About an unsolved problem involving normal numbers

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Abstract

We examine the discrepancy of various sequences created from the values of additive functions and exhibit connections with q-normal numbers.

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1 Introduction

Let $\mathcal{A}$ be the set of all additive functions and let $\mathcal{M}_1$ stand for the set of all multiplicative functions f such that $|f(n)| \le 1$ for all integers $n \ge 1$. Let $\wp$ be the set of all prime numbers. As usual, given a real number y, we set $e(y) := \exp\{2\pi i y\}$ and write $\{y\}$ for the fractional part of y.

Given a fixed integer $q \ge 2$, we say that a real number α is a *q-normal number* if the sequence $(\{q^n\alpha\})_{n\ge1}$ is *uniformly distributed modulo 1*. Moreover, given N real numbers $y_1, \ldots, y_N$, we define the discrepancy of these numbers as

$$D(y_1, \ldots, y_N) := \sum_{[\alpha,\beta)\subseteq[0,1]} \left| \frac{1}{N} \sum_{\{y_j\}\in[\alpha,\beta)} 1 - (\beta - \alpha) \right|.$$

In 1948, Erdős and Turán [3], [4] proved that, given any positive integer M,

$$D(y_1, \ldots, y_N) \le \frac{c}{N} \left| \sum_{k=1}^{M} \frac{1}{k} \left| \sum_{j=1}^{N} e(ky_j) \right| + \frac{1}{M} \right|. \tag{1.1}$$

Later, Daboussi and Delange [1], [2] proved that, if $h \in \mathcal{M}_1$ and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, then

$$\sum_{n\le x} h(n)e(n\alpha) = o(x) \qquad (x \to \infty). \tag{1.2}$$

By using a simple method, the second author [5] gave a generalization of Daboussi's result, namely the following.

Lemma 1. *Given a sequence of complex numbers $(a_n)_{n\ge1}$ such that $|a_n| \le 1$ for each integer $n \ge 1$ and letting $f \in \mathcal{M}_1$, set*

$$S(x) := \sum_{n\le x} f(n)a_n.$$

Let $\wp_x$ be a subset of primes all of whose elements do not exceed x and let $A_x := \sum_{p\in\wp_x} \frac{1}{p}$. Then,

$$|S(x)|^2 \le \frac{Cx^2}{A_x} + \frac{x}{A_x^2} \sum_{\substack{p_1,p_2\in\wp\\ p_1\neq p_2}} \left| \sum_{m\le \min(x/p_1,x/p_2)} a_{p_1m}\overline{a_{p_2m}} \right|, \tag{1.3}$$

where C is an absolute constant (so that the right hand side of (1.3) does not depend on f).

It follows from this that if $\alpha \in \mathbb{R}\setminus Q$, $h \in \mathcal{A}$ and $y_n(h,\alpha) = h(n) + n\alpha$ for $n = 1, 2, 3, \ldots$, then

$$\lim_{N\to\infty} \sup_{h\in\mathcal{A}} D(y_1(h,\alpha),\ldots,y_N(h,\alpha)) = 0. \tag{1.4}$$

2 Main results

Given $\alpha \in \mathbb{R}\setminus Q$ and $h \in \mathcal{A}$, let $z_n(h,\alpha) = h(n) + q^n\alpha$ for $n = 1, 2, 3, \ldots$.

Theorem 1. *For almost all $\alpha \in \mathbb{R}\setminus\mathbb{Q}$,*

$$\lim_{N\to\infty} \sup_{h\in\mathcal{A}} D(z_1(h,\alpha),\ldots,z_N(h,\alpha)) = 0. \tag{2.1}$$

Remark 1. *Considering the additive function h defined by $h(n) = 0$ for all $n \in \mathbb{N}$, one can easily see that (2.1) can only hold if α is a q-normal number.*

An interesting conjecture and an unsolved problem related to Theorem 1 are the following.

Conjecture. *If α is a q-normal number, then (2.1) holds.*

Open problem. *Construct a real number α for which (2.1) holds.*

Theorem 2. *Let $r_1 < r_2 < \cdots$ be an infinite sequence of positive integers satisfying the gap condition $\frac{r_{k+1}}{r_k} > \theta$ for all $k \ge k_0$, for some fixed real number $\theta > 1$, and let $w_n(h,\alpha) := h(n) + r_n\alpha$ for $n = 1, 2, 3 \ldots$. Then,*

$$\lim_{N\to\infty} \sup_{h\in\mathcal{A}} D(w_1(h,\alpha),\ldots,w_N(h,\alpha)) = 0. \tag{2.2}$$

3 Proof of the theorems

Since Theorem 1 is clearly a consequence of Theorem 2, we shall only prove Theorem 2.

Let $P, Q \in \wp$ with $P > Q$ and, for each $M \in \mathbb{N}$, set $L_M := [M^2, M^2 + 2M]$ and

$$T_M(\alpha) = \sum_{k \in L_M} e\left((r_{Pk} - r_{Qk})\alpha\right).$$

First observe that, for some positive constants C_1 and C_2, we have

$$\int_0^1 |T_M(\alpha)|^4 \, d\alpha \leq C_1 M^2 + C_2. \tag{3.1}$$

Now, since the left hand side of (3.1) represents the number of solutions (k_1, k_2, k_3, k_4) of the equation

$$r_{Pk_1} - r_{Qk_1} + r_{Pk_2} - r_{Qk_2} = r_{Pk_3} - r_{Qk_3} + r_{Pk_4} - r_{Qk_4} \tag{3.2}$$

and since

$$PM^2 - Q(M^2 + 2M) = (P - Q)M^2 - Q \cdot 2M,$$

it follows that

$$\frac{\max_{\nu \in L_M} r_{Q\nu}}{\max_{\mu \in L_M} r_{P\mu}} \leq \left(\frac{1}{\theta}\right)^{(P-Q)M^2 - Q \cdot 2M}.$$

First assuming that $k_1 > k_2$, $k_3 > k_4$ and $k_1 \geq k_3$, and dividing (3.2) by r_{Pk_1}, we obtain that

$$1 - \frac{r_{Qk_1}}{r_{Pk_1}} + \frac{r_{Pk_2}}{r_{Pk_1}} - \frac{r_{Qk_2}}{r_{Pk_1}} = \frac{r_{Pk_3}}{r_{Pk_1}} - \frac{r_{Qk_3}}{r_{Pk_1}} + \frac{r_{Pk_4}}{r_{Pk_1}} - \frac{r_{Qk_4}}{r_{Pk_1}}.$$

If $k_1 = k_3$, then $|k_2 - k_4| \leq c$, where c is a constant that may depend on θ if $QM^2 > k_0$. On the other hand, if $k_1 > k_3$, then $k_1 - k_3 \leq c$. We therefore have that if k_1, k_3 and k_2 are fixed, the number of different choices for k_4 cannot exceed c. It follows from this observation that equation (3.2) has no more than $C_1 M^2$ solutions.

For each $M \in \mathbb{N}$, consider the set

$$J_M := \left\{\alpha \in [0,1) : |T_M(\alpha)| \geq M^{3/4+\delta}\right\}.$$

From (3.1), it follows that $\lambda(J_M) \leq 1/M^{1+4\delta}$ (here λ stands for the Lebesgue measure) and therefore that $\sum_{M \geq 1} \lambda(J_M) < +\infty$. We may therefore apply the Borel-Cantelli Lemma and conclude that for almost $\alpha \in [0,1)$ there exists a positive integer M_0 such that

$$\alpha \notin \bigcup_{M \geq M_0} J_M.$$

Consequently, letting $M_1 := \lfloor x^{1/3} \rfloor \geq M_0$, we obtain that

$$\frac{1}{x}\left|\sum_{k \leq x} e(r_{Pk} - r_{Qk})\alpha\right| \leq \frac{1}{x}\left|\sum_{k \leq M_1^2} e(r_{Pk} - r_{Qk})\alpha\right| + \frac{1}{x}\sum_{M_1^2 < M \leq \sqrt{x}} |T_M(\alpha)| + O\left(\frac{1}{\sqrt{x}}\right)$$

$$\leq \frac{M_1^2}{x} + O\left(\frac{1}{\sqrt{x}}\right) + \frac{1}{x} \sum_{M \leq \sqrt{x}} M^{3/4+\delta}. \tag{3.3}$$

Observe that this last quantity tends to 0 as $x \to \infty$ and that the convergence is uniform with respect to h.

Now, let W_1 be the set of those α for which the last quantity in (3.3) does not tend to 0 for at least one prime pair $\{P, Q\}$, in which case we have that $\lambda(W_1) = 0$. From Lemma 1, we obtain that

$$\Delta(x, \alpha) := \sup_{h \in \mathcal{A}} \frac{1}{x} \left| \sum_{n \leq x} e(w_n(h, \alpha)) \right|$$

tends to zero as $x \to \infty$ whenever $\alpha \notin W_1$. Let us now replace α by $\ell\alpha$ (where $\ell \in \mathbb{N}$), and define W_ℓ to be the set of those α for which the last quantity in (3.3) does not tend to zero if α is replaced by $\ell\alpha$. We then have $\lambda(W_\ell) = 0$, so that $\lambda\left(\bigcup_{\ell \geq 1} W_\ell\right) = 0$.

Setting $S := \mathbb{R} \setminus \left(\bigcup_{\ell \geq 1} W_\ell\right)$, then for $\alpha \in S$, we have

$$\Delta(x, \ell\alpha) \to 0 \qquad (x \to \infty).$$

Then, using the Erdős-Turán inequality (1.1), we obtain that, given any positive integer K,

$$\begin{aligned} D_N(w_1(h, \alpha), \ldots, w_N(h, \alpha)) &\leq \frac{C}{K} + \sum_{\ell=1}^{K} \frac{1}{\ell} \frac{1}{N} \left| \sum_{n \leq N} e(w_n(\ell h, \ell\alpha) \right| \\ &\leq \frac{C}{K} \sum_{\ell=1}^{K} \frac{1}{\ell} \Delta(N, \ell\alpha). \end{aligned}$$

Hence,

$$\limsup_{N \to \infty} D_N(w_1(h, \alpha), \ldots, w_N(h, \alpha)) \leq \frac{C}{K}. \tag{3.4}$$

Since K can be chosen arbitrarily large, it follows that the left hand side of (3.4) is zero, thus completing the proof of Theorem 2.

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