On the n-th element of a set of positive integers

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Abstract

Given a set A of positive integers and its counting function $A(x) := \#\{n \leq x : n \in A\}$, we examine the size of the n-th element of A using the size of $A(x)$.

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1 Introduction and notation

Determining the size of the n-th element of a set of positive integers using the known size of the counting function of that set is a classical problem in analytic number theory. For example, letting $\pi(x)$ stand for the number of prime numbers $p \leq x$, by using the Prime Number Theorem in the form $\pi(x) \sim x/\log x$ as $x \to \infty$, one can easily show that the n-th prime number p_n satisfies

$$p_n = (1 + o(1))\, n \log n \qquad (n \to \infty)\,.$$

In fact, in 1902, by using the logarithmic integral function, Cipolla [3] improved this estimate by showing that there exists a unique sequence of polynomials $(Q_j)_{j\geq 1}$ with rational coefficients such that, for any given positive integer m,

(1.1)

$$p_n = n\left(\log n + \log_2 n - 1 + \sum_{j=1}^{m} \frac{(-1)^{j-1} Q_j(\log_2 n)}{\log^j n} + o\left(\frac{1}{\log^m n}\right)\right) \qquad (n \to \infty)\,.$$

Here and in what follows, we write $\log_2 x$ for $\max(1, \log\log x)$.

Another example is given by the search of an estimate for a_n, the n-th composite number. Bojarincev [2] and Shiu [12] showed that, for any given positive integer m,

$$a_n = n\left(1 + \frac{\beta_1}{\log n} + \frac{\beta_2}{\log^2 n} + \cdots + \frac{\beta_m}{\log^m n} + o\left(\frac{1}{\log^m n}\right)\right) \qquad (n \to \infty)\,, \tag{1.2}$$

where the β_i are computable constants.

Finally, recall that we say that a number is a *powerful number* (or a *square-full number*) if $p \mid n$ implies that $p^2 \mid n$. Let $\wp_n$ denote the n-th powerful number. In 1982, Ivić and Shiu [7] showed that

$$\wp_n = \left(\frac{\zeta(3)}{\zeta(3/2)}\right)^2 n^2 + O\left(n^{5/3}\right) \qquad (n \to \infty)\,. \tag{1.3}$$

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Here, we examine the problem of estimating the size of the n-th element of a given set A of positive integers using the size of $A(x) := \#\{n \le x : n \in A\}$, often called the counting function of A. We will do so in two particular cases. The first one is when $A(x) = b_1 x^{\lambda_1} + b_2 x^{\lambda_2} + R(x)$, where $R(x) = o(x^{\lambda_3})$, for some real constants $b_1 > 0$ and b_2, with $1 > \lambda_1 > \lambda_2 > \lambda_3 > 0$, from which we will then deduce an improvement of the estimate (1.3).

The second case is when $A(x) = \dfrac{x}{L(x)}\left(1 + O\left(\dfrac{1}{\varphi(x)}\right)\right)$ where φ is an increasing function which tends to $+\infty$ as $x \to \infty$ and L is a differentiable increasing slowly oscillating function. Recall that a function $L : [M, +\infty) \to \mathbb{R}$ continuous on $[M, +\infty)$, where M is a positive real number, is said to be a *slowly oscillating function* if for each positive real number $c > 0$,

$$\lim_{x\to\infty} \frac{L(cx)}{L(x)} = 1. \tag{1.4}$$

This class of functions was introduced by Karamata [8] in 1930. His paper, along with [9] as well as the book of Seneta [11], provide some interesting properties of slowly oscillating functions. In particular, it is possible to show that a differentiable function L is slowly oscillating if and only if

$$\frac{xL'(x)}{L(x)} = o(1) \qquad (x \to \infty) \tag{1.5}$$

and, in fact, that L is slowly oscillating if and only if there exists $x_0 > 0$ such that

$$L(x) = C(x) \exp\left\{\int_{x_0}^{x} \frac{\eta(t)}{t}\, dt\right\}, \tag{1.6}$$

where $\lim_{x\to\infty} C(x) = C$, for a certain constant $C \neq 0$, and $\eta(t) \to 0$ as $t \to \infty$.

We shall denote by $\mathcal{L}$ the set of increasing and differentiable slowly oscillating functions.

From here on, the letter c, with or without subscript, stands for an absolute positive constant, but not necessarily the same at each occurrence, while the letter p, with or without subscript, will always denote a prime number.

2 Main results

Theorem 1. *Given a sequence of positive integers $a_1 < a_2 < \cdots$, let $A = \{a_1, a_2, \ldots\}$ with counting function $A(x)$ satisfying*

$$A(x) = b_1 x^{\lambda_1} + b_2 x^{\lambda_2} + R(x), \tag{2.1}$$

where

$$R(x) = o(x^{\lambda_3}) \qquad (x \to \infty)$$

and where $b_1 > 0$ *and* b_2 *are real constants, with* $1 > \lambda_1 > \lambda_2 > \lambda_3 > 0$ *which satisfy*

$$\frac{3\lambda_2}{\lambda_1} - 3 < \frac{\lambda_3}{\lambda_1} - 1 \leq \frac{2\lambda_2}{\lambda_1} - 2.$$

Then

$$a_n = \frac{n^{\frac{1}{\lambda_1}}}{b_1^{\frac{1}{\lambda_1}}} - \frac{1}{\lambda_1} \frac{b_2}{b_1^{\frac{\lambda_2+1}{\lambda_1}}} n^{\frac{\lambda_2+1}{\lambda_1}-1} + \frac{1}{2}\left(\frac{2\lambda_2+1}{\lambda_1^2} - \frac{1}{\lambda_1}\right) \frac{b_2^2}{b_1^{\frac{2\lambda_2+1}{\lambda_1}}} n^{\frac{2\lambda_2+1}{\lambda_1}-2} + o\left(n^{\frac{\lambda_3+1}{\lambda_1}-1}\right). \tag{2.2}$$

Theorem 2. *Given a sequence of positive integers* $a_1 < a_2 < \cdots$, *let* $A = \{a_1, a_2, \dots\}$ *with counting function* $A(x)$ *satisfying*

$$A(x) = \frac{x}{L(x)} \left(1 + O\left(\frac{1}{\varphi(x)}\right)\right) \qquad (x \to \infty), \tag{2.3}$$

where φ *is an increasing function which tends to* $+\infty$ *as* $x \to \infty$ *and where* $L \in \mathcal{L}$ *with corresponding function* $\eta(t)$ *defined implicitly by (1.6). Moreover, assume that* $\eta(t)$ *is a decreasing function and that*

$$C(x) = C + O\left(\frac{1}{\psi(x)}\right) \qquad (x \to \infty)\,, \tag{2.4}$$

where $\psi(x)$ *is an increasing function which tends to* $+\infty$ *as* $x \to \infty$. *Then,*

$$a_n = n\,\frac{C(a_n)}{C(n)}\, L(n) \exp\left\{\int_n^{a_n} \frac{\eta(t)}{t}\, dt\right\} \left(1 + O\left(\frac{1}{\varphi(n)}\right)\right) \qquad (n \to \infty) \tag{2.5}$$

and

$$a_n = n\, L(n)^{1/(1-\delta(n))} \left(1 + O\left(\frac{1}{\varphi(n)} + \frac{1}{\psi(n)}\right)\right) \qquad (n \to \infty), \tag{2.6}$$

where δ *is some function satisfying* $\eta(a_n) < \delta(n) < \eta(n)$ *for all integers* $n \geq x_0$.

Moreover, if there exists a positive constant c *such that*

$$\eta(n) \int_{x_0}^{n} \frac{\eta(t)}{t}\, dt \to c \qquad (n \to \infty), \tag{2.7}$$

then

$$a_n = (e^c + o(1))\, nL(n) \qquad (n \to \infty). \tag{2.8}$$

3 Proof of Theorem 1

To prove Theorem 1, we use an approach already used by Copil and Panaitopol [4] to estimate the size of the n-th non powerful number.

First, observe that it follows from (2.1) that

$$n = A(a_n) = b_1 a_n^{\lambda_1} + b_2 a_n^{\lambda_2} + R(a_n),$$

so that

$$a_n^{\lambda_1} = \frac{n}{b_1}\left(1 - b_2 \frac{a_n^{\lambda_2}}{n} - \frac{R(a_n)}{n}\right)$$

thereby implying that

$$a_n = \frac{n^{1/\lambda_1}}{b_1^{1/\lambda_1}}\left(1 - b_2 \frac{a_n^{\lambda_2}}{n} - \frac{R(a_n)}{n}\right)^{1/\lambda_1}. \tag{3.1}$$

In particular, since both expressions $b_2 \frac{a_n^{\lambda_2}}{n}$ and $\frac{R(a_n)}{n}$ goes to 0 as $n \to \infty$, we have $a_n = \frac{n^{1/\lambda_1}}{b_1^{1/\lambda_1}}(1 + o(1))$ and

$$a_n = \frac{n^{1/\lambda_1}}{b_1^{1/\lambda_1}} + O\left(n^{\frac{\lambda_2+1}{\lambda_1}-1}\right). \tag{3.2}$$

Moreover, for any $\alpha > 0$,

$$(1-y)^\alpha = 1 - \alpha y + \frac{1}{2}\left(\alpha^2 - \alpha\right) y^2 + O\left(y^3\right) \qquad \text{as } y \to 0.$$

Thus

$$\left(1 - b_2 \frac{a_n^{\lambda_2}}{n} - \frac{R(a_n)}{n}\right)^{1/\lambda_1} = 1 - \frac{b_2}{\lambda_1}\frac{a_n^{\lambda_2}}{n} + \frac{1}{2}\left(\frac{1}{\lambda_1^2} - \frac{1}{\lambda_1}\right) b_2^2 \frac{a_n^{2\lambda_2}}{n^2} + O\left(\frac{R(a_n)}{n}\right). \tag{3.3}$$

Substituting this estimate in (3.1) yields

$$a_n = \frac{n^{1/\lambda_1}}{b_1^{1/\lambda_1}} - \frac{1}{\lambda_1}\frac{b_2}{b_1^{\frac{\lambda_2+1}{\lambda_1}}} n^{\frac{\lambda_2+1}{\lambda_1}-1} + O\left(n^{\frac{2\lambda_2+1}{\lambda_1}-2}\right). \tag{3.4}$$

Using this estimate and the fact that $R(a_n) = o\left(a_n^{\lambda_3}\right) = o\left(n^{\lambda_3/\lambda_1}\right)$, we can replace the RHS of (3.3) by

$$1 - \frac{1}{\lambda_1}\frac{b_2}{b_1^{\lambda_2/\lambda_1}} n^{\frac{\lambda_2}{\lambda_1}-1} + \frac{1}{2}\left(\frac{2\lambda_2+1}{\lambda_1^2} - \frac{1}{\lambda_1}\right)\frac{b_2^2}{b_1^{2\lambda_2/\lambda_1}} n^{\frac{2\lambda_2}{\lambda_1}-2} + o\left(n^{\frac{\lambda_3}{\lambda_1}-1}\right), \tag{3.5}$$

which substituted back in (3.4) yields (2.2).

4 Proof of Theorem 2

It follows from estimate (2.3) that

$$n = A(a_n) = \frac{a_n}{L(a_n)}\left(1 + O\left(\frac{1}{\varphi(a_n)}\right)\right) \qquad (n \to \infty) \tag{4.1}$$

and therefore that

$$a_n = nL(a_n)\left(1 + O\left(\frac{1}{\varphi(n)}\right)\right) \qquad (n \to \infty). \tag{4.2}$$

Since L is a slowly oscillating function, we have

$$L(a_n) = C(a_n)\exp\left(\int_{x_0}^{a_n} \frac{\eta(t)}{t}\mathrm{dt}\right) = \frac{C(a_n)}{C(n)}L(n)\exp\left(\int_{n}^{a_n} \frac{\eta(t)}{t}\mathrm{dt}\right). \tag{4.3}$$

Combining (4.2) and (4.3) proves (2.5).

Now, let $\alpha = \alpha(n)$ be the unique positive integer satisfying $2^{\alpha-1}n < nL(a_n) \leq 2^{\alpha}n$, so that $\alpha = \left\lceil \frac{\log L(a_n)}{\log 2} \right\rceil = \frac{\log L(a_n)}{\log 2} + \epsilon(n)$, where $0 \leq \epsilon(n) < 1$. On the one hand, since $\eta(t)$ is decreasing and positive, we have

$$\begin{aligned}
\int_n^{a_n} \frac{\eta(t)}{t}\,dt &\leq \int_n^{2n} \frac{\eta(t)}{t}\,dt + \int_{2n}^{2^2 n} \frac{\eta(t)}{t}\,dt + \cdots + \int_{2^{\alpha-1}n}^{2^{\alpha}n} \frac{\eta(t)}{t}\,dt \\
&< \eta(n)\log 2 + \eta(2n)\log 2 + \cdots + \eta(2^{\alpha-1}n)\log 2 \\
&\leq \eta(n)\alpha\log 2 = \eta(n)\log 2\left(\frac{\log L(a_n)}{\log 2} + \epsilon(n)\right).
\end{aligned} \tag{4.4}$$

On the other hand,

$$\begin{aligned}
\int_n^{a_n} \frac{\eta(t)}{t}\,dt &\geq \int_n^{2n} \frac{\eta(t)}{t}\,dt + \int_{2n}^{2^2 n} \frac{\eta(t)}{t}\,dt + \cdots + \int_{2^{\alpha-2}n}^{2^{\alpha-1}n} \frac{\eta(t)}{t}\,dt \\
&> \eta(2n)\log 2 + \eta(4n)\log 2 + \cdots + \eta(2^{\alpha-1}n)\log 2 \\
&> \eta(2^{\alpha-1}n)(\alpha-1)\log 2 \geq \eta(a_n)\log 2\left(\frac{\log L(a_n)}{\log 2} + \epsilon(n)\right).
\end{aligned} \tag{4.5}$$

It follows from (4.3), (4.4) and (4.5) that there exists a function δ satisfying $\eta(a_n) < \delta(n) < \eta(n)$ for all integers $n \geq x_0$ such that

$$\int_n^{a_n} \frac{\eta(t)}{t}\mathrm{dt} = \delta(n)\log L(a_n).$$

Combining this result with (2.4), we get

$$L(a_n) = L(n)^{1/(1-\delta(n))}\left(1 + O\left(\frac{1}{\psi(n)}\right)\right). \tag{4.6}$$

Substituting (4.6) in (4.2) proves (2.6). Finally, (2.8) follows easily from (2.7). Indeed, by (4.6), we have

$$L(a_n) = L(n)^{1+\delta(n)+O\left(\delta^2(n)\right)}. \tag{4.7}$$

We have

$$L(n)^{\delta(n)} \le L(n)^{\eta(n)} = C(n)^{\eta(n)} \exp\left(\eta(n)\int_{x_0}^{n} \frac{\eta(t)}{t}\mathrm{d}t\right) = e^c + o(1)$$

and

$$L(n)^{\delta(n)} \ge L(n)^{\eta(a_n)} = L(a_n)^{\eta(a_n)} \left(\frac{L(n)}{L(a_n)}\right)^{\eta(a_n)} = \left(\frac{L(n)}{L(a_n)}\right)^{\eta(a_n)} (e^c + o(1)).$$

Since

$$\log L(a_n) - \log L(n) = \int_n^{a_n} \frac{\eta(t)}{t}\mathrm{d}t\,(1+o(1)) = \delta(n)\log L(a_n)(1+o(1)),$$

it follows that

$$\left(\frac{L(n)}{L(a_n)}\right)^{\eta(a_n)} = 1 + o(1)$$

and thus

$$L(n)^{\delta(n)} = e^c + o(1). \tag{4.8}$$

Moreover, using (4.8),

$$L(n)^{\delta(n)^2} \le \left(L(n)^{\eta(n)}\right)^{\delta(n)} = (e^c + o(1))^{\delta(n)} = 1 + o(1).$$

Combining this last result with (4.8) and (4.7) gives

$$L(a_n) = L(n)(e^c + o(1)) \qquad (n \to \infty).$$

Substituting this estimate in (4.2) yields

$$a_n = (e^c + o(1))\, nL(n).$$

5 Applications of Theorem 1

We provide two applications.

First, we shall prove that there exists a positive constant C such that, as $n \to \infty$,

$$\wp_n = \left(\frac{\zeta(3)}{\zeta(3/2)}\right)^2 n^2 - 2\frac{\zeta(2/3)}{\zeta(2)}\left(\frac{\zeta(3)}{\zeta(3/2)}\right)^{\frac{8}{3}} n^{\frac{5}{3}} + \frac{7}{3}\left(\frac{\zeta(2/3)}{\zeta(2)}\right)^2\left(\frac{\zeta(3)}{\zeta(3/2)}\right)^{\frac{10}{3}} n^{\frac{4}{3}} + R_0(n), \tag{5.1}$$

where

$$(5.2) \qquad R_0(n) \ll n^{4/3} \exp\left(-C (\log n)^{3/5} (\log_2 n)^{-1/5}\right).$$

In order to prove (5.1), we first recall the 1958 result of Bateman and Grosswald [1]

$$(5.3) \qquad P_2(x) := \#\{n \le x : n \text{ powerful}\} = \frac{\zeta(3/2)}{\zeta(3)} x^{1/2} + \frac{\zeta(2/3)}{\zeta(2)} x^{1/3} + R(x),$$

where

$$(5.4) \qquad R(x) \ll x^{1/6} \exp\left(-C (\log x)^{3/5} (\log_2 x)^{-1/5}\right),$$

which is the best known error term and is due to Suryanarayana and Sitaramachandra Rao [14].

Then setting $\lambda_1 = 1/2$, $\lambda_2 = 1/3$, $\lambda_3 = 1/6$, $b_1 = \frac{\zeta(3/2)}{\zeta(3)}$ and $b_2 = \frac{\zeta(2/3)}{\zeta(2)}$ in Theorem 1, keeping track of the explicit error term given by (5.2), estimate (5.1) follows.

As a second application, we consider the general case of k-full numbers.

Recall that, given an integer $k \ge 2$, we say that a positive integer n is said to be *k-full* if $p \mid n$ implies that $p^k \mid n$. We denote by $P_k(x)$ the number of k-full integers $\le x$ and by $\wp_{n,k}$ the n-th k-full number.

Ivić and Shiu [7] obtained that

$$(5.5) \qquad P_k(x) = \gamma_{0,k} x^{1/k} + \gamma_{1,k} x^{1/(k+1)} + \cdots + \gamma_{k-1,k} x^{1/(2k-1)} + \Delta_k(x),$$

where the constants $\gamma_{i,k}$ are given explicitly and $\Delta_k(x)$ is a suitable error term.

Using (5.5) in the particular case $k = 3$ and Theorem 1, we can prove that there exists a positive constant C and constants A_3, A_4 and A_5 such that, as $n \to \infty$,

$$\wp_{n,3} = (A_3)^{-3} n^3 - 3 (A_3)^{\frac{-15}{4}} A_4 n^{\frac{11}{4}} - 3 (A_3)^{\frac{-18}{5}} A_5 n^{\frac{13}{5}} + \frac{21}{4} (A_3)^{\frac{-9}{2}} (A_4)^2 n^{\frac{5}{2}} + R(n),$$

where

$$R(n) \ll n^{\frac{19}{8}} \exp\left(-C (\log n)^{3/5} (\log_2 n)^{-1/5}\right).$$

Remark 1. *Observe that explicit values for the constants A_i were obtained by Shiu [12]. Moreover, for $k \ge 4$, as $n \to \infty$, one can prove that*

$$(5.6) \qquad \wp_{n,k} = \left(\frac{n}{\gamma_{0,k}}\right)^k - k \frac{\gamma_{1,k}}{(\gamma_{0,k})^{\frac{k(k+2)}{k+1}}} n^{\frac{k^2+k-1}{k+1}} - k \frac{\gamma_{2,k}}{(\gamma_{0,k})^{\frac{k(k+3)}{k+2}}} n^{\frac{k^2+2k-2}{k+2}} + R_k(n),$$

where $R_k(n) \ll n^{\frac{k^2+k-2}{k+1}}$.

Remark 2. *Observe that explicit values for the constants $\gamma_{i,k}$ are given in Bateman and Grosswald [1] and Erdős and Szekeres [6]. Moreover, for $k > 4$, additional terms on the right hand side of (5.6) can be provided.*

6 Applications of Theorem 2

We provide three applications.

1. Fix a positive integer k and let

$$A = A_k = \{n \in \mathbb{N} : \omega(n) = k\} = \{a_n : n \in \mathbb{N}\},$$

where $\omega(n)$ stands for the number of distinct prime factors of n. It is well known that, as $x \to \infty$,

$$A(x) = \frac{x}{L(x)}\left(1 + O\left(\frac{1}{\log\log x}\right)\right),$$

where $L(x) = \frac{(k-1)!\log x}{(\log_2 x)^{k-1}}$ (see for instance Theorem 10.4 in the book of De Koninck and Luca [5]). It follows from Theorem 2 that

$$a_n = n\frac{(k-1)!\log n}{(\log_2 n)^{k-1}}\left(1 + O\left(\frac{1}{\log\log n}\right)\right) \qquad (n \to \infty).$$

2. Consider the set A of those integers $n \geq 2$ such that $\lfloor z(n)\rfloor - \lfloor z(n-1)\rfloor = 1$, where $z(n) = n/e^{\sqrt{\log n}}$. Using a computer, we easily obtain the first elements of A, so that we may write

$$A = \{3, 9, 16, 24, 33, 42, 51, 61, 71, 82, 93, \ldots\} = \{a_n : n \in \mathbb{N}\}.$$

Clearly $|A(x) - z(x)| \leq 1$ for all $x \geq 2$. Then, since condition (2.7) of Theorem 2 is satisfied with $c = 1/2$, we get from (2.8) that

$$a_n = \left(\sqrt{e} + o(1)\right) n e^{\sqrt{\log n}} \qquad (n \to \infty).$$

3. Let $W = \{a_1, a_2, \ldots\}$ be the set of those positive integers which can be written as the sum of two squares. It has been known since Euler that a positive integer can be represented as a sum of two squares if and only if each of its prime factors of the form $4k + 3$ occurs with an even power, so that

$$W = \{2, 5, 8, 10, 13, 17, 18, 20, 25, 26, 29, 32, 34, 37, 40, 41, 45, 50, \ldots\}.$$

In 1908, Landau [10] showed that

$$W(x) = (B + o(1))\frac{x}{\sqrt{\log x}} \qquad (x \to \infty),$$

where $B = \frac{1}{\sqrt{2}} \prod_{p \equiv 3 \pmod 4} \left(\sqrt{1 - \frac{1}{p^2}}\right)^{-1} = 0.7642236\ldots$. In 1986, Shiu [13] showed that

$$W(x) = \frac{Bx}{\sqrt{\log x}}\left(1 + O\left(\frac{1}{\log x}\right)\right). \tag{6.1}$$

Since one can show that

$$L(x) := \frac{1}{B}\sqrt{\log x} = \frac{1}{B}\exp\left\{\int_e^x \frac{1}{2\log t}\frac{dt}{t}\right\}, \tag{6.2}$$

it follows from (6.1) and (6.2) that the corresponding functions $\varphi(x)$, $\eta(x)$, $C(x)$ and $\psi(x)$ from the statement of Theorem 2 are given by

$$\varphi(x) = \log x, \qquad \eta(x) = \frac{1}{2\log x}, \qquad C(x) = 1/B, \qquad \psi(x) = \infty.$$

Hence, it follows from (2.5) that

$$a_n = \frac{1}{B} n\sqrt{\log a_n}\left(1 + O\left(\frac{1}{\log n}\right)\right).$$

Thus, using the logarithm on this formula, one can improve it to

$$a_n = \frac{1}{B} n\sqrt{\log n}\left(1 + \frac{1}{4}\frac{\log\log n}{\log n} + O\left(\frac{1}{\log n}\right)\right) \qquad (n \to \infty).$$

References

[1] P. Bateman and E. Grosswald, *On a theorem of Erdős and Szekeres*, Illinois J. Math. **2** (1958), 88–98.

[2] A.E. Bojarincev, *Asymptotic expressions for the* nth *composite number*, Ural. Gos. Univ. Mat. Zap., **6** (1987), 21–43.

[3] M. Cipolla, *La determinazione dell' n^{imo} numero primo*, Rend. Acad. Sci. Fis. Mat. Napoli **8** (1902), 132–166.

[4] V. Copil and L. Panaitopol, *Properties of non powerful numbers*, J. Inequal. Pure Appl. Math. **9** (2008), no. 1, Article 29, 8.

[5] J.M. De Koninck and Florian Luca, *Analytic Number Theory: Exploring the Anatomy of Integers*, Graduate Studies in Mathematics, Vol. 134, American Mathematical Society, Providence, Rhode Island, 2012.

[6] P. Erdős and G. Szekeres, *Über die Anzahl der Abelschen Gruppen gegebener Ordnung und über ein verwandtes zahlentheoretisches Problem (in German)*, Acta Litt. Sci. Szeged, (1934), 95–102.

[7] A. Ivić and P. Shiu, *The distribution of powerful integers*, Illinois J. Math. **26** (1982), no. 4, 576–590.

[8] J. Karamata, *Sur un mode de croissance régulière des fonctions*, Mathematica (Cluj) **4** (1930), 38–53.

[9] J. Karamata, *Sur un mode de croissance régulière. Théorèmes fondamentaux*, Bull. Soc. Math. France **61** (1933), 55–62.

[10] E. Landau, *Über die Einteilung der positiven ganzen Zahlen in vier Klassen nach der Mindestzahl der zu ihrer additiven Zusammensetzung erforderlichen Quadrate*, Arch. Math. Phys., (3) 13 (1908), 305-312; Collected works, Vol. 4, Thales Verlag, Essen, 1986, 59-66.

[11] E. Seneta, *Regularly varying functions*, Springer-Verlag, Berlin, 1976.

[12] P. Shiu, *The distribution of cube-full numbers*, Glasgow Math. J. **33** (1991), no. 3, 287–295.

[13] P. Shiu, *Counting sums of two squares: the Meissel-Lehmer method*, Math. Comp. **47** (1986), no. 175, 351–360.

[14] D. Suryanarayana and R.R. Sitaramachandra, *The distribution of square-full integers*, Arkiv für Matematik **11** (1973), 195–201.

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