1)).$$

More generally, if

$$\phi_{j-1}(p-1) = T_{j-1}(p-1)S_{j-1}(p-1),$$

then $P(T_{j-1}(p-1)) \le L_{j-1}(x)$ and $p(S_{j-1}(p-1)) > L_{j-1}(x)$, $S_{j-i}(p-1)$ is square-free and

$$\phi_j(p-1) = T_j(p-1)S_j(p-1)$$

and

$$\begin{aligned} T_j(p-1) &= \phi(T_{j-1}(p-1)) \prod_{\substack{\pi^\alpha \| \phi(S_{j-1}(p-1)) \\ \pi \le L_j(x)}} \pi^\alpha, \\ S_j(p-1) &= \prod_{\substack{\pi | \phi(S_{j-1}(p-1)) \\ \pi > L_j(x)}} \pi \quad \text{(a square-free number)}. \end{aligned}$$

Let us now estimate the expression

$$K_j(p) := \prod_{\substack{\pi^\alpha \| \phi(S_{j-1}(p-1)) \\ \pi \le L_j(x)}} \pi^\alpha.$$

For this, let us assume that $\pi^{\ell_\pi} \mid \phi(S_{j-1}(p-1))$ with $\pi \le L_j(x)$. Since $\phi(S_{j-1}(p-1))$ is a divisor of $\phi_j(p-1)$ and since $\omega(\phi_j(p-1)) < a_j(1+\kappa)x_2^{j+1}$, it follows that there exists a prime q_0 such that $q_0 \mid \phi_{j-1}(p-1)$ and $\pi^{r_\pi} \mid q_0 - 1$ with

$$r_\pi \ge \frac{\ell_\pi}{\omega(\phi_j(p-1))} \ge \frac{\ell_\pi}{a_j(1+\kappa)x_2^{j+1}}.$$

Thus, for fixed π^{r_π} and using Lemma 8 along with inequality (2.4), it follows that the number of possible primes $p \in \mathcal{N}_x^{(3)}$ for which $\pi^{\ell_\pi} \mid \phi(S_{j-1}(p-1))$ is less than

$$\sum_{(\pi^{r_\pi} \to) q_0 \to \cdots \to q_{j-1}} \pi(x; q_{j-1}, 1) \le \frac{C_{12}\mathrm{li}(x) \cdot x_2^j}{\pi^{r_\pi}}.$$

Letting ℓ_π be sufficiently large so that

$$\pi^{\frac{\ell_\pi}{a_j(1+\kappa)x_2^{j+1}}} > x_2^{j+1}, \tag{2.11}$$

it follows that

$$\begin{aligned} &\frac{1}{\pi(x)}\#\{p \in \mathcal{N}_x^{(3)} : \text{ there exists one } \pi \le L_j(x) \text{ and } \ell_\pi \text{ satisfying (2.11)} \\ &\qquad \text{such that } \pi^{\ell_\pi} \mid \phi(S_{j-1}(p-1))\} = o(1) \qquad (x \to \infty). \end{aligned} \tag{2.12}$$

Hence, if $\pi^{m_\pi} \mid \phi(S_{j-1}(p-1))$ and it is not counted in the set appearing in (2.12), then

$$\pi^{m_\pi} < (x_2^{j+1})^{a_j(1+\kappa)x_2^{j+1}}$$

and so

$$K_j(p) \le \prod_{\pi \le L_j(x)} \pi^{m_\pi} \le (x_2^{j+1})^{a_j(1+\kappa)x_2^{j+1}x_2^{2j}} \le \exp\{x_2^{3j+2}\}, \tag{2.13}$$

say, provided x is large enough.

Now, since

$$T_j(p-1) = \phi(T_{j-1}(p-1))K_j(p), \tag{2.14}$$

and since $\phi(n) \le n$, it follows that, in light of (2.13) and (2.14)

$$T_j(p-1) < \exp\{2x_2^{3j+2}\} \qquad (j = 0, 1, \ldots, k).$$

when $p \in \mathcal{N}_x^{(3)}$ with the possible exception of $o(\mathrm{li}(x))$ primes.

This completes the proof of Lemma 10.

3 Proof of Theorem 1

We are now in a position to prove our main theorem.

We first write

$$\begin{aligned}
&\#\{p \le x : \lambda(\phi_k(p-1)) \ge 1+\varepsilon\} \\
&\le \#\{p \in \mathcal{N}_x^{(1)} : \lambda(\phi_k(p-1)) \ge 1+\varepsilon\} + \#\{p \in \mathcal{N}_x : P(p-1) > x^{1-\delta}\} \\
&\le \#\{p \in \mathcal{N}_x^{(3)} : \lambda(\phi_k(p-1)) \ge 1+\varepsilon\} + \#\{p \in \mathcal{N}_x : P(p-1) > x^{1-\delta}\} + \#(\mathcal{N}_x^{(1)} \setminus \mathcal{N}_x^{(3)}) \\
&\quad = S_1(x) + S_2(x) + S_3(x),
\end{aligned}$$

say.

Using Lemma 10, we have that $S_1(x) = o(\mathrm{li}(x))$ as $x \to \infty$. On the other hand, using Lemma 1, we get that $S_2(x) \le C_1\delta\,\mathrm{li}(x)$, while it is clear that $S_3(x) = o(\mathrm{li}(x))$ as $x \to \infty$.

We have therefore established that, for some constant $c > 0$,

$$\limsup_{x\to\infty} \frac{1}{\pi(x)} \#\{p \le x : \lambda(\phi_k(p-1)) \ge 1+\varepsilon\} \le c\delta. \tag{3.1}$$

But since δ can be chosen arbitrarily small, the right hand side of (3.1) is equal to 0.

The proof of our main theorem is therefore complete.

4 Final remarks

Let σ^* and ϕ^* be the unitary analogues of σ and ϕ. These are multiplicative functions defined on prime powers p^α by

$$\sigma^*(p^\alpha) = p^\alpha + 1 \qquad \text{and} \qquad \phi^*(p^\alpha) = -1.$$

Using the same methods as those above, we can prove the following.

Theorem 2. *For every $\varepsilon > 0$ and each $k = 0, 1, \ldots,$ we have*

$$\frac{1}{\pi(x)} \#\{p \le x : \lambda(\phi_k^*(p-1)) \ge 1+\varepsilon\} \to 0 \qquad (x \to \infty)$$

and

$$\frac{1}{\pi(x)} \#\{p \le x : \lambda(\sigma_k^*(p-1)) \ge 1+\varepsilon\} \to 0 \qquad (x \to \infty).$$

Perhaps, Theorem 2 is true also for $\lambda(\sigma_k(p-1))$ for a general k, but we could only prove the case $k = 1$.

□

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