On a property of non Liouville numbers

Jean-Marie De Koninck
Département de mathématiques
et de statistique
Université Laval
Québec G1V 0A6
Canada
jmdk@mat.ulaval.ca

Imre Kátai
Computer Algebra Department
Eötvös Loránd University
1117 Budapest
Pázmány Péter Sétány I/C
Hungary
katai@inf.elte.hu

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Dedicated to the memory of Professor Ferenc Gécseg

Abstract

Let α be a non Liouville number and let $f(x) = \alpha x^r + a_{r-1}x^{r-1} + \cdots + a_1 x + a_0 \in \mathbb{R}[x]$ be a polynomial of positive degree r. We consider the sequence $(y_n)_{n\geq 1}$ defined by $y_n = f(h(n))$, where h belongs to a certain family of arithmetic functions and show that $(y_n)_{n\geq 1}$ is uniformly distributed modulo 1.

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1 Introduction and notation

Let $t(n)$ be an arithmetic function and let $f \in \mathbb{R}[x]$ be a polynomial. Under what conditions is the sequence $(f(t(n)))_{n\geq 1}$ uniformly distributed modulo 1 ? In the particular case where f is of degree one, the problem is partly solved. For instance, it is known that, if α is an irrational number and if $t(n) = \omega(n)$ or $\Omega(n)$, where $\omega(n)$ stands for the number of distinct prime factors of n and $\Omega(n)$ for the number of prime factors of n counting their multiplicity, with $\omega(1) = \Omega(1) = 0$, then the sequence $(\{\alpha t(n)\})_{n\geq 1}$ is uniformly distributed modulo 1 (here $\{y\}$ stands for the fractional part of y). In 2005, we [1] proved that if α is a positive irrational number such that for each real number $\kappa > 1$ there exists a positive constant $c = c(\kappa, \alpha)$ for which the inequality

$\|\alpha q\| > c/q^\kappa$ holds for every positive integer q, then the sequence $(\{\alpha\sigma(n)\})_{n\geq 1}$ is uniformly distributed modulo 1. (Here $\|x\|$ stands for the distance between x and the nearest integer and $\sigma(n)$ stands for the sum of the positive divisors of n.) Observe that one can construct an irrational number α for which the corresponding sequence $(\{\alpha\sigma(n)\})_{n\geq 1}$ is not uniformly distributed modulo 1. On the other hand, given an integer $q \geq 2$ and letting $s_q(n)$ stand for the sum of the digits of n expressed in base q, it is not hard to prove that, if α is an irrational number, the sequence $(\{\alpha s_q(n)\})_{n\geq 1}$ is uniformly distributed modulo 1. In fact, in the past 15 years, important results have been obtained concerning the topic of the so-called q-ary arithmetic functions. For instance, it was proved that the sequence $(\{\alpha s_q(p)\})_{p\in\wp}$ (here $\wp$ is the set of all primes) is uniformly distributed modulo 1 if and only if $\alpha \in \mathbb{R}\setminus\mathbb{Q}$. In 2010, answering a problem raised by Gelfond [10] in 1968, Mauduit and Rivat [13] proved that the sequence $(\{\alpha s_q(n^2)\})_{n\geq 1}$ is uniformly distributed modulo 1 if and only if $\alpha \in \mathbb{R}\setminus\mathbb{Q}$.

Recall that an irrational number β is said to be a *Liouville number* if for all integers $m \geq 1$, there exist two integers t and $s > 1$ such that

$$0 < \left|\beta - \frac{t}{s}\right| < \frac{1}{s^m}.$$

Hence, Liouville numbers are those real numbers which can be approximated "quite closely" by rational numbers.

Here, if α is a non Liouville number and

$$f(x) = \alpha x^r + a_{r-1}x^{r-1} + \cdots + a_1 x + a_0 \in \mathbb{R}[x] \quad \text{is of degree } r \geq 1, \tag{1.1}$$

we prove that $(f(t(n)))_{n\geq 1}$ is uniformly distributed modulo 1, for those arithmetic functions $t(n)$ for which the corresponding function $a_{N,k} := \frac{1}{N}\#\{n \leq N : t(n) = k\}$ is "close" to the normal distribution as N becomes large.

Given $\mathcal{P} \subseteq \wp$, let $\Omega_{\mathcal{P}}(n) = \sum_{\substack{p^r\|n \\ p\in\mathcal{P}}} r$. From here on, we let $q \geq 2$ stand for a fixed integer. Now, consider the sequence $(y_n)_{n\geq 1}$ defined by $y_n = f(h(n))$, where $h(n)$ is either one of the five functions

$$\omega(n), \qquad \Omega(n), \qquad \Omega_{\mathcal{P}}(n), \qquad s_q(n), \qquad s_q(n^2). \tag{1.2}$$

Here, we show that the sequence $(y_n)_{n\geq 1}$ is uniformly distributed modulo 1.

For the particular case $h(n) = s_q(n)$, we also examine an analogous problem, as n runs only through the primes. Finally, we consider a problem involving strongly normal numbers.

Recall that the *discrepancy* of a set of N real numbers $x_1, \ldots, x_N$ is the quantity

$$D(x_1, \ldots, x_N) := \sup_{[a,b)\subseteq[0,1)} \left| \frac{1}{N} \sum_{\{x_\nu\}\in[a,b)} 1 - (b-a) \right|.$$

For each positive integer N, let

$$M = M_N = \lfloor \delta_N \sqrt{N} \rfloor, \qquad \text{where } \delta_N \to 0 \text{ and } \delta_N \log N \to \infty \text{ as } N \to \infty. \tag{1.3}$$

We shall say that an infinite sequence of real numbers $(x_n)_{n\geq 1}$ is *strongly uniformly distributed* mod 1 if

$$D(x_{N+1}, \ldots, x_{N+M}) \to 0 \qquad \text{as} \qquad N \to \infty$$

for every choice of M (and corresponding δ_N) satisfying (1.3). Then, given a fixed integer $q \geq 2$, we say that an irrational number α is a *strongly normal number* in base q (or a strongly q-normal number) if the sequence $(x_n)_{n\geq 1}$, defined by $x_n = \{\alpha q^n\}$, is strongly uniformly distributed modulo 1. The concept of strong normality was recently introduced by De Koninck, Kátai and Phong [2].

We will at times be using the standard notation $e(x) := \exp\{2\pi i x\}$. Finally, we let φ stand for the Euler totient function.

2 Background results

The sum of digits function $s_q(n)$ in a given base $q \geq 2$ has been extensively studied over the past decades. Delange [4] was one of the first to study this function. Drmota and Rivat [7], [14] studied the function $s_q(n^2)$ and then, very recently, Drmota, Mauduit and Rivat [9] analyzed the distribution of the function $s_q(P(n))$, where $P \in \mathbb{Z}[x]$ is a polynomial of a certain type.

Here, we state as propositions some other results and recall two relevant results of Halász and Kátai.

First, given an integer $q \geq 2$, we set

$$\mu_q = \frac{q-1}{2}, \qquad \sigma_q^2 = \frac{q^2-1}{12}.$$

Proposition 1. *Let $\delta > 0$ be an arbitrary small number and let $\varepsilon > 0$. Then, uniformly for $\left|k - \mu_q \log_q N\right| < \frac{1}{\delta}\sqrt{\log_q N}$,*

$$\#\{n \leq N : s_q(n) = k\} = \frac{N}{\sqrt{2\pi\sigma_q^2 \log_q N}} \left(\exp\left\{ -\frac{(k - \mu_q \log_q N)^2}{2\sigma_q^2 \log_q N} \right\} + O\left(\frac{1}{\log^{\frac{1}{2}-\varepsilon} N} \right) \right).$$

Proof. This result is in fact a particular case of Proposition 3 below. □

Proposition 2. *Let $\varepsilon > 0$. Uniformly for all integers $k \geq 0$ such that $(k, q-1) = 1$,*

$$\#\{p \leq N : s_q(p) = k\} = \frac{q-1}{\varphi(q-1)} \frac{\pi(N)}{\sqrt{2\pi\sigma_q^2 \log_q N}} \left(\exp\left\{ -\frac{(k - \mu_q \log_q N)^2}{2\sigma_q^2 \log_q N} \right\} + O\left(\frac{1}{\log^{\frac{1}{2}-\varepsilon} N} \right) \right).$$

Proof. This is Theorem 1.1 in the paper of Drmota, Mauduit and Rivat [8]. □

Let $G = (G_j)_{j\geq 0}$ be a strictly increasing sequence of integers, with $G_0 = 1$. Then, each non negative integer n has a unique representation as $n = \sum_{j\geq 0} \epsilon_j(n) G_j$ with integers $\epsilon_j(n) \geq 0$ provided that $\sum_{j<k} \epsilon_j(n) G_j < G_k$ for all integers $k \geq 1$. Then, the sum of digits function $s_G(n)$ is given by

$$s_G(n) = \sum_{j\geq 0} \epsilon_j(n). \tag{2.1}$$

Setting $a_{N,k} := \#\{n \leq N : s_G(n) = k\}$, consider the related sequence $(X_N)_{N\geq 1}$ of random variables defined by

$$P(X_N = k) = \frac{a_{N,k}}{N},$$

so that the expected value of X_N and its variance are given by

$$E[X_N] = \frac{1}{N} \sum_{n\leq N} s_G(n) \qquad \text{and} \qquad V[X_N] = \frac{1}{N} \sum_{n\leq N} (s_G(n) - E[X_N])^2. \tag{2.2}$$

Let us choose the sequence $(G_j)_{j\geq 0}$ as the particular sequence

$$G_0 = 1, \qquad G_j = \sum_{i=1}^{j} a_i G_{j-1} + 1 \quad (j > 0), \tag{2.3}$$

where the a_i's are simply the positive integers appearing in the Parry α-expansion (here $\alpha > 1$ is a real number) of 1, that is

$$1 = \frac{a_1}{\alpha} + \frac{a_2}{\alpha^2} + \frac{a_3}{\alpha^3} + \cdots$$

It can be shown (see Theorem 2.1 of Drmota and Gajdosik [5]) that, for such a sequence $(G_j)_{j\geq 0}$, setting

$$G(z, u) := \sum_{j=1}^{\infty} \left(\sum_{\ell=0}^{a_j - 1} z^{\ell} \right) z^{a_1 + \cdots + a_{j-1}} u^j$$

and letting $1/\alpha(z)$ denote the analytic solution $u = 1/\alpha(z)$ of the equation $G(z, u) = 1$ for z in a sufficiently small (complex) neighbourhood of $z_0 = 1$ such that $\alpha(1) = \alpha$, then,

$$E[X_N] = \mu \frac{\log N}{\log \alpha} + O(1)$$

and

$$V[X_N] = \sigma^2 \frac{\log N}{\log \alpha} + O(1),$$

where

$$\mu = \frac{\alpha'(1)}{\alpha} \qquad \text{and} \qquad \sigma^2 = \frac{\alpha''(1)}{\alpha} + \mu - \mu^2.$$

Proposition 3. *Let $G = (G_j)_{j\geq 0}$ be as in (2.3). If $\sigma^2 \neq 0$, then, given an arbitrary small $\varepsilon > 0$, uniformly for all integers $k \geq 0$,*

$$\#\{n \leq N : s_G(n) = k\} = \frac{N}{\sqrt{2\pi V[X_N]}} \left(\exp\left\{ -\frac{(k - E[X_N])^2}{2V[X_N]} \right\} + O\left(\frac{1}{\log^{\frac{1}{2}-\varepsilon} N} \right) \right).$$

Proof. This is Theorem 2.2 in the paper of Drmota and Gajdosik [5]. □

Let a be a positive integer. Let $q = -a + i$ (or $q = -a - i$) and set $Q = a^2 + 1$ and $\mathcal{N} = \{0, 1, \ldots, Q - 1\}$. It is well known that every Gaussian integer z can be written uniquely as

$$z = \sum_{\ell \geq 0} \epsilon_\ell(z) q^\ell \qquad \text{with each } \epsilon_\ell \in \mathcal{N}.$$

Then, define the sum of digits function $s_q(z)$ of $z \in \mathbb{Z}[i]$ in base q as

$$s_q(z) = \sum_{\ell \geq 0} \epsilon_\ell(z).$$

Proposition 4. *Let $\mathcal{A}$ be the set of those positive integers a for which if $p \mid q = -a \pm i$ and $|p| \neq 1$, then $|p|^2 \geq 689$. Let $\mathcal{D}_N = \{z \in \mathbb{C} : |z| \leq \sqrt{N}\} \cap \mathbb{Z}[i]$ or $\mathcal{D}_N = \{z \in \mathbb{C} : |\Re(z)| \leq \sqrt{N}, |\Im(z)| \leq \sqrt{N}\} \cap \mathbb{Z}[i]$. Then, uniformly for all integers $k \geq 0$, we have*

$$\frac{1}{\#\mathcal{D}_N} \#\{z \in \mathcal{D}_N : s_q(z^2) = k\} = \frac{Q(k, q-1)}{\sqrt{2\pi\sigma_Q^2 \log_Q(N^2)}} \left(\exp\{-\frac{\Delta_k^2}{2}\} + O\left(\frac{(\log\log N)^{11}}{\sqrt{\log N}} \right) \right),$$

where

$$\Delta_k = \frac{k - \mu_Q \log_Q(N^2)}{\sqrt{\sigma_Q^2 \log_Q(N^2)}}, \quad \mu_Q = \frac{Q-1}{2}, \quad \sigma_Q^2 = \frac{Q^2 - 1}{12}.$$

Proof. This result is a simplified version of Theorem 4 in Morgenbesser [15]. □

Let $a \in \mathbb{N}$ and $q = -a + i \in \mathbb{Z}[i]$. Set $\mathcal{N} = \{0, 1, \ldots, a^2\}$. Then, every $z \in \mathbb{Z}[i]$ can be written uniquely as

$$z = \sum_{j \geq 0} \epsilon_j(z) q^j \quad \text{with each } \epsilon_j(z) \in \mathcal{N}.$$

Let L be a non negative integer and consider a function $F : \mathcal{N}^{L+1} \to \mathbb{Z}$ satisfying $F(0, 0, \ldots, 0) = 0$ and set

$$s_F(z) = \sum_{j=-L}^{\infty} F(\epsilon_j(z), \epsilon_{j+1}(z), \ldots, \epsilon_{j+L}(z)).$$

The following is due to Drmota, Grabner and Liardet [6].

Proposition 5. *Under certain conditions on F stated in Corollary 3 in Drmota, Grabner and Liardet [6],*

$$\#\{z \in \mathbb{Z}[i] : |z|^2 < N,\ s_F(z) = k\} = \frac{\pi N}{\sqrt{2\pi\sigma^2 \log_{|q|^2} N}} \exp\left\{-\frac{(k-\mu \log_{|q|^2} N)^2}{2\sigma^2 \log_{|q|^2} N}\right\}\left(1 + O\left(\frac{1}{\sqrt{\log N}}\right)\right)$$

uniformly for $|k - \mu \log_{|q|^2} N| \le c\sqrt{\log_{|q|^2} N}$, where c can be taken arbitrarily large.

For any particular set of primes $\mathcal{P}$, let $E(x) = E_{\mathcal{P}}(x) := \sum_{\substack{p \le x \\ p \in \mathcal{P}}} \frac{1}{p}$.

The following two results, which we state as propositions, are due respectively to Halász [11] and Kátai [12].

Proposition 6. (Halász) *Let $0 < \delta \le 1$ and let $\mathcal{P}$ be a set of primes with corresponding functions $\Omega_{\mathcal{P}}(n)$ and $E(x) = E_{\mathcal{P}}(x)$. Then, assuming that $E(x) \to \infty$ as $x \to \infty$, the estimate*

$$\sum_{\substack{n \le x \\ \Omega_{\mathcal{P}}(n) = k}} 1 = \frac{xE(x)^k}{k!} e^{-E(x)} \left\{1 + O\left(\frac{|k - E(x)|}{E(x)}\right) + O\left(\frac{1}{\sqrt{E(x)}}\right)\right\}$$

holds uniformly for all positive integers k and real numbers $x \ge 3$ satisfying

$$E(x) \ge \frac{8}{\delta^3} \qquad \text{and} \qquad \delta \le \frac{k}{E(x)} \le 2 - \delta.$$

Proposition 7. (Kátai) *For $1 \le h \le x$, let*

$$A_k(x,h) := \sum_{\substack{x \le n \le x+h \\ \omega(n) = k}} 1, \qquad B_k(x) := \sum_{\substack{n \le x \\ \omega(n) = k}} 1,$$

$$\delta_k(x,h) := \frac{A_k(x,h)}{h} - \frac{B_k(x)}{x}, \qquad E(x,h) := \sum_{k=1}^{\infty} \delta_k^2(x,h).$$

Letting $\varepsilon > 0$ be an arbitrarily small number and $x^{7/12+\varepsilon} \le h \le x$, then

$$E(x,h) \ll \frac{1}{\log^2 x \cdot \sqrt{\log\log x}}.$$

3 Main results

Theorem 1. *Let $f(x)$ be as in (1.1), $h(n)$ be one of the five functions listed in (1.2) and $y_n := f(h(n))$. Then, the sequence $(y_n)_{n \ge 1}$ is uniformly distributed modulo 1.*

Theorem 2. *Let $f(x)$ be as in (1.1). Then, the sequence $(z_p)_{p\in\wp}$, where $z_p := f(s_q(p))$, is uniformly distributed modulo 1.*

Theorem 3. *Let $Q \geq 2$ and $q \geq 2$ be fixed integers. Let α be a strongly Q-normal number. Let g be a real valued continuous function defined on $[0,1]$ such that $\int_0^1 g(x)\,dx = 0$. Then,*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} g(\alpha Q^{h(n)}) = 0, \tag{3.1}$$

where $h(n) = s_q(n)$ or $s_q(n^2)$. Moreover, letting $\pi(N)$ stand for the number of prime numbers not exceeding N, we have

$$\lim_{N\to\infty} \frac{1}{\pi(N)} \sum_{p\leq N} g(\alpha Q^{s_q(p)}) = 0. \tag{3.2}$$

The following corollary follows from estimate (3.1) of Theorem 3.

Corollary 1. *With α and $h(n)$ as in Theorem 3, the sequence $(\alpha Q^{h(p)})_{p\in\wp}$ is uniformly distributed modulo 1.*

In light of Proposition 3, we have the following two corollaries.

Corollary 2. *Let G be as in (2.1). Then, letting f be as in (1.1), the sequence $(\{f(s_G(n))\})_{n\geq 0}$ is uniformly distributed modulo 1.*

Corollary 3. *Let G be as in (2.1). Then, if α is a strongly normal number in base Q, the sequence $(\{\alpha \cdot Q^{s_G(n)}\})_{n\geq 0}$ is uniformly distributed modulo 1.*

As a direct consequence of the Main Lemma and of Proposition 4, we have the following result.

Theorem 4. *Let $\mathcal{D}_N$ be as in Proposition 4. Let f be as in (1.1). For each $z \in \mathcal{D}_N$, set $y_z := f(s_q(z^2))$. Then, the discrepancy of the sequence y_z tends to 0 as $N \to \infty$, that is*

$$D(y_z : z \in \mathcal{D}_N) \to 0 \qquad \text{as } N \to \infty.$$

Theorem 5. *Let $\mathcal{D}_N$ be as in Proposition 4. Let α be a strongly normal number in base Q and consider the sequence $(y_z)_{z\in\mathcal{D}_N}$. Then*

$$D(y_z : z \in \mathcal{D}_N) \to 0 \qquad \text{as } N \to \infty.$$

In line with Proposition 7, we have the following.

Theorem 6. *Let $\varepsilon > 0$ be a fixed number. Let $H = \lfloor x^{7/12+\varepsilon} \rfloor$ and set*

$$\pi_k([x, x+H]) := \#\{n \in [x, x+H] : \omega(n) = k\}.$$

Let f be as in (1.1) and set

$$S(x) = \sum_{x \le n \le x+H} e(f(\omega(n))).$$

Then

$$\frac{S(x)}{H} \to 0 \qquad \text{as } x \to \infty.$$

4 Preliminary lemmas

Lemma 1. *Let α be a non Liouville number and let $f(x)$ be as in (1.1). Then,*

$$\sup_{U \ge 1} \frac{1}{N} \left| \sum_{n=U+1}^{U+N} e(f(n)) \right| \to 0 \qquad \text{as } N \to \infty.$$

Proof. Since α is a non Liouville number, there exists a positive integer ℓ such that if τ is a fixed positive number and

$$\left| \alpha - \frac{t}{s} \right| \le \frac{1}{s\tau}, \quad (t, s) = 1, \quad s \le \tau,$$

then $\tau^{1/\ell} < s$.

Vaughan ([16], Lemma 2.4) proved that if $\left|\alpha - \frac{t}{s}\right| < \frac{1}{s^2}$ and $K = 2^{t-1}$, then, given any small number $\varepsilon > 0$,

$$\sum_{n=U+1}^{U+N} e(f(n)) \ll_\varepsilon N^{1+\varepsilon} \left(\frac{1}{s} + \frac{1}{N} + \frac{s}{N^t} \right)^{1/K}. \tag{4.1}$$

Now, choose $\tau = N^{t/2}$ so that $N^{t/2^\ell} < s < \tau$. It then follows from (4.1) that

$$\sum_{n=U+1}^{U+N} e(f(n)) \ll N^{1-\delta},$$

for some $\delta > 0$ which depends only on ε and ℓ, thus completing the proof of Lemma 1. □

Using this result, we can establish our Main Lemma.

Lemma 2. *(**Main Lemma**) For each positive integer N, let $(E_N(k))_{k \ge 1}$ be a sequence of non negative integers called weights which, given any $\delta > 0$, satisfies the following three conditions:*

(a) $\sum_{k=1}^{\infty} E_N(k) = 1$;

(b) there exists a sequence $(L_N)_{N\geq 1}$ *which tends to infinity as* $N\to\infty$ *such that*

$$\limsup_{N\to\infty} \sum_{\substack{k=1\\ \frac{|k-L_N|}{\sqrt{L_N}}>\frac{1}{\delta}}}^{\infty} E_N(k) \to 0 \qquad \text{as } \delta\to 0;$$

(c) $\lim_{N\to\infty} \max_{\frac{|k-L_N|}{\sqrt{L_N}}\leq\frac{1}{\delta}} \max_{1\leq\ell\leq\delta^{3/2}} \left| \frac{E_N(k+\ell)}{E_N(k)} - 1 \right| = 0.$

Moreover, let α *and* f *be as in (1.1) and let*

$$T_N(f) := \sum_{k=1}^{\infty} e(f(k)) E_N(k).$$

Then,

$$T_N(f) \to 0 \qquad \text{as } N\to\infty. \tag{4.2}$$

Proof. Let $\delta>0$ be fixed and set

$S := \lfloor \delta^{3/2}\sqrt{L_N}\rfloor$, $\quad t_m = \lfloor L_N\rfloor + mS$ $(m=1,2,\ldots)$, $\quad U_m = [t_m, t_{m+1}-1]$ $(m=1,2,\ldots)$.

Let us now write

$$T_N(f) = S_1(N) + S_2(N), \tag{4.3}$$

where

$$\begin{aligned} S_2(N) &= \sum_{|k-L_N|>\frac{1}{\delta}\sqrt{L_N}} E_k(N) e(f(k)),\\ S_1(N) &= \sum_{|m|\leq 1/\delta^{5/2}} \sum_{k\in U_m} E_k(N) e(f(k)) = \sum_{|m|\leq 1/\delta^{5/2}} S_1^{(m)}(N), \end{aligned}$$

say.

First observe that, by condition (b) above,

$$|S_2(N)| \leq \sum_{\frac{|k-L_N|}{\sqrt{L_N}}>\frac{1}{\delta}} E_N(k) = o(1) \qquad \text{as } N\to\infty. \tag{4.4}$$

On the other hand, it follows from condition (c) above and Lemma 1 that, as $N\to\infty$,

$$\left|S_1^{(m)}(N)\right| \;\leq\; E_{t_m}(N) \left| \sum_{k\in U_m} e(f(k)) \right| + o(1) \sum_{k\in U_m} E_k(N)$$

$$= o(1) SE_{t_m}(N) + o(1) \sum_{k \in U_m} E_k(N),$$

while

$$\left| SE_{t_m}(N) - \sum_{k \in U_m} E_k(N) \right| = o(1) \sum_{k \in U_m} E_k(N).$$

Gathering these two estimates, we obtain that

$$S_1(N) \to 0 \quad \text{as } N \to \infty. \tag{4.5}$$

Using (4.4) and (4.5) in (4.3), conclusion (4.2) follows. □

Lemma 3. *For each integer $k \geq 1$, let*

$$\begin{aligned} \pi_k(x) &:= \#\{n \leq x : \omega(n) = k\}, \\ \pi_k^*(x) &:= \#\{n \leq x : \Omega(n) = k\} \end{aligned}$$

Then, the relations

$$\begin{aligned} \pi_k(x) &= (1 + o(1)) \frac{x}{\log x} \frac{(\log \log x)^{k-1}}{(k-1)!}, \\ \pi_k^*(x) &= (1 + o(1)) \frac{x}{\log x} \frac{(\log \log x)^{k-1}}{(k-1)!} \end{aligned}$$

hold uniformly for

$$|k - \log \log x| \leq \frac{1}{\delta_x} \sqrt{\log \log x}, \tag{4.6}$$

where δ_x is some function of x chosen appropriately and which tends to 0 as $x \to \infty$.

Proof. This follows from Theorem 10.4 stated in the book of De Koninck and Luca [3]. □

5 Proof of Theorem 1

We first consider the case when $h(n)$ is one of the three functions $\omega(n)$, $\Omega(n)$ and $\Omega_E(n)$. Set

$$\begin{aligned} \pi_k(N) &= \#\{n \leq N : \omega(n) = k\}, \\ \pi_k^*(N) &= \#\{n \leq N : \Omega(n) = k\}, \\ T_k(N) &= \#\{n \leq N : \Omega_E(n) = k\}. \end{aligned}$$

In light of Lemma 3 and Proposition 6, the corresponding weights of the sequences $(\pi_k(N))_{k \geq 1}$, $(\pi_k^*(N))_{k \geq 1}$ and $(T_k(N))_{k \geq 1}$ are $\pi_k(N)/N$, $\pi_k^*(N)/N$ and $T_k(N)/N$, respectively.

Now, in order to obtain the conclusion of the Theorem, we only need to prove that, for each non zero integer m,

$$\frac{1}{N}\sum_{n\le N} e(mf(h(n))) \to 0 \qquad \text{as } N\to\infty.$$

But this is guaranteed by Lemma 1 if we take into account the fact that since α is a non Liouville number, the number $m\alpha$ is also non Liouville for each $m\in\mathbb{Z}\setminus\{0\}$. Hence, the theorem is proved.

6 Proof of Theorem 2

We cannot make a direct use of Lemma 2 because the estimate in that lemma only holds for those positive integers k such that $(k, q-1) = 1$. To avoid this obstacle, we shall subdivide the positive integers k according to their residue class modulo $q-1$. Observe that there are $\varphi(q-1)$ such classes. Hence, we write each k as

$$k = t(q-1)+\ell, \qquad (\ell, q-1) = 1.$$

Hence, for each positive integer ℓ such that $(\ell, q-1) = 1$, we set

$$(6.1)\quad \wp_\ell := \{p\in\wp : s_q(p)\equiv \ell \pmod{q-1}\}, \qquad \Pi_\ell(N) := \#\{p\le N : p\in\wp_\ell\}.$$

It is easy to verify that

$$(6.2)\qquad \frac{\Pi_\ell(N)}{\pi(N)} = (1+o(1))\frac{1}{\varphi(q-1)} \qquad (N\to\infty).$$

Thus, in order to prove Theorem 2, we need to show that the sum

$$U_\ell(N) := \sum_{\substack{p\le N\\ s_q(p)\equiv\ell \pmod{q-1}}} e(mf(s_q(p))),$$

where m is any fixed non zero integer, satisfies

$$(6.3)\qquad U_\ell(N) = o(1) \quad \text{as } N\to\infty.$$

Setting

$$\sigma_N(k) := \#\{p\le N : s_q(p) = k\},$$

we have

$$\begin{aligned} U_\ell(N) &= \sum_{k\equiv\ell \pmod{q-1}} e(mf(k))\sigma_N(k) \\ &= \sum_{t\ge 0} e(mf(t(q-1)+\ell))\sigma_N(t(q-1)+\ell). \end{aligned} \qquad (6.4)$$

Observe that the leading coefficient of the above polynomial $f(t(q-1)+\ell)$ is $\alpha(q-1)^k$, which is a non Liouville number as well (as we mentioned in the proof of Theorem 1), and also that the functions

$$w_N(t) := \frac{1}{\Pi_\ell(N)}\sigma_N(t(q-1)+\ell)$$

may be considered as weights (since $\sum_{k=1}^{\infty} w_N(t) = 1$). Thus, applying Lemma 2, we obtain (6.3), thereby completing the proof of Theorem 2.

7 Proof of Theorem 3

We shall skip the proof of estimate (3.1), since it can be obtained along the same lines as that of the main theorem in De Koninck, Kátai and Phong [2].

In order to obtain (3.2), we separate the set $\wp$ into $\varphi(q-1)$ distinct sets $\wp_\ell$, with corresponding counting function $\Pi_N(\ell)$ defined in (6.1).

Observe that

$$g(\alpha Q^{t(q-1)+\ell})\sigma_N(t(q-1)+\ell) = g((\alpha Q^\ell)\cdot Q^{t(q-1)})\sigma_N(t(q-1)+\ell)$$

Now, since α is a strongly Q-normal number, then so is αQ^ℓ, a number which is strongly Q^{q-1}-normal.

We then have

$$\begin{aligned}
\sum_{p\le N} g(\alpha Q^{s_q(p)}) &= \sum_{k\ge 1}\sum_{\substack{p\le N\\ s_q(p)=k}} g(\alpha Q^k)\\
&= \sum_{\substack{\ell=1\\ (\ell,q-1)=1}}^{q-1}\sum_{\substack{p\le N\\ p\in\wp_\ell}} g(\alpha Q^{t(q-1)+\ell})\sigma_N(t(q-1)+\ell)\\
&= \sum_{\substack{\ell=1\\ (\ell,q-1)=1}}^{q-1}\sum_{\substack{p\le N\\ p\in\wp_\ell}} g((\alpha Q^\ell)\cdot Q^{t(q-1)})\sigma_N(t(q-1)+\ell).
\end{aligned}$$

Since we then have

$$\lim_{N\to\infty}\frac{1}{\Pi_\ell(N)}\sum_{\substack{p\le N\\ p\in\wp_\ell}} g(\alpha Q^{s_q(p)}) = 0 \quad \text{for each } \ell \text{ with } (\ell,q-1)=1,$$

summing up over all ℓ's such that $(\ell,q-1)=1$, estimate (3.2) follows immediately.

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