On the proximity of additive and multiplicative functions

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Abstract

Given an additive function f and a multiplicative function g, let $E(f,g;x) = \#\{n \le x : f(n) = g(n)\}$. We study the size of $E(f,g;x)$ for functions f such that $f(n) \ne 0$ for at least one integer $n > 1$. In particular, we show that for those additive functions f whose values $f(n)$ are concentrated around their mean value $\lambda(n)$, one can find a multiplicative function g such that, given any $\varepsilon > 0$, then $E(f,g;x) \gg x/\lambda(x)^{1+\varepsilon}$. We also show that given any additive function satisfying certain regularity conditions, no multiplicative function can coincide with it on a set of positive density. It follows that if $\omega(n)$ stands for the number of distinct prime factors of n, then, given any $\varepsilon > 0$, there exists a multiplicative function g such that $E(\omega,g;x) \gg x/(\log\log x)^{1+\varepsilon}$, while for all multiplicative functions g, we have $E(\omega,g;x) = o(x)$ as $x \to \infty$.

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1 Introduction

We investigate the deceptively simple question of whether the behavior of a multiplicative function can resemble that of an additive function. More precisely, letting $\mathbb{N}$ stand for the set of all positive integers, we ask the following question: can one find an additive function f and a multiplicative function g such that $f(n) = g(n)$ for all $n \in R$?

For convenience, let us write $\mathcal{A}$ for the set of all additive functions f such that $f(1) = 0$ and $\mathcal{M}$ for the set of all multiplicative functions g such that $g(1) = 1$. From here on, we only consider real arithmetic functions. Given $f \in \mathcal{A}$ and $g \in \mathcal{M}$, we define

$$E(f,g;x) := \#\{n \le x : f(n) = g(n)\}.$$

A *strongly additive function* (resp. *strongly multiplicative function*) h is a function in $\mathcal{A}$ (resp. in $\mathcal{M}$) such that $h(p^a) = h(p)$ for all integers $a \ge 1$ and all primes p. We shall write $\mathcal{A}^*$ (resp. $\mathcal{M}^*$) for the set of strongly additive functions (resp. strongly multiplicative functions). Given an integer $n \ge 2$, we let $\omega(n)$ stand for the number of distinct prime factors of n, and set $\omega(1) = 0$.

Of course, we are not interested in the trivial case $f(n) = 0$ for all $n \geq 1$ and $g(n) = 0$ for all $n \geq 2$, since in this case we simply get $E(f, g; x) = \lfloor x \rfloor - 1$. We are instead interested in studying the size of $E(f, g; x)$ when the functions $f \in \mathcal{A}$ and $g \in \mathcal{M}$ are such that $f(n) \neq 0$ for at least one integer $n > 1$, in which case one can prove that they differ infinitely often.

Here we show that for those additive functions f whose values $f(n)$ are concentrated around their mean value $\lambda(n)$ in the sense that, given any $\varepsilon > 0$,

$$\lim_{x \to \infty} \frac{1}{x} \#\{n < x : |f(n) - \lambda(x)| > \lambda(x)^{1-\varepsilon}| = 0,$$

then, one can find a multiplicative function g such that

$$E(f, g; x) \gg \frac{x}{\lambda(x)^{1+\varepsilon}}.$$

We also show that given an additive function f satisfying certain regularity conditions, no function $g \in \mathcal{M}$ can coincide with f on a set of positive density.

In the case where the additive function f is chosen to be ω, it follows that, given any $\varepsilon > 0$, there exists a multiplicative function g such that

$$E(\omega, g; x) \gg \frac{x}{(\log \log x)^{1+\varepsilon}},$$

while for all $g \in \mathcal{M}$ we have

$$E(\omega, g; x) = o(x) \qquad (x \to \infty).$$

2 Notation and preliminary results

Before stating our results, we introduce additional notation and state preliminary results.

We shall write $\wp$ for the set of all prime numbers, while the letter p will always stand for a prime number. The letter c, with or without subscript, always denotes a constant, but not necessarily the same at each occurrence. By $\log_2 x$, we mean $\max(1, \log \log x)$; by $\log_3 x$, we mean $\max(1, \log \log \log x)$; and so on. We will use $P^+(n)$ for the largest prime factor of $n \geq 2$ and $P^-(n)$ for the smallest prime factor of $n \geq 2$. For convenience, we set $P^+(1) = P^-(1) = 1$. We also define as is customary, for $2 \leq y \leq x$,

$$\Psi(x, y) := \#\{n \leq x : P^+(n) \leq y\} \quad \text{and} \quad \Phi(x, y) := \#\{n \leq x : P^-(n) > y\}.$$

Lemma 1. *Let $D \geq 3$ be a fixed integer and let $0 < a_1 < a_2 < 1$ with $a_2 > 2a_1$. Then,*

$$S_D(x) := \sum_{\substack{a_1 x \leq n \leq a_2 x \\ P^-(n) > D}} \frac{1}{n} \ll \frac{\log(a_2/a_1)}{\log D}. \tag{2.1}$$

Proof. Using the Stieltjes integral representation of the sum in (2.1), we obtain that

$$S_D(x) = \int_{a_1x}^{a_2x} \frac{1}{t} d\Phi(t,D) = \frac{\Phi(t,D)}{t}\bigg|_{a_1x}^{a_2x} + \int_{a_1x}^{a_2x} \frac{\Phi(t,D)}{t^2} dt. \tag{2.2}$$

Setting $u = \log t/\log D$ and using the estimate

$$\Phi(t,D) = \frac{t\omega_0(u)}{\log D} - \frac{D}{\log D} + O\left(\frac{t}{\log^2 D}\right),$$

where $\omega_0(u)$ stands for the Buchstab function (see Theorem 6.4 in the book of Tenenbaum [7]) and recalling that $\frac{1}{2} \le \omega_0(u) \le 1$ for all $u \ge 1$, it follows from (2.2) that

$$\begin{aligned} S_D(x) &= \frac{\omega_0(u)}{\log D}\bigg|_{a_1x}^{a_2x} + O\left(\frac{1}{\log^2 D}\right) + \int_{a_1x}^{a_2x} \left(\frac{\omega_0(u)}{t\log D} + O\left(\frac{1}{t\log^2 D}\right)\right) dt \\ &= O\left(\frac{1}{\log D}\right) + O\left(\frac{1}{\log D}\int_{a_1x}^{a_2x}\frac{dt}{t}\right) \\ &\ll \frac{\log(a_2/a_1)}{\log D}, \end{aligned}$$

thus establishing our claim. □

Lemma 2. *There exists a positive constant* c_3 *such that*

$$\Psi(x,y) \le c_3 x \exp\left\{-\frac{1}{2}\frac{\log x}{\log y}\right\} \qquad (2 \le y \le x).$$

Proof. For a proof, see Theorem 5.1 in the book of Tenenbaum [7]. □

Lemma 3. *Uniformly for* $x \ge 3$ *and* $0 \le \xi(x) \le \sqrt{\log\log x}$,

$$\#\{n \le x : |\omega(n) - \log\log x| > \xi(x)\sqrt{\log\log x}\} \ll x e^{-\xi(x)^2/3}.$$

Proof. This result follows immediately from Theorem 3.8 in the book of Tenenbaum [7]. □

Lemma 4. *For each positive integer* k*, let* $\pi_k(x) := \#\{n \le x : \omega(n) = k\}$*. Then the maximum value of* $\pi_k(x)$ *is attained when* $k = k_0 = \log\log x + O(1)$*, in which case we have* $\pi_{k_0}(x) = (1+o(1))x/\sqrt{\log\log x}$ *as* $x \to \infty$.

Proof. This follows from a result of Balazard [1]. □

Lemma 5. *Let* f *be a complex-valued additive function and set*

$$A(x) = A_f(x) := \sum_{p^\alpha \le x} \frac{f(p^\alpha)}{p^\alpha}\left(1 - \frac{1}{p}\right) \quad \text{and} \quad B(x)^2 = B_f(x)^2 := \sum_{p^\alpha \le x} \frac{|f(p^\alpha)|^2}{p^\alpha}. \tag{2.3}$$

Then there exists an absolute constant $C>0$ such that for all $x\geq 2$, we have

$$\frac{1}{x}\sum_{n\leq x}|f(n)-A(x)|^2\leq CB(x)^2,$$

where the constant C can be replaced by $\dfrac{3}{2}+O\left(\dfrac{1}{\sqrt{\log x}}\right)$.

Proof. See Kubilius [5] and Hildebrand [4]. □

To each real number $D>1$ we associate the multiplicative function

$$G_D(n)=\prod_{\substack{p^\alpha\| n\\ p\leq D}}p^\alpha.$$

Lemma 6. *Let $\varepsilon>0$ be a small number and let D be a large number satisfying*

$$D>e^{12/\varepsilon}. \tag{2.4}$$

Then, there exist two positive constants $c_1=c_1(\varepsilon)$ and $c_2=c_2(\varepsilon)$, with $c_1<1<c_2$, and a real number $x_0=x_0(\varepsilon)$ such that

$$\#\{n\leq x: D^{c_1}<G_D(n)<D^{c_2}\}\geq(1-\varepsilon)x \qquad (x\geq x_0). \tag{2.5}$$

Proof. In order to prove (2.5), we will prove that for some large $x_1=x_1(\varepsilon)$,

$$\#\{n\leq x: G_D(n)\leq D^{c_1}\}\leq\frac{\varepsilon}{2}x \qquad (x\geq x_1) \tag{2.6}$$

and that

$$\#\{n\leq x: G_D(n)\geq D^{c_2}\}\leq\frac{\varepsilon}{2}x \qquad (x\geq 1). \tag{2.7}$$

The result will then follow by choosing $x_0=x_1$.

First recall that, for some positive constant c_4,

$$\sum_{p\leq x}\frac{1}{p}=\log\log x+c_4+\frac{\theta_x}{\log^2 x} \qquad (x\geq 286), \tag{2.8}$$

where $|\theta_x|\leq 1/2$, an estimate due to Rosser and Schoenfeld [6].

With $c_1<1$, it follows from (2.8) that there exists $x_1=x_1(\varepsilon)\geq 286$ such that for all $x\geq x_1$,

$$\begin{aligned}\#\{n\leq x: G_D(n)\leq D^{c_1}\} &\leq \#\{n\leq x:\ \text{there exists no prime } p\mid n \text{ with } D^{c_1}<p\leq D\}\\ &\leq \frac{3}{2}x\prod_{D^{c_1}<p\leq D}\left(1-\frac{1}{p}\right)=\frac{3}{2}x\exp\sum_{D^{c_1}<p\leq D}\log\left(1-\frac{1}{p}\right)\end{aligned}$$

$$
\begin{aligned}
&< \frac{3}{2}x\exp\left(-\sum_{D^{c_1}<p\le D}\frac{1}{p}+\frac{1}{2D^{c_1}}\right)\\
&\le \frac{3}{2}x\exp\left(\log\log D^{c_1}-\log\log D+\frac{1}{2D^{c_1}}+\frac{1}{2\log^2 D^{c_1}}\right)\\
&= \frac{3}{2}xc_1\exp\left(\frac{1}{2D^{c_1}}+\frac{1}{\log^2 D^{c_1}}\right)\\
&< \frac{3}{2}xc_1 2 = 3c_1x, \qquad (2.9)
\end{aligned}
$$

where we used (2.4). Choosing $c_1=\varepsilon/6$, then (2.9) yields (2.6).

On the other hand, in order to prove (2.7), first apply Lemma 5 to the additive function $f(n)=\log G_D(n)$ for which the corresponding sums $A(x)$ and $B(x)$ satisfy

$$
A(x)=\log D+O(1) \qquad \text{and} \qquad B^2(x)=\frac{\log^2 D}{2}+O(1),
$$

from which it follows that, for all $x\ge 1$,

$$
\begin{aligned}
\#\{n\le x: G_D(n)\ge D^{c_2}\} &= \#\{n\le x: f(n)\ge c_2\log D\}\\
&\le \#\{n\le x: |f(n)-\log D|^2\ge (c_2-1)^2\log^2 D\}\\
&\le \sum_{n\le x}\frac{|f(n)-\log D|^2}{(c_2-1)^2\log^2 D}\\
&\le \frac{1}{(c_2-1)^2\log^2 D}Cx\frac{\log^2 D}{2}\\
&< \frac{\varepsilon}{2}x,
\end{aligned}
$$

provided we choose $c_2>\sqrt{\dfrac{C}{\varepsilon}}+1$, thus establishing (2.7). □

3 Lower bounds

It is easy to show that for any $f\in\mathcal{A}$, there exists $g\in\mathcal{M}$ for which $E(f,g;x)>x/\log x$ for all $x\ge 11$. Indeed, since any additive or multiplicative function is entirely determined by its values at prime powers, we only need to choose $g(p^\alpha)=f(p^\alpha)$ for all $p\in\wp$ and $\alpha\in\mathbb{N}$. Then, letting R stand for the set of all prime powers, it is clear that $g(n)=f(n)$ for all $n\in R$, whose counting function $R(x)$ satisfies $R(x)>x/\log x$ for all $x\ge 11$.

We now turn to a general question: Given $f\in\mathcal{A}$, can one find $g\in\mathcal{M}$ such that $E(f,g;x)$ is maximal? The answer is of course highly dependant on the particular function f considered and we will show that the size of $E(x)$ can range from $E(x)=0$ (simply choose $f\in\mathcal{A}^*$ defined by $f(p)=-1$

for all $p \in \wp$ and $g \in \mathcal{M}^*$ defined by $g(p) = 1$ for all $p \in \wp$) to $E(x) \geq cx$ for any positive constant $c < 1$.

To show that for any given positive constant $c < 1$, there exist a function $f \in \mathcal{A}$ and a corresponding function $g \in \mathcal{M}$ which yields $E(f, g; x) \geq cx$, as $x \to \infty$, we first let $\mathcal{S} \subset \wp$ be a set satisfying

$$\sum_{p \in \mathcal{S}} \frac{1}{p} < \infty \tag{3.1}$$

and define $f \in \mathcal{A}^*$ by

$$f(p) = \begin{cases} 1 & \text{if} \quad p \in \mathcal{S}, \\ 0 & \text{if} \quad p \in \wp \setminus \mathcal{S}. \end{cases}$$

It is easy to see that $f(n)$ will be zero on a subset of $\mathbb{N}$ of positive density $c = \prod_{p \in \mathcal{S}} \left(1 - \frac{1}{p}\right)$. It follows that the strongly multiplicative function g defined by $f(p) = g(p)$ on all primes p will give $E(f, g; x) \geq cx$. It is also straightforward, given the construction of $\mathcal{S}$, that the constant c can be chosen arbitrarily close to 1.

More generally, if f is a strongly additive function such that $f(p) \neq 0$ on a set of primes $\mathcal{S}$ satisfying (3.1), then one can construct a strongly multiplicative function g such that $E(f, g; x) \geq c_1 x$ for some positive constant c_1 depending on f. One way to construct this corresponding function g is to choose

$$g(p) = \begin{cases} f(p) & \text{if} \quad p \in \mathcal{S}, \\ 1 & \text{if} \quad p \in \wp \setminus \mathcal{S}. \end{cases}$$

This will give $g(n) = f(n) = f(p)$ for a certain $p \in \mathcal{S}$, that is provided that

$$\prod_{\substack{q \mid n \\ q \in \mathcal{S}}} q = p,$$

and thus

$$E(f, g; x) \gtrsim \sum_{p \in \mathcal{S}} \sum_{a \geq 1} \frac{x}{p^a} \prod_{p \in \mathcal{S}} \left(1 - \frac{1}{p}\right) = c \cdot x \sum_{p \in \mathcal{S}} \sum_{a \geq 1} \frac{1}{p^a} =: c_1 x.$$

In some instances, it is easy to show that this strategy is optimal. Indeed, consider the additive function f defined by $f(2) = b$, where b is any fixed real number, and $f(p) = 0$ for all primes $p \geq 3$, and take the strongly multiplicative function g as $g(2) = b$ and $g(p) = 1$ for all primes $p \geq 3$. Then, since $f(n)$ and $g(n)$ agree at even integers, we easily see that $E(f, g; x) = \lfloor x/2 \rfloor$. But we cannot hope to do any better. Indeed, for an arbitrary multiplicative function g, let κ be the proportion of those odd integers m for which $g(m) = f(m) = 0$. Then, since $g(2^a m) \neq f(2^a m)$ for all $a \in \mathbb{N}$, the proportion of even integers for which g agrees with f is at most $1 - \kappa$. It follows from this observation that

$$E(f, g; x) \lesssim x \left(\frac{1}{2}\kappa + \frac{1}{2}(1 - \kappa)\right) = \frac{x}{2}.$$

If one considers a more common additive function, say $f = \omega$, then determining the maximal size of $E(\omega, g; x)$ for an optimal g is much more difficult. A natural way to proceed is the following. Fix an arbitrary integer $k \geq 2$. One can prove that for $f = \omega \in \mathcal{A}$, there exists $g \in \mathcal{M}$ such that, for some positive constant $C > 0$,

$$E(\omega, g; x) > C \frac{x}{\log x} \cdot \frac{(\log\log x)^{k-1}}{(k-1)!} \qquad \text{(for } x \text{ sufficiently large).} \tag{3.2}$$

Indeed, consider the strongly multiplicative function g defined on primes p by $g(p) = k^{1/k}$. Then, clearly, for those integers n such that $\omega(n) = k$, we have $g(n) = (k^{1/k})^{\omega(n)} = k$.

It remains to count the number of integers $n \leq x$ such that $\omega(n) = k$. But this is well known. Indeed, letting $\pi_r(x)$ stand for the number of positive integers $n \leq x$ such that $\omega(n) = r$, it is known since Landau (see Theorem 10.3 in De Koninck and Luca [3]) that, for each fixed $r \in \mathbb{N}$,

$$\pi_r(x) = (1 + o(1)) \frac{x}{\log x} \cdot \frac{(\log\log x)^{r-1}}{(r-1)!} \qquad (x \to \infty).$$

It is also known that $\pi(x) > x/\log x$ for all $x \geq 11$ (see for instance Rosser and Schoenfeld [6]). Using this, one can easily prove by induction that for each integer $r \geq 2$, there exists a positive constant c_r and a number x_r such that

$$\pi_r(x) > c_r \frac{x}{\log x} \cdot \frac{(\log\log x)^{r-1}}{(r-1)!} \qquad \text{(for all } x > x_r\text{).} \tag{3.3}$$

Hence, choosing $r = k$ in (3.3), estimate (3.2) follows immediately.

Does this provide an optimal result ? Can we construct a multiplicative function g which agrees with ω on a larger set ? Let us first consider a multiplicative function g which has "natural" properties. For example, since the mean and normal orders of $\omega(n)$ are $\log\log n$, it would seem natural to look for a multiplicative function $g(n)$ which also has mean and normal orders of $\log\log n$. But this is impossible. Indeed, Birch [2] has proven that the only unbounded multiplicative functions $g(n)$ with a non-decreasing normal order are the powers of n. Let us therefore consider a seemingly unnatural multiplicative function g that agrees with ω on a large set of values. One key idea to construct such a tricky function is to choose g such that $g(p) \neq 1$ only if $p \in \mathcal{S}$, where $\mathcal{S}$ is an infinite set of primes satisfying

$$\sum_{p \in \mathcal{S}} \frac{1}{p} < \infty.$$

Given an integer $n \geq 2$, we shall write

$$n = \prod_{\substack{p^a \| n \\ p \in \mathcal{S}}} p^a \cdot \prod_{\substack{p^a \| n \\ p \notin \mathcal{S}}} p^a = s \cdot m,$$

say. This means that the value of $g(n)$ will be determined by s while the value of $\omega(n)$ will essentially be determined by $\omega(m)$. Thus, in order for $g(n)$ to be equal to $\omega(n)$ for many $n \leq x$, we will arrange for $g(s)$ to be close to $\log\log x$ as often as possible. This idea is expressed in the following theorem, where we consider a more general strongly additive function f.

Theorem 1. *Let $\mathcal{S} := \{s_1, s_2, \ldots\}$ be an infinite set of primes such that $s_1 < s_2 < \cdots$ and satisfying $\sum_{j=1}^{\infty} \frac{1}{s_j} < \infty$. Assume that f is a non negative integer valued strongly additive function and $\lambda(x)$ a function which tends to infinity with x such that*

$$\#\left\{n \leq \frac{x}{s_{\lceil \lambda(x) \rceil}} : f(n) > \lambda(x)\right\} = o\left(\frac{x}{s_{\lceil \lambda(x) \rceil}}\right) \qquad (x \to \infty). \tag{3.4}$$

Then, letting g be the strongly multiplicative function defined on the primes p by

$$g(p) = \begin{cases} j + f(s_j) & \text{if } p = s_j \text{ for some } s_j \in \mathcal{S}, \\ 1 & \text{if } p \notin \mathcal{S}, \end{cases}$$

we have

$$E(f, g; x) \gg \frac{x}{s_{\lceil \lambda(x) \rceil}}. \tag{3.5}$$

Remark 1. *Clearly Theorem 1 implies in particular that, given any $\varepsilon > 0$,*

$$E(f, g; x) \gg \frac{x}{\lambda(x)^{1+\varepsilon}}.$$

Proof of Theorem 1. In order to find a lower bound for $E(f, g; x) = \#\{n \leq x : f(n) = g(n)\}$, we will only count those positive integers $n \leq x$ which can be written as $n = m \cdot s_j$ with an m such that $s_k \nmid m$ for all integers $k \geq 1$. Hence,

$$E(f, g; x) \geq \sum_{j \geq 1} \#\{n = m \cdot s_j \leq x : s_k \nmid m \ \forall\, k,\ f(n) = g(n)\}. \tag{3.6}$$

Given that $g(n) = g(m \cdot s_j) = g(s_j) = j + f(s_j)$ and that $f(n) = f(m \cdot s_j) = f(m) + f(s_j)$, the condition $f(n) = g(n)$ is equivalent to $f(m) = j$. Inequality (3.6) can therefore be replaced by

$$E(f, g; x) \geq \sum_{j \geq 1} \#\{n = m \cdot s_j \leq x : s_k \nmid m \ \forall\, k,\ f(m) = j\},$$

which is equivalent to

$$E(f, g; x) \geq \sum_{j \geq 1} \#\left\{m \leq \frac{x}{s_j} : s_k \nmid m \ \forall\, k,\ f(m) = j\right\}. \tag{3.7}$$

It follows from (3.7) that

$$
\begin{aligned}
E(f,g;x) &\geq \sum_{j\leq\lambda(x)} \#\left\{m \leq \frac{x}{s_j} : s_k \nmid m\ \forall\, k, f(m)=j\right\} \\
&\geq \sum_{j\leq\lambda(x)} \#\left\{m \leq \frac{x}{s_{\lceil\lambda(x)\rceil}} : s_k \nmid m\ \forall\, k, f(m)=j\right\} \\
&\geq \#\left\{m \leq \frac{x}{s_{\lceil\lambda(x)\rceil}} : s_k \nmid m\ \forall\, k\right\} \\
&\qquad + O\left(\#\left\{m \leq \frac{x}{s_{\lceil\lambda(x)\rceil}} : f(m) > \lambda(x)\right\}\right).
\end{aligned} \tag{3.8}
$$

Setting $C(\mathcal{S}) := \prod_{j=1}^{\infty}\left(1-\frac{1}{s_j}\right)$, which is positive because $\sum_{j=1}^{\infty}\frac{1}{s_j} < \infty$, we obtain, in light of inequality (3.8) and hypothesis (3.4),

$$E(f,g;x) \gg C(\mathcal{S})\frac{x}{s_{\lceil\lambda(x)\rceil}} + o\left(\frac{x}{s_{\lceil\lambda(x)\rceil}}\right),$$

from which conclusion (3.5) follows immediately. □

A direct application of this theorem is the following.

Corollary 1. *Given an arbitrary integer $k \geq 1$ and any real number $\delta > 0$, there exists a multiplicative function g such that*

$$E(\omega,g;x) \gg \frac{x}{\left(\prod_{r=1}^{k}\log_{r+1} x\right)(\log_{k+2} x)^{1+\delta}}. \tag{3.9}$$

In particular, given any $\varepsilon > 0$, there exists a multiplicative function g such that

$$E(\omega,g;x) \gg \frac{x}{(\log\log x)^{1+\varepsilon}}. \tag{3.10}$$

Proof. In Theorem 1, take $f = \omega$. Then, for any positive integer k and fixed $\delta > 0$, let $\mathcal{S} := \{s_1, s_2, \ldots\}$ be an infinite set of primes defined as follows: first set $s_1 = 2$ and, for each $j \geq 2$, let s_j be the smallest prime number larger than

$$\max\left(s_{j-1}, j\left(\prod_{r=1}^{k}\log_r j\right)(\log_{k+1} j)^{1+\delta}\right).$$

Then, choosing $\lambda(x) = \log\log x + \xi(x)\sqrt{\log\log x}$, where $\xi(x)$ is any function tending to infinity with x, we immediately obtain (3.9).

Finally, (3.10) is an easy consequence of (3.9). □

Can we improve the lower bound obtained in (3.10) ? While an optimal answer to this question is difficult to provide, the next theorem and the remark that follows provide a partial answer.

Theorem 2. *Given any small number $\varepsilon > 0$, there exists a multiplicative function g and an infinite sequence of integers r such that*

$$E(\omega, g; r) \gg \frac{r}{(\log\log r)^{1/2+\varepsilon}}.$$

Proof. Let $\delta > 0$ be a fixed small number. Let $s_1 = 2$, and for each integer $j \geq 2$, let s_j be the smallest prime number larger than $\max(s_{j-1}, j^{1+\delta})$ and set $\mathcal{S} := \{s_1, s_2, \ldots\}$. Define the sequence of positive real numbers $r_1, r_2, \ldots$ by $r_j = e^{e^{2^j}}$. Further consider the sequence of positive integers (z_j), where each z_j is the integer maximizing the quantity

$$\#\left\{m \leq \frac{r_j}{s_j} : s_k \nmid m \text{ for each } s_k \in \mathcal{S},\ \omega(m) = z_j - 1\right\}.$$

Consider the strongly multiplicative function g defined on the primes p by

$$g(p) = \begin{cases} z_j & \text{if} \quad p = s_j \text{ for some } s_j \in \mathcal{S}, \\ 1 & \text{if} \quad p \notin \mathcal{S}. \end{cases}$$

Setting $I_j := [\log\log r_j - (\log\log r_j)^{1/2+\varepsilon}, \log\log r_j + (\log\log r_j)^{1/2+\varepsilon}]$, we have

$$\#\left\{m \leq \frac{r_j}{s_j} : \omega(m) \notin I_j\right\} = o\left(\frac{r_j}{s_j}\right) \qquad (j \to \infty), \tag{3.11}$$

which allows us to write that, as $j \to \infty$,

$$\begin{aligned} &\#\left\{m \leq \frac{r_j}{s_j} : s_k \nmid m \text{ for each } s_k \in \mathcal{S}\right\} \\ &= \sum_{\nu \in I_j} \#\left\{m \leq \frac{r_j}{s_j} : s_k \nmid m \text{ for each } s_k \in \mathcal{S}, \omega(m) = \nu\right\} + o\left(\frac{r_j}{s_j}\right). \end{aligned} \tag{3.12}$$

Since by the very nature of the sequence (z_j), we have that

$$\begin{aligned} &\sum_{\nu \in I_j} \#\left\{m \leq \frac{r_j}{s_j} : s_k \nmid m \text{ for each } s_k \in \mathcal{S},\ \omega(m) = \nu\right\} \\ &\qquad \leq 2(\log\log r_j)^{1/2+\varepsilon} \#\left\{m \leq \frac{r_j}{s_j} : s_k \nmid m, \text{ for each } s_k \in \mathcal{S}, \omega(m) = z_j - 1\right\}, \end{aligned}$$

it follows from (3.12), using (3.11), that

$$\#\left\{m \leq \frac{r_j}{s_j} : s_k \nmid m \text{ for each } s_k \in \mathcal{S}, \omega(m) = z_j - 1\right\}$$

$$\geq \frac{\#\{m \leq \frac{r_j}{s_j} : s_k \nmid m \text{ for each } s_k \in \mathcal{S}\} + o(\frac{r_j}{s_j})}{2(\log\log r_j)^{1/2+\varepsilon}}. \tag{3.13}$$

On the other hand, letting $C(\delta) := \prod_{j=1}^{\infty}\left(1 - \frac{1}{s_j}\right)$, it is easy to see that

(3.14)

$$\#\left\{m \leq \frac{r_j}{s_j} : s_k \nmid m \text{ for each } s_k \in \mathcal{S}\right\} = (1+o(1))C(\delta)\frac{r_j}{s_j} \qquad (j \to \infty).$$

We thus obtain, combining (3.13) and (3.14), that as $j \to \infty$,

$$\begin{aligned}\#&\left\{m \leq \frac{r_j}{s_j} : s_k \nmid m \text{ for each } s_k \in \mathcal{S},\, \omega(m) = z_j - 1\right\}\\ &\geq \left(\frac{1}{2} + o(1)\right) C(\delta)\frac{r_j}{s_j(\log\log r_j)^{1/2+\varepsilon}}.\end{aligned} \tag{3.15}$$

Now, it is clear that, writing each integer $n \leq r_j$ as $n = s_j \cdot m$, we have

$$\begin{aligned} E(\omega, g; r_j) &\geq \#\{n \leq r_j : g(n) = z_j,\ \omega(n) = z_j\}\\ &\geq \#\{n \leq r_j : s_j | n,\ s_k \nmid n \text{ for } k \neq j,\ \omega(n) = z_j\}\\ &\geq \#\left\{m \leq \frac{r_j}{s_j} : s_k \nmid m \text{ for each } s_k \in \mathcal{S},\, \omega(m) = z_j - 1\right\},\end{aligned}$$

which implies, in light of (3.15), that as $j \to \infty$,

$$E(\omega, g; r_j) \geq \left(\frac{1}{2} + o(1)\right) C(\delta)\frac{r_j}{s_j(\log\log r_j)^{1/2+\varepsilon}}. \tag{3.16}$$

Now, using the fact that, for j sufficiently large, we have

$$s_j < j^{1+2\delta} < (2^j)^{\varepsilon} = (\log\log r_j)^{\varepsilon},$$

it follows from (3.16) that

$$E(\omega, g; r_j) \gg \frac{r_j}{(\log\log r_j)^{1/2+2\varepsilon}},$$

thus completing the proof of Theorem 2. □

Remark 2. *The lower bound of Theorem 2 is better than the one in Corollary 1 but only over a thin subset of the integers, while Theorem 1 provides a stronger uniform bound.*

The next theorem states the limitation of the strategy used to construct the "optimal" function g of Theorem 1.

Theorem 3. *Let g be a multiplicative function for which the corresponding set $\mathcal{S}_g := \{p \in \wp : g(p) \neq 1\}$ is such that*

$$\sum_{p\in\mathcal{S}_g} \frac{1}{p} < \infty. \tag{3.17}$$

Then, for any $\varepsilon > 0$, there exists a sequence of real number x_k tending to infinity such that

$$E(\omega, g; x_k) \leq \frac{x_k}{(\log\log x_k)^{1-\varepsilon}} \qquad (k = 1, 2, \ldots).$$

Proof. First, we introduce the sets

$$\mathfrak{A} := \{n \in \mathbb{N} : p|n \Rightarrow p \in \mathcal{S}_g\} \quad \text{ and } \quad \mathfrak{B} := \{n \in \mathbb{N} : p|n \Rightarrow p \notin \mathcal{S}_g\}.$$

Observe that the condition (3.17) implies that there exists a real number C_g such that

$$C_g = \sum_{a\in\mathfrak{A}} \frac{1}{a}.$$

For each positive integer j, define the numbers c_j implicitly by the relation

$$\sum_{\substack{a\in\mathfrak{A}\\ g(a)=j}} \frac{1}{a} = c_j C_g, \tag{3.18}$$

so that in particular we have

$$\sum_{j=1}^{\infty} c_j \leq 1. \tag{3.19}$$

Let us now introduce the additive function ω^* defined by

$$\omega^*(n) = \sum_{\substack{p|n\\ p\in\mathcal{S}_g}} 1.$$

If we apply Lemma 5 to the function ω^*, we obtain that the corresponding sums $A(x)$ and $B(x)$ defined in (2.3) satisfy $A(x) \ll 1$ and $B(x) \ll 1$, thus yielding

$$\#\{n \leq x : \omega^*(n) > (\log\log x)^{1-\delta}\} \ll \frac{x}{(\log\log x)^{2-2\delta}}$$

for any $\delta > 0$. Thus from now on we can assume that $\omega^*(n) \leq (\log\log x)^{1-\delta}$.

Now fix an arbitrary $\varepsilon \in (0, 1)$. For each integer $k \geq 1$, let $y_k = k^{2+6\varepsilon}$ and consider the intervals

$$J_k := \left(y_k - y_k^{1/2+\varepsilon}, y_k + y_k^{1/2+\varepsilon}\right] \qquad (k = 1, 2, 3, \ldots).$$

Observe that these intervals do not overlap for k large enough (say $k > k_0$) since

$$\begin{aligned} y_{k+1} - y_{k+1}^{1/2+\varepsilon} &= (k+1)^{2+6\varepsilon} - (k+1)^{(2+6\varepsilon)(1/2+\varepsilon)} \\ &> k^{2+6\varepsilon} + (2+6\varepsilon)k^{1+6\varepsilon} - (k+1)^{1+5\varepsilon+6\varepsilon^2} \\ &= y_k + (2+6\varepsilon)k^{1+6\varepsilon} - k^{\frac{\log(k+1)}{\log k}(1+5\varepsilon+6\varepsilon^2)} > y_k + k^{1+6\varepsilon} \\ &> y_k + k^{(2+6\varepsilon)(1/2+\varepsilon)} = y_k + y_k^{1/2+\varepsilon}. \end{aligned}$$

Finally define the sequence (D_k) by

$$D_k := \sum_{j\in J_k} c_j \qquad (k > k_0). \tag{3.20}$$

The fact that $\sum_{k>k_0} D_k \le 1$ (because of (3.19)) implies that $D_k \le \frac{1}{k}$ for infinitely many integers k. For each such integer k, let $x_k = e^{e^{y_k}}$. Then, for each such k,

$$E(\omega, g; x_k) = \sum_{j\in J_k} \#\{n \le x_k : \omega(n) = j, g(n) = j\} + O\left(x_k \exp(-(\log\log x_k)^{2\varepsilon}/3)\right),$$

where for the error term we used Lemma 3 to account for those integers $n \le x$ counted by $E(\omega, g; x_k)$ for which the corresponding $\omega(n)$ value lies outside J_k.

On the other hand, for each $j \in J_k$, we have

$$\#\{n \le x_k : \omega(n) = j, g(n) = j\} \le \sum_{\substack{a\in\mathfrak{A} \\ g(a)=j}} \#\left\{b \le \frac{x_k}{a} : b \in \mathfrak{B}, \omega(b) = j - \omega(a)\right\}. \tag{3.21}$$

In the above sum, first consider those $a \in \mathfrak{A}$ such that $a < \frac{x_k}{e^{\sqrt{\log x_k}}}$. Then, in light of Lemma 4, there exists a positive constant c such that

$$\begin{aligned} \#\left\{b \le \frac{x_k}{a} : b \in \mathfrak{B}, \omega(b) = j - \omega(a)\right\} &\le c\frac{x_k}{a\sqrt{\log\log(x_k/a)}} \\ &\le \frac{2c}{a\sqrt{\log\log x_k}}. \end{aligned} \tag{3.22}$$

Secondly, for those $a \in \mathfrak{A}$ with $a \ge \frac{x_k}{e^{\sqrt{\log x_k}}}$, we have that $b \le \frac{x_k}{a} \le e^{\sqrt{\log x_k}}$, which implies that $\log\log b \le \frac{1}{2}\log\log x_k$ and that $j - \omega(a) \ge \frac{2}{3}\log\log x_k$. Applying one more time Lemma 5 yields

$$\begin{aligned} \#\left\{b \le \frac{x_k}{a} : b \in \mathfrak{B}, \omega(b) = j - \omega(a)\right\} &\lesssim \frac{2x_k}{a}\frac{\log\log(x_k/a)}{((1/6)\log\log x_k)^2} \\ &\lesssim \frac{6x_k}{a\log\log x_k}. \end{aligned} \tag{3.23}$$

Using (3.22) and (3.23) in (3.21), we get that

$$\#\{n \le x_k : \omega(n) = j,\ g(n) = j\} \le c \sum_{\substack{a\in\mathfrak{A}\\ g(a)=j}} \frac{x_k}{a\sqrt{\log\log x_k}}.$$

We thus obtain, recalling (3.18) and (3.20),

$$\begin{aligned} E(\omega, g; x_k) &\le c\, C_g \sum_{j\in J_k} c_j \frac{x_k}{\sqrt{\log\log x_k}} + O\left(x_k \exp(-(\log\log x_k)^{2\varepsilon}/3)\right) \\ &= c\, C_g D_k \frac{x_k}{\sqrt{\log\log x_k}} + O\left(x_k \exp(-(\log\log x_k)^{2\varepsilon}/3)\right) \\ &\le \frac{c\, C_g\, x_k}{k\sqrt{\log\log x_k}} + O\left(x_k \exp(-(\log\log x_k)^{2\varepsilon}/3)\right). \qquad (3.24) \end{aligned}$$

Since $k = y_k^{1/(2+6\varepsilon)} \ge y_k^{1/2-2\varepsilon} = (\log\log x_k)^{1/2-2\varepsilon}$, we obtain from (3.24) that

$$E(\omega, g; x_k) \le \frac{c\, C_g\, x_k}{(\log\log x_k)^{1-2\varepsilon}} + O\left(x_k \exp(-(\log\log x_k)^{2\varepsilon}/3)\right),$$

thus completing the proof of Theorem 3. □

This last theorem essentially confirms that if one hopes to obtain significant improvements of the results in this section, it can only be possible using a different approach.

4 An upper bound

In this section, we show that, given an additive function f satisfying certain basic conditions, no function $g \in \mathcal{M}$ can agree with f on a set of positive density. More precisely, our goal in this section is to prove the following result.

Theorem 4. *Let f be an integer valued additive function for which the corresponding sums $A(x)$ and $B(x)$ defined in (2.3) satisfy the conditions*

(i) $\varphi(x) = \varphi_f(x) := \dfrac{B(x)}{A(x)} \to 0$ *as* $x \to \infty$,

(ii) $\max_{z\in\mathbb{N}} \#\{n \le x : f(n) = z\} = O\left(\dfrac{x}{H(x)}\right)$,
where $H(x) = H_f(x) \to \infty$ *as* $x \to \infty$.

Then, given any multiplicative function g, we have $E(f, g; x) = o(x)$ as $x \to \infty$.

Remark 3. *Observe that in the case $f = \omega$, we have $A(x) = (1+o(1)) \log\log x$ and $B(x) = (1+o(1))\sqrt{\log\log x}$ as $x \to \infty$, so that $\varphi(x) = (1+o(1))/\sqrt{\log\log x}$ as $x \to \infty$, while in light of Lemma 4 we have $H(x) = \sqrt{\log\log x}$.*

Before we start the proof of Theorem 4, observe that it is clear that we can discard those integers $n \le x$ for which

$$|f(n) - A(x)| > \varphi(x)^{-1/3} B(x)$$

since, in light of Lemma 5, their contribution to $E(f, g; x)$ is $o(x)$. This means that throughout the rest of the proof we only need to consider those $n \le x$ for which

$$|f(n) - A(x)| \le \varphi(x)^{-1/3} B(x) = \varphi(x)^{2/3} A(x). \tag{4.1}$$

We now introduce a key set of integers. Let D and k be positive integers. Later, D and k will be chosen large. Define the set $\mathcal{T}_{D,k}$ by

$$\mathcal{T}_{D,k} := \left\{ n \in \mathbb{N} : P^+(n) \le D \text{ and } p^{k+1} \nmid n \text{ for any prime } p \right\}.$$

Observe that for any given D and k, the set $\mathcal{T}_{D,k}$ is finite, which implies in particular that if $g(n) \ne 1$ for some positive integer n, then, for D large enough and depending only on g, the quantity

$$V_{D,k} := \min_{\substack{m,n \in \mathcal{T}_{D,k} \\ |g(n)| > |g(m)|}} \frac{|g(n)|}{|g(m)|}$$

is well defined, in which case $V_{D,k} > 1$. Now choose x large enough so that

$$1 + \varphi(x)^{1/3} < V_{D,k}. \tag{4.2}$$

Let us further introduce the set $\mathcal{U}_D$ defined by

$$\mathcal{U}_\mathcal{D} := \{ n \in \mathbb{N} : P^-(n) > D \}.$$

Note that the counting function of the set $\mathcal{U}_D$ is the function $\Phi(x, D)$. Finally, for each positive integer k, let $\mathcal{S}_k$ stand for the set of those positive integers n such that $p^{k+1} | n$ for some prime number p, and let $\mathcal{S}_k(x)$ be the counting function of $\mathcal{S}_k$. Observe that

$$\mathcal{S}_k(x) \le \sum_{p \le x} \left\lfloor \frac{x}{p^{k+1}} \right\rfloor \le x \int_2^x \frac{dt}{t^{k+1}} < \frac{x}{2^k}. \tag{4.3}$$

Observe also that, given any fixed positive integers D and k, the prime powers dividing an arbitrary integer $n \ge 2$ either belong to $\mathcal{T}_{D,k}$ or to $\mathcal{U}_\mathcal{D}$ or to $\mathcal{S}_k$.

Before we move on to the proof of Theorem 4, we need an additional result which essentially says that if we gather the prime powers p^a dividing n into two parts, namely the part $u(n)$ made up of the product of those p^a which are not in $\mathcal{T}_{D,k}$ and the part $t(n)$ made up of the product of those p^a which belong to $\mathcal{T}_{D,k}$, and if two integers $n \le x$ such that $g(n) = f(n)$ happen to have an identical $u(n)$ part, then their $t(n)$ parts have the same image under f. This is formally expressed in the following lemma.

Lemma 7. *Let g be a multiplicative function. Under the assumption (4.1), write each integer $n \in [2, x]$ as*

$$n = \prod_{\substack{p^a \| n \\ p \notin \mathcal{T}_{D,k}}} p^a \cdot \prod_{\substack{p^a \| n \\ p \in \mathcal{T}_{D,k}}} p^a = u(n) \cdot t(n). \tag{4.4}$$

Let n_1 and n_2 be two integers such that $g(n_1) = f(n_1)$ and $g(n_2) = f(n_2)$. Write $u_j = u(n_j)$ and $t_j = t(n_j)$ for $j = 1, 2$. If $u_1 = u_2$, then $f(t_1) = f(t_2)$.

Proof. We will first show that $g(t_1) = g(t_2)$. Indeed, it is clear that

$$\frac{f(n_1)}{f(n_2)} = \frac{g(n_1)}{g(n_2)} = \frac{g(u_1 t_1)}{g(u_2 t_2)} = \frac{g(t_1)}{g(t_2)}. \tag{4.5}$$

Hence, assuming that $g(t_1) \neq g(t_2)$, we have by the definition of $V_{D,k}$ and in light of (4.2), that

$$\max\left(\frac{g(t_1)}{g(t_2)}, \frac{g(t_2)}{g(t_1)}\right) \geq V_{D,k} > 1 + \varphi(x)^{1/3}. \tag{4.6}$$

Taking into account (4.5), (4.6) and assumption (4.1), we have, assuming say that $f(n_2) \geq f(n_1)$ and that x is sufficiently large,

$$\begin{aligned} f(n_2) &= f(n_1) \cdot \frac{f(n_2)}{f(n_1)} > \left(A(x) - \varphi(x)^{-1/3} B(x)\right)\left(1 + \varphi(x)^{1/3}\right) \\ &> A(x) + \frac{1}{2}\varphi(x)^{1/3} A(x) \end{aligned}$$

while

$$\begin{aligned} f(n_1) &= f(n_2) \cdot \frac{f(n_1)}{f(n_2)} < \left(A(x) + \varphi(x)^{-1/3} B(x)\right)\left(1 + \varphi(x)^{1/3}\right)^{-1} \\ &< A(x) - \frac{3}{4}\varphi(x)^{1/3} A(x), \end{aligned}$$

from which it follows that $f(n_2) - f(n_1) > \frac{5}{4}\varphi(x)^{1/3} A(x)$, thus contradicting our assumption (4.1). It follows that we must have $g(t_1) = g(t_2)$. Hence, by (4.5), we have that $f(n_1) = f(n_2)$ and therefore that $f(t_1) = f(t_2)$, as requested. □

We are now ready to begin the proof of Theorem 4.

Proof. As we did in the statement of Lemma 7, write those numbers $n \leq x$ for which $g(n) = f(n)$ as in (4.4). Then, Lemma 7 guarantees that, for each fixed positive integer $u \leq x$, we only need to count the number of positive integers $t \leq x/u$, with $t \in \mathcal{T}_{D,k}$, knowing that these t's have the same value under f, say the value z.

This allows us to write that

(4.7)
$$E(f,g;x) \le \sum_{\substack{u\le x\\ u\in\mathcal{U}_{\mathcal{D}}}} \#\left\{t \le \frac{x}{u} : t\in \mathcal{T}_{D,k}, g(u\cdot t) = f(u\cdot t)\right\} + O\left(\frac{x}{2^k}\right) + o(x),$$

where the first error term is a consequence of (4.3) while the second one comes from the fact that we chose to discard those $n \le x$ for which $|f(n) - A(x)| > \varphi(x)^{-1/3}B(x)$.

Suppose that for a fixed value of $u \le x$, the numbers $t_1, t_2, \ldots, t_N$ are the integers counted by the counting function

$$\#\left\{t \le \frac{x}{u} : t\in \mathcal{T}_{D,k}, g(u\cdot t) = f(u\cdot t)\right\},$$

then by Lemma 7, there exists an integer z such that

$$f(t_1) = f(t_2) = \cdots = f(t_N) = z.$$

We thus obtain from (4.7) that

$$\begin{aligned} E(f,g;x) &\le \sum_{\substack{u\le x\\ u\in\mathcal{U}_{\mathcal{D}}}} \max_{z\ge1} \#\left\{t \le \frac{x}{u} : t\in \mathcal{T}_{D,k},\ f(t) = z\right\} + O\left(\frac{x}{2^k}\right) + o(x) \\ &= R(x) + O\left(\frac{x}{2^k}\right) + o(x), \qquad (4.8)\end{aligned}$$

say.

Lemma 6 allows us to discard those $n \le x$ for which $t(n) \le D^{c_1}$ or $t(n) \ge D^{c_2}$, that is those $n \le x$ for which $u(n) \ge n/D^{c_1}$ or $u(n) \le n/D^{c_2}$, and therefore it follows that

(4.9)
$$R(x) = \sum_{\substack{\frac{x}{D^{c_2}}<u<\frac{x}{D^{c_1}}\\ P^-(u)>D}} \max_{z\ge1} \#\left\{t \le \frac{x}{u} : t\in \mathcal{T}_{D,k},\ f(t) = z\right\} + o(x) = R_0(x) + o(x),$$

say. Using condition (ii) and Lemma 1 with $a_1 = 1/D^{c_2}$ and $a_2 = 1/D^{c_1}$, we obtain that

$$\begin{aligned} R_0(x) &\le \sum_{\substack{\frac{x}{D^{c_2}}<u<\frac{x}{D^{c_1}}\\ P^-(u)>D}} \max_{z\ge1} \#\left\{t \le \frac{x}{u} : f(t) = z\right\} \\ &\le \frac{x}{H(D^{c_1})} \sum_{\substack{\frac{x}{D^{c_2}}<u<\frac{x}{D^{c_1}}\\ P^-(u)>D}} \frac{1}{u} \\ &\ll \frac{x}{H(D^{c_1})} \frac{(c_2-c_1)}{\log D} \log D \ll \frac{x}{H(D^{c_1})}. \qquad (4.10)\end{aligned}$$

Substituting (4.9) and (4.10) in (4.8), it follows that

$$E(f,g;x) \le \frac{x}{H(D^{c_1})} + O\left(\frac{x}{2^k}\right) + o(x).$$

Since both D and k can be chosen to be arbitrarily large, this completes the proof of Theorem 4. □

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