The number of prime factors function on shifted primes and normal numbers

Jean-Marie De Koninck and Imre Kátai

Abstract In a series of papers, we have constructed large families of normal numbers using the concatenation of the values of the largest prime factor $P(n)$, as n runs through particular sequences of positive integers. A similar approach using the smallest prime factor function also allowed for the construction of normal numbers. Letting $\omega(n)$ stand for the number of distinct prime factors of the positive integer n, we show that the concatenation of the successive values of $|\omega(n) - \lfloor \log\log n \rfloor|$, as n runs through the integers $n \geq 3$, yields a normal number in any given basis $q \geq 2$. We show that the same result holds if we consider the concatenation of the successive values of $|\omega(p+1) - \lfloor \log\log(p+1) \rfloor|$, as p runs through the prime numbers.

1 Introduction

Given an integer $q \geq 2$, we say that an irrational number η is a *q-normal number* if the q-ary expansion of η is such that any preassigned sequence of length $k \geq 1$, taken within this expansion, occurs with the expected limiting frequency, namely $1/q^k$.

In a series of papers, we have constructed large families of normal numbers using the distribution of the values of $P(n)$, the largest prime factor function (see [1], [2], [3] and [4]). Recently [5], we showed how the concatenation of the successive values of the smallest prime factor $p(n)$, as n runs through the positive integers, can also yield a normal number.

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Let $\omega(n)$ stand for the number of distinct prime factors of the positive integer n. One can easily show that the concatenation of the successive values of $\omega(n)$, say by considering the real number $\xi := 0.\overline{\omega(2)}\,\overline{\omega(3)}\,\overline{\omega(4)}\,\overline{\omega(5)}\ldots$, where each $\overline{m}$ stands for the q-ary expansion of the integer m, will not yield a normal number. Indeed, since the interval $I := [e^{e^{r-1}}, e^{e^r}]$, where $r := \lfloor \log\log x \rfloor$, covers most of the interval $[1,x]$ and since $\left| \frac{\omega(n)}{r} - 1 \right| < \frac{1}{r^{1/4}}$, say, with the exception of a small number of integers $n \in I$, it follows that ξ cannot be normal in basis q.

Recently, Vandehay [9] used another approach to yet create normal numbers using certain small additive functions. He considered irrational numbers formed by concatenating some of the base q digits from additive functions $f(n)$ that closely resemble the prime counting function $\Omega(n) := \sum_{p^\alpha \| n} \alpha$. More precisely, he used the concatenation of the last $\lceil y \frac{\log\log\log n}{\log q} \rceil$ digits of each $f(n)$ in succession and proved that the number thus created turns out to be normal in basis q if and only if $0 < y \le 1/2$.

In this paper, we show that the concatenation of the successive values of $|\omega(n) - \lfloor \log\log n \rfloor|$, as n runs through the integers $n \ge 3$, yields a normal number in any given basis $q \ge 2$. We show that the same result holds if we consider the concatenation of the successive values of $|\omega(p+1) - \lfloor \log\log(p+1) \rfloor|$, as p runs through the prime numbers.

2 Notation

Let $\wp$ stand for the set of all the prime numbers. The letter p, with or without subscript, will always denote a prime number. The letter c, with or without subscript, will always denote a positive constant, but not necessarily the same at each occurrence.

At times, we will use the notation $x_1 = \log x$, $x_2 = \log\log x$, $x_3 = \log\log\log x$.

Let $q \ge 2$ be a fixed integer. Given an integer $t \ge 1$, an expression of the form $i_1 i_2 \ldots i_t$, where each i_j is one of the numbers $0, 1, \ldots, q-1$, is called a *word* of length t. Given a word α, we shall write $\lambda(\alpha) = t$ to indicate that α is a *word* of length t. We shall also use the symbol Λ to denote the *empty word*.

Let $A = A_q = \{0, 1, 2, \ldots, q-1\}$. Then, A^t will stand for the set of words of length t over A, while A^* will stand for the set of all words over A regardless of their length, including the empty word Λ. Observe that the concatenation of two words $\alpha, \beta \in A^*$, written $\alpha\beta$, also belongs to A^*. Finally, given a word α and a subword β of α, we will denote by $F_\beta(\alpha)$ the number of occurrences of β in α, that is, the number of pairs of words μ_1, μ_2 such that $\mu_1 \beta \mu_2 = \alpha$.

Given a positive integer n, we write its q-ary expansion as

$$n = \varepsilon_0(n) + \varepsilon_1(n) q + \cdots + \varepsilon_t(n) q^t,$$

where $\varepsilon_i(n) \in A$ for $0 \le i \le t$ and $\varepsilon_t(n) \neq 0$. To this representation, we associate the word

$$\overline{n} = \varepsilon_0(n)\varepsilon_1(n)\ldots\varepsilon_t(n) \in A^{t+1}$$

For convenience, if $n \le 0$, we write $\overline{n} = \Lambda$.

Finally, the number of digits of such a number $\overline{n}$ will be

$$\lambda(\overline{n}) = \left\lfloor \frac{\log n}{\log q} \right\rfloor + 1.$$

Finally, given a sequence of integers $a(1), a(2), a(3), \ldots$, we will say that the concatenation of their q-ary digit expansions $\overline{a(1)}\,\overline{a(2)}\,\overline{a(3)}\ldots$, denoted by Concat$(\overline{a(n)} : n \in \mathbb{N})$, is a *normal sequence* if the number $0.\overline{a(1)}\,\overline{a(2)}\,\overline{a(3)}\ldots$ is a q-normal number.

For each integer $n \ge 2$, we let $\omega(n)$ stand for the number of distinct prime factors of n. We then introduce the arithmetic function $\delta(n) := |\omega(n) - \lfloor \log\log n \rfloor|$.

3 Main results

Theorem 1. *Let $R \in \mathbb{Z}[x]$ be a polynomial such that $R(y) \ge 0$ for all $y \ge 0$. Let*

$$\eta = Concat(\overline{R(\delta(n))} : n = 3, 4, 5, \ldots).$$

Then, η is a normal sequence in any given basis $q \ge 2$.

Theorem 2. *Let*

$$\xi = Concat(\overline{\delta(p+1)} : p \in \wp).$$

Then, ξ is a normal sequence in any given basis $q \ge 2$.

Remark 1. We shall only provide the proof of Theorem 2, the reason being that it is somewhat harder than that of Theorem 1. Indeed, for the proof of Theorem 1, one can use the fact that

$$\pi_k(n) := \#\{\le x : \omega(n) = k\} = (1 + o(1))\frac{x}{x_1}\frac{x_2^{k-1}}{(k-1)!}$$

uniformly for $|k - x_2| \le \sqrt{x_2}\,x_3$, say, and also the Hardy-Ramanujan inequality

$$\pi_k(x) < c_1 \frac{x}{x_1}\frac{(x_2 + c_2)^{k-1}}{(k-1)!}$$

which is valid uniformly for $1 \le k \le 10x_2$ and $x \ge x_0$ (see for instance the book of De Koninck and Luca [6], p. 157). Hence, using these estimates, one can easily prove Theorem 1 essentially as we did to prove that Concat$(P(m) : m \in \mathbb{N})$ is a normal sequence in any given basis $q \ge 2$ (see [1]). Now, since there are no known estimate for the asymptotic behavior of $\#\{p \le x : \omega(p+1) = k\}$, we need to find another approach for the proof of Theorem 2.

Remark 2. It will be clear from our approach that if $\omega(n)$ is replaced by $\Omega(n)$ or by $\delta_2(n) := |\lfloor \log \tau(n) \rfloor - \lfloor \log\log n \rfloor|$, the same results hold.

4 Preliminary results

For each real number $u > 0$, let $\Phi(u) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{u} e^{-t^2/2}\, dt$.

Lemma 1. *(a) As $x \to \infty$,*

$$\frac{1}{\pi(x)} \# \left\{ p \le x : \frac{\delta(p+1)}{\sqrt{x_2}} < u \right\} = (1 + o(1)) \left(\Phi(u) - \Phi(-u) \right).$$

(b) Letting ε_x a function which tends to 0 as $x \to \infty$. Then, as $x \to \infty$,

$$\frac{1}{\pi(x)} \# \left\{ p \le x : \delta(p+1) \le \varepsilon_x \sqrt{\log\log x} \right\} \to 0.$$

Proof. For a proof of part (a), see the book of Elliott [7]. Part (b) is an immediate consequence of part (a).

Let x be a fixed large number. For each integer $n \ge 2$, we now introduce the function

$$\delta^*(n) := |\omega(n) - \lfloor \log\log x \rfloor| .$$

Lemma 2. *For all $x \ge 2$,*

$$\sum_{p \le x} (\delta^*(p+1))^2 \le c\pi(x) \log\log x.$$

Proof. For a proof, see the book of Elliott [7].

Lemma 3. *Given an arbitrary $\kappa \in (0, 1/2)$, then, for all $x \ge 2$,*

$$\#\{p \le x : P(p+1) < x^{\kappa}\} + \#\{p \le x : P(p+1) > x^{1-\kappa}\} \le c\kappa\pi(x).$$

Proof. For a proof see Theorem 4.2 in the book of Halberstam and Richert [8].

Lemma 4. *Let a and b be two non zero co-prime integers, one of which is even. Then, as $x \to \infty$, we have, uniformly in a and b,*

$$\#\{p \le x : ap + b \in \wp\} \le 8 \prod_{p > 2} \left(1 - \frac{1}{(p-1)^2} \right) \prod_{\substack{p > 2 \\ p \mid ab}} \frac{p-1}{p-2} \frac{x}{\log^2 x} \left(1 + O\left(\frac{\log\log x}{\log x} \right) \right).$$

Proof. This is Theorem 3.12 in the book of Halberstam and Richert [8] for the particular case $k = 1$.

Lemma 5. *Let $M \geq 2k$, $\beta_1, \beta_2 \in A_q^k$. Set $\Delta(\alpha) = \left|F_{\beta_1}(\alpha) - F_{\beta_2}(\alpha)\right|$. Then,*

$$\sum_{\alpha \in A_q^{M+1}} \Delta^2(\alpha) \leq cMq^M.$$

Proof. Let $\beta = b_{k-1} \dots b_0 \in A_q^k$. Consider the function $f_\beta : A_q^k \to \{0,1\}$ defined by

$$f_\beta(u_{k-1}, \dots, u_0) = \begin{cases} 1 & \text{if } u_{k-1} \dots u_0 = \beta, \\ 0 & \text{otherwise.} \end{cases}$$

Let $M \in \mathbb{N}$, $M \geq 2k$. Let $\alpha = \varepsilon_M \dots \varepsilon_0$ run over elements of A_q^{M+1}. It is clear that

$$\begin{aligned} A &:= \sum_{\alpha \in A_q^{M+1}} F_\beta(\alpha) \\ &= \sum_{\nu=0}^{M+1-k} \#\{\alpha \in A_q^{M+1} : \varepsilon_{\nu+k-1} \dots \varepsilon_\nu = \beta\} \\ &= (M+1-k)q^{M+1-k}. \end{aligned} \tag{1}$$

On the other hand,

$$\begin{aligned} B &:= \sum_{\alpha \in A_q^{M+1}} F_\beta^2(\alpha) \\ &= \sum_{\nu_1=0}^{M+1-k} \sum_{\nu_2=0}^{M+1-k} \sum_{\varepsilon_0,\dots,\varepsilon_M} f_\beta(\varepsilon_{\nu_1+k-1}, \dots, \varepsilon_{\nu_1}) f_\beta(\varepsilon_{\nu_2+k-1}, \dots, \varepsilon_{\nu_2}) \\ &= A + 2 \sum_{\substack{\nu_1,\nu_2=0 \\ \nu_1<\nu_2}}^{M+1-k} \sum_{\varepsilon_0,\dots,\varepsilon_M} f_\beta(\varepsilon_{\nu_1+k-1}, \dots, \varepsilon_{\nu_1}) f_\beta(\varepsilon_{\nu_2+k-1}, \dots, \varepsilon_{\nu_2}) \\ &= A + 2 \sum_{\substack{\nu_1,\nu_2=0 \\ \nu_1<\nu_2\leq\nu_1+k}}^{M+1-k} \sum_{\varepsilon_0,\dots,\varepsilon_M} f_\beta(\varepsilon_{\nu_1+k-1}, \dots, \varepsilon_{\nu_1}) f_\beta(\varepsilon_{\nu_2+k-1}, \dots, \varepsilon_{\nu_2}) \\ &\quad + 2 \sum_{\substack{\nu_1,\nu_2=0 \\ \nu_2>\nu_1+k}}^{M+1-k} \sum_{\varepsilon_0,\dots,\varepsilon_M} f_\beta(\varepsilon_{\nu_1+k-1}, \dots, \varepsilon_{\nu_1}) f_\beta(\varepsilon_{\nu_2+k-1}, \dots, \varepsilon_{\nu_2}). \end{aligned} \tag{2}$$

Now, on the one hand we have

$$\sum_{\substack{\nu_1,\nu_2=0 \\ \nu_1<\nu_2\leq\nu_1+k}}^{M+1-k} \sum_{\varepsilon_0,\dots,\varepsilon_M} f_\beta(\varepsilon_{\nu_1+k-1}, \dots, \varepsilon_{\nu_1}) f_\beta(\varepsilon_{\nu_2+k-1}, \dots, \varepsilon_{\nu_2}) \leq ckMq^{M+1-k}, \tag{3}$$

while on the other hand,

$$\sum_{\substack{\nu_1,\nu_2=0\\ \nu_2>\nu_1+k}}^{M+1-k} \sum_{\varepsilon_0,\dots,\varepsilon_M} f_\beta(\varepsilon_{\nu_1+k-1},\dots,\varepsilon_{\nu_1}) f_\beta(\varepsilon_{\nu_2+k-1},\dots,\varepsilon_{\nu_2})$$
$$= \sum_{\substack{\nu_1,\nu_2=0\\ \nu_1+k<\nu_2}}^{M+1-k} q^{M+1-2k} = q^{M+1-2k}\left((M+1)^2 - O(kM)\right). \tag{4}$$

Combing (1) and (2), using estimates (3) and (4), we conclude that

$$\sum_{\alpha=\varepsilon_M\dots\varepsilon_0} \left(F_\beta(\alpha) - \frac{M+1}{q^k}\right)^2 \le cMq^M. \tag{5}$$

Note that here we summed over those $\varepsilon_M = 0$ as well. But (5) remains true if we drop those $\varepsilon_M = 0$. This allows us to conclude that

$$\sum_{\alpha\in A_q^{M+1}} \left(F_\beta(\alpha) - \frac{M+1}{q^k}\right)^2 \le cMq^M,$$

thus completing the proof of Lemma 5.

5 Proof of Theorem 2

Let

$$\xi_x = \mathrm{Concat}(\overline{\delta(p+1)} : p \le x).$$

Our goal is to prove that there exist two positive constants c_1 and c_2 such that

$$c_1 \le \frac{\lambda(\xi_x)}{\pi(x)x_3} \le c_2, \tag{6}$$

provided x is sufficiently large.

We first establish the size of $\lambda(\xi_x)$. We have

$$\lambda(\xi_x) = \sum_{\substack{p\le x\\ \delta(p+1)\ne 0}} \left\lfloor \frac{\log\delta(p+1)}{\log q} \right\rfloor + \pi(x) = \Sigma_1 + \Sigma_2 + \pi(x), \tag{7}$$

say, where the sum in Σ_1 runs over the primes $p \le x/x_2$, while that of Σ_2 runs over the primes located in the interval $J_x := (x/x_2, x]$. From this observation, it follows that

$$\Sigma_1 \le 2\pi(x/x_2)x_2 = O(\pi(x)). \tag{8}$$

On the other hand, it follows from Lemma 1 that, for each $u > 0$ there exists $c(u) > 0$ such that

$$\#\left\{p \le x : \frac{\delta(p+1)}{\sqrt{x_2}} > u\right\} > c(u)\pi(x).$$

From this, we may conclude that

$$\Sigma_2 \ge c\pi(x)x_3. \tag{9}$$

Combining (8) and (9) in (7), we obtain that, if $x > x_0$, the inequality $\frac{\lambda(\xi_x)}{\pi(x)x_3} > c$ holds for some positive constant c, thereby establishing the first inequality in (6).

Now, from the definitions of the functions δ and δ^*, it is clear that

$$|\delta^*(p+1) - \delta(p+1)| \le 1 \qquad \text{for all } p \in J_x.$$

From the trivial estimate $\delta(p+1) \le c\log x$, we obtain that $\log\delta(p+1) \le x_2 + c_1$, so that

$$\Sigma_2 \le c\pi(x)x_3 + \sum_{\frac{\delta^*(p+1)}{\sqrt{x_2}} > 4} (\log 2)\,\delta^*(p+1) = c\pi(x)x_3 + \Sigma_3, \tag{10}$$

say.

From Lemma 2, we obtain that for every $A \ge 1$, we have

$$\#\left\{p \in J_x : A < \frac{\delta^*(p+1)}{\sqrt{x_2}} < 2A\right\} \le \frac{c\pi(x)}{A^2}. \tag{11}$$

We now apply (11) successively with $A = 2^j$, $j = 2, 3, \ldots$, thus obtaining

$$\begin{aligned}
\Sigma_3 &\le c\pi(x)\sum_{j\ge 2}\frac{\log 2^{j+2}\sqrt{x_2}}{2^{j+2}} \\
&\le c\pi(x)\left[\frac{1}{2}x_3\sum_{j\ge 2}\frac{1}{2^{j+2}} + c\sum_{j\ge 2}\frac{j}{2^j}\right] \\
&\le c_1\pi(x)x_3,
\end{aligned}$$

from which we may conclude that, in light of (7), (8) and (10), the right hand side of (6) follows as well.

We will prove that, given any fixed integer $k \ge 1$, $\beta_1, \beta_2 \in A_q^k$, and setting $\Delta(\alpha) := F_{\beta_1}(\alpha) - F_{\beta_2}(\alpha)$ for each word $\alpha \in A_q^*$,

$$\lim_{x\to\infty}\frac{|\Delta(\xi_x)|}{\lambda(\xi_x)} = 0. \tag{12}$$

In order to achieve this, now that we know (from (6)) that the true size of $\lambda(\xi_x)$ is $\pi(x)x_3$, we essentially need to prove that $\Delta(\xi_x)$ is of smaller order than $\pi(x)x_3$.

Let θ_x be an arbitrary function which tends monotonically to 0 very slowly. Then consider the sets

$$\begin{aligned} D_1 &= \{p \in \wp : p \le x/x_2\}, \\ D_2 &= \{p \in \wp : p \le x \text{ and } \delta(p+1) \le \theta_x \sqrt{x_2}\}, \\ D_3 &= \{p \in \wp : p \le x \text{ and } \delta(p+1) > \frac{1}{\theta_x}\sqrt{x_2}\}, \end{aligned}$$

and let $D = D_1 \cup D_2 \cup D_3$.

Because $\Delta(\delta(p+1)) \le cx_3$ if $p \in D_1$ and $p \le cx_2$, and since (11) holds for $p \in D_3$, it follows from Lemma 1 and (8) that

$$\begin{aligned} \sum_{p\in D} |\Delta(\delta(p+1))| &\le cx_3\pi(x)\left(\Phi(\theta_x) - \Phi(-\theta_x)\right) + c\pi(x/x_2)x_2 \\ &\quad + \sum_{j=0}^{\infty} \#\left\{p \in J_x : \frac{\delta^*(p+1)}{\sqrt{x_2}} \in \left[\frac{2^j}{\theta_x}, \frac{2^{j+1}}{\theta_x}\right]\right\} \cdot \log\left(\sqrt{x_2}\cdot\frac{2^{j+1}}{\theta_x}\right) \end{aligned} \tag{13}$$

Since this last sum is less than

$$\pi(x)\sum_{j\ge 0}(x_3 + j + \log(1/\theta_x))\cdot\frac{\theta_x^2}{2^{2j}} \le c\left(\log(1/\theta_x) + x_3\right)\theta_x^2\pi(x),$$

it follows that (13) yields

$$\sum_{p\in D} |\Delta(\delta(p+1))| = o(\pi(x)x_3) \qquad (x\to\infty). \tag{14}$$

Now, from (14), we have that

$$\Delta(\xi_x) = \sum_{p\notin D}\Delta(\delta(p+1)) + o(\pi(x)x_3) = \Sigma_A + o(\pi(x)x_3), \tag{15}$$

say.

From Lemma 3, we obtain, using the fact that $p \notin D_3$ (since $p \notin D$), that

$$\sum_{\substack{p\notin D \\ P(p+1)\notin[x^\kappa, x^{1-\kappa}]}} |\Delta(\delta((p+1))| \le c\pi(x)\log\left(\frac{1}{\theta_x}\sqrt{x_2}\right) \le c\pi(x)x_3, \tag{16}$$

provided that θ_x is chosen so that $1/\theta_x < x_2$, say.

Now let $K = \lfloor x_2 \rfloor$ and then, for ℓ satisfying $\varepsilon_x\sqrt{K} \le |\ell| \le \frac{1}{\varepsilon_x}\sqrt{K}$, let

$$R_\kappa(\ell) := \#\{p \in J_x : P(p+1) \in (x^\kappa, x^{1-\kappa}) \text{ and } \omega(p+1) = K + \ell\}.$$

Using Lemma 4, we obtain that

$$\begin{aligned} R_\kappa(\ell) &\le \#\{p \in J_x : p+1 = aq,\ a < x^{1-\kappa},\ q > x^\kappa/x_2,\ \omega(a) = K+\ell-1\} \\ &\le \frac{x}{\kappa^2}\frac{x}{\log^2 x}\sum_{\omega(n)=K+\ell-1}\frac{1}{a}\prod_{\substack{p>2 \\ p\mid a}}\frac{p-1}{p-2} + O\left(x^{1-\kappa}\right), \end{aligned} \tag{17}$$

where the $O(\ldots)$ term accounts for the contribution of those q such that $q^2 \mid p+1$.

It follows from (17) that

$$\begin{aligned} R_\kappa(\ell) &\le \frac{c_1 x}{\kappa^2 \log^2 x} \left(\sum_{p \le x} \frac{1}{p} + c \right)^{K+\ell-1} \frac{1}{(K+\ell-1)!} + O\left(x^{1-\kappa}\right) \\ &\le \frac{c_2 x}{\kappa^2 \log^2 x} \frac{(K+c)^{K+\ell-1}}{(K+\ell-1)!}. \end{aligned} \tag{18}$$

Now, observe that, if $\omega(p+1) = K+\ell$, $p \in J_x$, then $\delta(p+1) \in \{|\ell|-1, |\ell|, |\ell|+1\}$. Thus,

$$\begin{aligned} |\Sigma_A| \le & \sum_{\varepsilon_x \sqrt{K} \le |\ell| \le \frac{1}{\varepsilon_x}\sqrt{K}} \left(\Delta(|\bar{\ell}|) + \Delta(|\bar{\ell}|-1) + \Delta(|\bar{\ell}|+1)\right) \cdot \left(R_\kappa(-\ell) + R_\kappa(\ell)\right) \\ & \qquad + c\kappa\pi(x)x_3 \\ = & \ \Sigma_B + c\kappa\pi(x)x_3, \end{aligned} \tag{19}$$

say.

Using (18), we obtain that

$$\Sigma_B \le \frac{c_2 x}{\kappa^2 \log^2 x} \sum_{\varepsilon_x \le \frac{\ell}{\sqrt{K}} \le \frac{1}{\varepsilon_x}} \left(\Delta(\bar{l}) + \Delta(\bar{l}-1) + \Delta(\bar{l}+1)\right) \cdot \left(\frac{(K+c)^{K+\ell-1}}{(K+\ell-1)!} + \frac{(K+c)^{K-\ell-1}}{(K-\ell-1)!} \right). \tag{20}$$

Since we can easily establish that

$$\max_{0 \le \ell \le \frac{1}{\varepsilon_x \sqrt{K}}} \left(\frac{(K+c)^{K+\ell-1}}{(K+\ell-1)!} + \frac{(K+c)^{K-\ell-1}}{(K-\ell-1)!} \right) < \frac{(K+c)^{K-1}}{(K-1)!} \exp\left\{ c_3 \left(\frac{1}{\varepsilon_x} \right)^2 \right\},$$

it follows from (20) that

$$\Sigma_B \le \frac{c_2 x}{\kappa^2 \log^2 x} \exp\left\{ c_3 \left(\frac{1}{\varepsilon_x} \right)^2 \right\} \frac{(K+c)^{K-1}}{(K-1)!} \Sigma_C, \tag{21}$$

where

$$\begin{aligned} \Sigma_C &= \sum_{\varepsilon_x \le \frac{\ell}{\sqrt{K}} \le \frac{1}{\varepsilon_x}} \left(\Delta(\bar{\ell}) + \Delta(\bar{\ell}-1) + \Delta(\bar{\ell}+1)\right) \\ &\le 3 \sum_{\varepsilon_x \le \frac{\ell}{\sqrt{K}} \le \frac{1}{\varepsilon_x}} \Delta(\bar{\ell}) + O(x_3) = 3\Sigma_D + O(x_3), \end{aligned} \tag{22}$$

say.

To estimate Σ_D, we will use Lemma 5. Indeed, let M_0 be the largest integer for which $q^{M_0} \le \varepsilon_x \sqrt{K}$ and let M_1 be the smallest integer for which $q^{M_1} > \frac{1}{\varepsilon_x}\sqrt{K}$. Set $\mathscr{K}_M = [q^M, q^{M+1}-1]$. With this set up, we clearly have that

$$\Sigma_D \le \sum_{M_0 \le M \le M_1} T_M, \tag{23}$$

where $T_M = \sum_{\ell \in \mathscr{K}_M} \Delta(\ell)$. Now, it follows from Lemma 5 that

$$T_M \le c(q^{M+1})^{1/2}(Mq^{M-k})^{1/2} \le c\sqrt{M}q^M. \tag{24}$$

Using (24) in (23), we obtain that

$$\Sigma_D \le c\sqrt{M_1}q^{M_1}\left(1+\frac{1}{q}+\frac{1}{q^2}+\cdots\right) < \frac{c_1}{\varepsilon_x}\sqrt{K}\sqrt{\log K} < \frac{c_1 x_2^{1/2}\sqrt{x_3}}{\varepsilon_x}. \tag{25}$$

Gathering (21), (22) and (25), we have that

$$\Sigma_B \le \frac{cx}{\kappa^2 \log^2 x}\exp\left\{c_3\left(\frac{1}{\varepsilon_x}\right)^2\right\}\frac{(K+c)^{K-1}}{(K-1)!}\cdot\frac{\sqrt{x_2}\sqrt{x_3}}{\varepsilon_x}. \tag{26}$$

Since it is well known that $n! = n^n e^{-n}\sqrt{2\pi n}(1+O(1/n))$, it follows that by setting $\ell_K = \dfrac{(K+c)^{K-1}}{(K-1)!}$,

$$\begin{aligned}\log \ell_K &= (K-1)\log(K+c) - (K-1)\log\left(\frac{K-1}{e}\right) - \frac{1}{2}\log K + O(1)\\ &= (K-1)\log\frac{K+c}{K-1} - \frac{1}{2}\log K + O(1) + K - 1,\end{aligned}$$

from which it follows that

$$\ell_K \sim \frac{x_1}{\sqrt{x_2}}.$$

Using this last estimate in (26), it follows that

$$\Sigma_B \le \frac{\exp\left\{c_3/\varepsilon_x^2\right\}}{\kappa^2\varepsilon_x}\pi(x)x_3. \tag{27}$$

Choosing $\varepsilon_x = x_5$, say, we get from (27) that

$$\limsup_{x\to\infty}\frac{\Sigma_B}{\lambda(\xi_x)} = 0. \tag{28}$$

Combining (28), (19) and (15), we obtain that

$$\limsup_{x\to\infty}\frac{\Delta(\xi_x)}{\lambda(\xi_x)} \le c\kappa. \tag{29}$$

Since κ can be taken arbitrarily small, we may finally conclude that (12) holds, thus completing the proof of Theorem 1.

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