Multidimensional sequences uniformly distributed modulo 1 created from normal numbers

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Abstract

Let $q \geq 3$ be a prime number. We create an infinite sequence $\alpha_1, \alpha_2, \ldots$ of normal numbers in base $q-1$ such that, for any fixed positive integer r, the r-dimensional sequence $(\{\alpha_1(q-1)^n\}, \ldots, \{\alpha_r(q-1)^n\})$ is uniformly distributed on $[0,1)^r$.

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1 Introduction

In previous papers, we used the factorization of integers to generate large families of normal numbers; see for instance [1] and [2]. Here, we go a step further. But first let us mention that it is well known that if α is an irrational number, then the sequence $(\alpha n)_{n\geq 1}$ is uniformly distributed modulo 1 (see for instance Example 2.1 in the book of Kuipers and Neiderreiter [3]). Here, given a prime number $q \geq 3$, we construct an infinite sequence of normal numbers in base $q-1$ which, for any fixed positive integer r, yields an r-dimensional sequence which is uniformly distributed on $[0,1)^r$.

2 Main result

Let $q \geq 3$ be a prime number. Our main result will consist in creating an infinite sequence $\alpha_1, \alpha_2, \ldots$ of normal numbers in base $q-1$ such that, for any fixed positive integer r, the r-dimensional sequence $(\{\alpha_1(q-1)^n\}, \ldots, \{\alpha_r(q-1)^n\})$ is uniformly distributed on $[0,1)^r$, where $\{y\}$ stands for the fractional part of y.

Let $A_q := \{0, 1, \ldots, q-1\}$. Given an integer $t \geq 1$, an expression of the form $i_1 i_2 \ldots i_t$, where each $i_j \in A_q$, is called a *finite word* of length t. The symbol Λ will denote the *empty word*, so that if we concatenate the words α, Λ, β, then, instead of writing $\alpha\Lambda\beta$, we may simply write $\alpha\beta$.

Fix a positive integer r. For each integer $j \in \{1, \ldots, r\}$, write the $(q-1)$-ary expansion of α_j as

$$\alpha_j = 0.a_{j,1}a_{j,2}a_{j,3}\ldots$$

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To prove our claim we only need to prove that for every positive integer k and arbitrary integers $b_{j,\ell} \in A_q$ (for $1 \le j \le r$, $1 \le \ell \le k$), the proportion of those positive integers $n \le x$ for which $a_{j,n+\ell} = b_{j,\ell}$ simultaneously for $j = 1, \dots, r$ and $\ell = 1, \dots, k$ is asymptotically equal to $1/(q-1)^{kr}$.

To do so, we first construct the proper set up. For each positive integer N, consider the semi-open interval $J_N := [x_N, x_{N+1})$, where $x_N = e^N$. For each integer $N \ge 3$, we introduce the expression $\lambda_N = \log\log N$ and define the corresponding interval $K_N := [N, N^{\lambda_N}]$. Given an integer $n \in J_N$, we define the function $q_N(n)$ as the smallest prime factor of n which belongs to K_N, while we let $q_N(n) = 1$ if $(n, p) = 1$ for all primes $p \in K_N$.

Further let $\pi_1 \le \pi_2 \le \cdots \le \pi_{h(n)}$ be the prime factors of n which belong to K_N (written with multiplicity). With this definition, we clearly have $(n/\pi_1 \cdots \pi_{h(n)}, p) = 1$ for each prime $p \in K_N$.

For each positive integer ℓ and each $n \in K_N$, we let

$$q_N^{(\ell)}(n) = \begin{cases} \pi_\ell & \text{if } \ 1 \le \ell \le h, \\ 1 & \text{if } \ \ell > h, \end{cases}$$

where $h = h(n)$, so that in particular $q_N(n) = q_N^{(1)}(n)$.

We further set

$$f_q(m) = \begin{cases} \ell - 1 & \text{if } \ m \equiv \ell \pmod q \text{ and } \ell \ne 0, \\ \Lambda & \text{if } \ q \mid m. \end{cases}$$

Let r and k be fixed positive integers. Let $Q_{i,\ell}$, for $i = 1, \dots, r$ and $\ell = 1, \dots, k$ be distinct primes belonging to K_N such that $Q_{1,\ell} < Q_{2,\ell} < \cdots < Q_{r,\ell}$. For a given interval $J = [x, x+y] \subseteq J_N$, we let $S_J(Q_{i,\ell} \mid i = 1, \dots, r,\ \ell = 1, \dots, k)$ be the number of those integers $n \in J$ for which $q_N^{(i)}(n+\ell) = Q_{i,\ell}$.

For each integer $r \ge 1$, let $\sigma(1), \dots, \sigma(k)$ be the permutation of the set $\{1, \dots, k\}$ which allows us to write

$$Q_{r,\sigma(1)} < Q_{r,\sigma(2)} < \cdots < Q_{r,\sigma(k)}.$$

Using the Eratosthenian sieve, we obtain that, as $N \to \infty$,

$$\begin{aligned} & S_J(Q_{i,\ell} \mid i = 1, \dots, r,\ \ell = 1, \dots, k) \\ & \quad = (1 + o(1)) \frac{y}{\prod\limits_{\substack{1 \le i \le r \\ 1 \le \ell \le k}} Q_{i,\ell}} \cdot \prod_{N \le \pi < Q_{r,\sigma(k)}} \left(1 - \frac{\rho(\pi)}{\pi}\right) + o(x_N), \end{aligned} \tag{2.1}$$

where

$$\rho(\pi) = \begin{cases} k & \text{if } \ N \le \pi < Q_{r,\sigma(1)}, \\ k-1 & \text{if } \ Q_{r,\sigma(1)} < \pi < Q_{r,\sigma(2)}, \\ \vdots & \qquad \vdots \\ 1 & \text{if } \ Q_{r,\sigma(k-1)} < \pi < Q_{r,\sigma(k)}, \\ 0 & \text{if } \ \pi \in \{Q_{i,\ell} : i = 1, \dots, r,\ \ell = 1, \dots, k\}. \end{cases}$$

Let $t_{i,\ell}$ $(i = 1, \ldots, r,\ \ell = 1, \ldots, k)$ be any collection of the (non zero) reduced residues modulo q and set

(2.2)
$$B_J(t_{i,\ell} \mid i = 1, \ldots, r,\ \ell = 1, \ldots, k) := \sum_{\substack{Q_{i,\ell} \equiv t_{i,\ell} \pmod q \\ N < Q_{i,\ell} < N^{\lambda_N}}} S_J(Q_{i,\ell} \mid i = 1, \ldots, r,\ \ell = 1, \ldots, k).$$

Now, letting $\pi(x; \ell, k)$ stand for the number of primes $p \le x$ such that $p \equiv \ell \pmod k$, it follows from the Prime Number Theorem in arithmetical progressions that, with $2 \le v \le u$, as $u \to \infty$,

$$\pi(u+v; \ell, q) - \pi(u; \ell, q) = (1+o(1)) \frac{1}{q-1} \left(\pi(u+v) - \pi(u)\right) + O\left(\frac{u}{\log^{10} u}\right),$$

from which we obtain that

$$\sum_{\substack{u \le p \le u+v \\ p \equiv \ell \pmod q}} \frac{1}{p \log p} = (1+o(1)) \frac{1}{q-1} \frac{1}{\log u} \log \frac{\log(u+v)}{\log u} + O\left(\frac{1}{\log^{11} u}\right) \tag{2.3}$$

and

$$\sum_{\substack{u \le p \le u+v \\ p \equiv \ell \pmod q}} \frac{1}{p} = (1+o(1)) \frac{1}{q-1} \log \frac{\log(u+v)}{\log u} + O\left(\frac{1}{\log^{10} u}\right) \tag{2.4}$$

Substituting (2.3) and (2.4) in (2.1), we obtain

$$\begin{aligned} & S_J(Q_{i,\ell} \mid i = 1, \ldots, r,\ \ell = 1, \ldots, k) \\ &= (1+o(1)) \frac{y}{\prod_{1 \le i \le r, 1 \le \ell \le k} Q_{i,\ell}} \exp\{k \log\log N - k \log\log Q_{r,\sigma(1)} \\ &\qquad -(k-1) \log\log Q_{r,\sigma(2)} + (k-1) \log\log Q_{r,\sigma(1)} - \ldots - \log\log Q_{r,\sigma(k)}\} \\ &= (1+o(1)) \frac{y}{\prod_{1 \le i \le r, 1 \le \ell \le k} Q_{i,\ell}} \prod_{\ell=1}^{k} \frac{\log N}{\log Q_{r,\ell}} + o(x_N). \qquad (2.5) \end{aligned}$$

Using (2.5) and definition (2.2), we obtain that, as $y \to \infty$,

$$B_J(t_{i,\ell} \mid i = 1, \ldots, r,\ \ell = 1, \ldots, k) = (1+o(1)) \frac{y}{(q-1)^{kr}} \sum_{\pi_{i,\ell}} \frac{1}{\prod \pi_{i,\ell}} \prod_{\ell=1}^{k} \frac{\log N}{\log \pi_{r,\ell}}, \tag{2.6}$$

where the summation runs over those subsets of primes $\pi_{i,\ell}$ for which

$$N < \pi_{1,\ell} < \pi_{2,\ell} < \cdots < \pi_{r,\ell} < N^{\lambda_N} \quad (\ell = 1, \ldots, k).$$

Now, observe that, as $N \to \infty$,

$$\sum_{N < \pi_{1,\ell} < \cdots < \pi_{r-1,\ell} < \pi_{r,\ell} < N^{\lambda_N}} \frac{1}{\pi_{1,\ell} \cdots \pi_{r-1,\ell}} \cdot \frac{1}{\pi_{r,\ell} \log \pi_{r,\ell}}$$

$$= (1+o(1)) \sum_{\pi_{r,\ell}} \frac{1}{(r-1)!} \left(\sum_{N<\pi<\pi_{r,\ell}} \frac{1}{\pi} \right)^{r-1} \cdot \frac{1}{\pi_{r,\ell} \log \pi_{r,\ell}}$$

$$= (1+o(1)) \sum_{\pi_{r,\ell}} \frac{1}{(r-1)!} \left(\log \left(\frac{\log \pi_{r,\ell}}{\log N} \right) \right)^{r-1} \cdot \frac{1}{\pi_{r,\ell} \log \pi_{r,\ell}}$$

$$= (1+o(1)) \int_N^{N^{\lambda_N}} \frac{1}{(r-1)!} \left(\log \left(\frac{\log u}{\log N} \right) \right)^{r-1} \frac{du}{u \log^2 u}$$

$$= (1+o(1)) \int_{\log N}^{\lambda_N \log N} \frac{1}{(r-1)!} \left(\log \left(\frac{v}{\log N} \right) \right)^{r-1} \frac{dv}{v^2}. \tag{2.7}$$

Setting $v = y \log N$ in this last integral, we obtain that the above expression can be replaced by

$$\frac{(1+o(1))}{\log N} \int_1^{\lambda_N} \frac{1}{(r-1)!} \frac{(\log y)^{r-1}}{y^2}\, dy = \frac{(1+o(1))}{\log N} \frac{1}{(r-1)!} \int_1^{\infty} \frac{(\log y)^{r-1}}{y^2}\, dy,$$

which in turn, after setting $z = \log y$, becomes

$$\frac{(1+o(1))}{\log N} \int_0^{\infty} \frac{e^{-z} z^{r-1}}{(r-1)!}\, dz = \frac{(1+o(1))}{\log N},$$

which substituted in (2.7) yields

$$\sum_{N<\pi_{1,\ell}<\cdots<\pi_{r-1,\ell}<\pi_{r,\ell}<N^{\lambda_N}} \frac{1}{\pi_{1,\ell} \cdots \pi_{r-1,\ell}} \cdot \frac{1}{\pi_{r,\ell} \log \pi_{r,\ell}} = \frac{(1+o(1))}{\log N} \qquad (N \to \infty). \tag{2.8}$$

Using (2.8) in (2.6), we obtain that

$$B_J(t_{i,\ell} \mid i = 1, \ldots, r,\ \ell = 1, \ldots, k) = (1+o(1)) \frac{y}{(q-1)^{kr}} \qquad (y \to \infty). \tag{2.9}$$

We now define, for each integer $N \in \mathbb{N}$,

$$\theta_N^{(i)} = \text{Concat}\{f_q(q_N^{(i)}(n)) : n \in J_N\} \qquad (i = 1, 2, \ldots).$$

Then consider the number

$$\theta^{(i)} = \theta_1^{(i)} \theta_2^{(i)} \ldots$$

and from these numbers, introduce the number

$$\alpha_i := 0.\theta^{(i)},$$

that is the number whose q-ary expansion is $0.\theta^{(i)}$.

Recall that, for $n \in J_N$, we defined $h(n)$ as the number of prime divisors of n located in the interval $[N, N^{\lambda_N}]$. Thus, setting

$$U_N := \sum_{N<p<N^{\lambda_N}} \frac{1}{p} = \log \lambda_N + o(1) \qquad (N \to \infty),$$

we obtain, using the Turán-Kubilius inequality, that for some absolute constant $c > 0$,

$$\sum_{n\in J_N} (h(n) - U_N)^2 \le c x_N \log \lambda_N. \tag{2.10}$$

On the one hand, it follows from (2.10) that for each integer $r \ge 1$, there exists a constant $c_r > 0$ such that

$$\#\{n \in J_N : h(n) \le r\} \le \frac{c_r x_N}{\log \lambda_N}. \tag{2.11}$$

On the other hand, it is easy to see that, as $y \to \infty$,

$$\#\{n \in J_N : p^2|n \text{ for some prime } p > N\} \le c x_N \sum_{p>N} \frac{1}{p^2} = O\left(\frac{x_N}{N}\right). \tag{2.12}$$

We therefore have, in light of (2.9), keeping in mind (2.11) and (2.12), that, as $y \to \infty$ (and thus $N \to \infty$),

$$\#\{n \in J_N : f_q(q_N^{(i)}(n+\ell)) = t_{i,\ell}-1 : i = 1,\dots,r,\ \ell = 1,\dots,k\} = (1+o(1))\frac{y}{(q-1)^{kr}}+o(x_N). \tag{2.13}$$

Now, to prove the normality of α_i in base $q-1$, we need to estimate the quantity

$$H(x) := \#\{n \le x : f_q(q_N^{(i)}(n+\ell)) = t_{i,\ell} - 1 : i = 1,\dots,r,\ \ell = 1,\dots,k\}.$$

For this, let us set

$$K_N := \#\{n \in J_N : f_q(q_N^{(i)}(n+\ell)) = t_{i,\ell} - 1 : i = 1,\dots,r,\ \ell = 1,\dots,k\}.$$

Let x be a large number. Then, $x \in J_{N_0}$ for some N_0. Hence, applying (2.13), we get

$$\begin{aligned} H(x) &= O(1) + K_3 + K_4 + \cdots + K_{N_0-1} \\ &\quad +\#\{J_{N_0-1} \le x : f_q(q_{N_0-1}^{(i)}(n+\ell)) = t_{i,\ell} - 1 : i = 1,\dots,r,\ \ell = 1,\dots,k\} \\ &= \frac{(1+o(1))}{(q-1)^{kr}}\left((x_2-x_1)+(x_3-x_2)+\cdots+(x_{N_0}-x_{N_0-1})+(x-x_{N_0})\right)+O(1) \\ &= (1+o(1))\frac{x-x_1}{(q-1)^{kr}} = (1+o(1))\frac{x}{(q-1)^{kr}}, \end{aligned}$$

thus completing the proof of our main result.

3 Final remarks

The method we used will also lead to the construction of normal numbers if, for each $N \in \mathbb{N}$, instead of choosing the smallest prime factor of $n \in J_N$ which belongs to K_N, we choose the largest prime factor of n which is smaller than $y_N := x_N^{1/\sqrt{N}}$, where $x_N = e^N$. Even more generally, instead of choosing the largest prime factor, we may pick the second largest prime factor, or the third, and so on.

Similarly, instead of working with $y_N := x_N^{1/\sqrt{N}}$, where $x_N = e^N$, we could also choose any two sequences $(w_N)_{N\geq 1}$ and $(z_N)_{N\geq 1}$ which satisfy the conditions

$$w_N \to \infty, \qquad \frac{\log z_N}{\log x_N} \to \infty, \qquad \frac{\log z_N}{\log w_N} \to \infty \qquad (N \to \infty).$$

Finally, one can easily see that, letting ϕ stand for the Euler totient function, our main result will still hold for any base $b > 2$ of the form $b = \phi(q)$ for some integer $q > 4$.

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