# On the Powerful and Squarefree Parts of an Integer

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**Abstract**

Any integer $n \geq 2$ can be written in a unique way as the product of its powerful part and its squarefree part, that is as $n = mr$ where $m$ is a powerful number and $r$ a squarefree number, with $(m, r) = 1$. We denote these two parts of an integer $n$ by $\mathrm{pow}(n)$ and $\mathrm{sq}(n)$ respectively, setting for convenience $\mathrm{pow}(1) = \mathrm{sq}(1) = 1$. We first examine the behavior of the counting functions $\sum_{n\leq x,\, \mathrm{sq}(n)\leq y} 1$ and $\sum_{n\leq x,\, \mathrm{pow}(n)\leq y} 1$. Letting $P(n)$ stand for the largest prime factor of $n$, we then provide asymptotic values of $A_y(x) := \sum_{n\leq x,\, P(n)\leq y} \mathrm{pow}(n)$ and $B_y(x) := \sum_{n\leq x,\, P(n)\leq y} \mathrm{sq}(n)$ when $y = x^{1/u}$ with $u \geq 1$ fixed. We also examine the size of $A_y(x)$ and $B_y(x)$ when $y = (\log x)^\eta$ for some $\eta > 1$. Finally, we prove that $A_y(x)$ will coincide with $B_y(x)$ in the sense that $\log(A_y(x)/x) = (1 + o(1)) \log(B_y(x)/x)$ as $x \to \infty$ if we choose $y = 2\log x$.

## 1 Introduction and notation

A *powerful number* (or *square-full number*) is a positive integer $n$ such that $p^2 \mid n$ for every prime factor $p$ of $n$. In 1934, Erdős and Szekeres [5] were the first to study the distribution of these numbers. In 1958, Bateman and Grosswald [1] established a very accurate estimate for the number $N(x)$ of powerful numbers not exceeding $x$, namely

$$N(x) = C_2 x^{1/2} + C_3 x^{1/3} + o(x^{1/6}) \qquad (x \to \infty),$$

where $C_2 = \dfrac{\zeta(3/2)}{\zeta(3)} \approx 2.1732$ and $C_3 = \dfrac{\zeta(2/3)}{\zeta(2)} \approx -1.4879$. Here $\zeta$ stands for the Riemann Zeta Function.

Further estimates and generalizations concerning powerful numbers were later obtained by Ivić and Shiu [8].

On the other hand, a *squarefree number* is a positive integer $n$ such that $p^2 \nmid n$ for every prime factor $p$ of $n$. It is well known (see for instance Hardy and Wright [6], pp. 169-170) that the number $Z(x)$ of squarefree numbers not exceeding $x$ can be written as

$$Z(x) = \sum_{n \le x} \mu^2(n) = \frac{6}{\pi^2} x + O\left(x^{1/2}\right),$$

where $\mu$ stands for the Moebius function.

Interestingly, any integer $n \ge 2$ can be written in a unique way as the product of its powerful part and its squarefree part, that is as $n = mr$ where $m$ is a *powerful number* and $r$ a squarefree number, with $(m, r) = 1$. We denote these two parts of an integer $n$ by $\text{pow}(n)$ and $\text{sq}(n)$ respectively, setting for convenience $\text{pow}(1) = \text{sq}(1) = 1$.

In his master's thesis, Cloutier [2] obtained asymptotic values for $\sum_{n \le x} \text{pow}(n)^a \, \text{sq}(n)^b$, where $a$ and $b$ are fixed integers. Particular cases of his results yield the following estimates.

$$\sum_{n \le x} \text{pow}(n) = \frac{d_1}{3} x^{3/2} + O(x^{4/3}), \tag{1.1}$$

$$\sum_{n \le x} \text{sq}(n) = \frac{d_2}{2} x^2 + O(x^{3/2}), \tag{1.2}$$

$$\sum_{n \le x} \frac{1}{\text{pow}(n)} = d_2 x + O(x^{1/2}), \tag{1.3}$$

$$\sum_{n \le x} \frac{1}{\text{sq}(n)} = d_1 x^{1/2} + O(x^{1/3}), \tag{1.4}$$

where

$$d_1 = \prod_p \left(1 + \frac{2}{p^{3/2}} - \frac{1}{p^{5/2}}\right) \approx 3.52, \tag{1.5}$$

$$d_2 = \prod_p \left(1 - \frac{1}{p^2} + \frac{1}{p^3(p+1)}\right) \approx 0.65. \tag{1.6}$$

In particular, it follows from (1.1) and (1.2) that $B(x) := \sum_{n \le x} \text{sq}(n)$ is much larger than $A(x) := \sum_{n \le x} \text{pow}(n)$ for large values of $x$. However, letting $P(n)$ stand for the largest prime factor of $n$, with $P(1) = 1$, one can easily check that if we set

$$A_y(x) := \sum_{\substack{n \le x \\ P(n) \le y}} \text{pow}(n) \qquad \text{and} \qquad B_y(x) = \sum_{\substack{n \le x \\ P(n) \le y}} \text{sq}(n), \tag{1.7}$$

then, for small values of $y = y(x)$, we have $B_y(x) = o(A_y(x))$ as $x \to \infty$, while for large values of $y = y(x)$, we have $A_y(x) = o(B_y(x))$.

Here, after examining the comparative sizes of $\#\{n \le x : \text{sq}(n) \le y\}$ and $\#\{n \le x : \text{pow}(n) \le y\}$, we show that $\#\{n \le x : \text{pow}(n) > \text{sq}(n)\}$ is asymptotic to $cx^{3/4}$ for some computable constant $c > 0$. We then provide asymptotic values for $A_y(x)$ and $B_y(x)$ when $y = x^{1/u}$ with $u \geq 1$ fixed. We also examine the size of $A_y(x)$ and $B_y(x)$ when $y = (\log x)^\eta$ for some $\eta > 1$. Finally, we establish that if $y = 2\log x$, then $\log(A_y(x)/x) = (1 + o(1))\log(B_y(x)/x)$ as $x \to \infty$.

Throughout this paper, we let $\phi$ stand for the Euler totient function, $\omega(n)$ for the number of distinct prime factors of $n$ (with $\omega(1) = 0$), $\Omega(n)$ for the number of prime factors of $n$ counting their multiplicity (with $\Omega(1) = 0$) and $p(n)$ for the smallest prime factor of $n$ (with $p(1) = 1$). We shall also be using the function $\Psi(x, y) := \#\{n \leq x : P(n) \leq y\}$ defined for $2 \leq y \leq x$.

Denoting by $\pi(x)$ the number of prime numbers not exceeding $x$, we shall be using the prime number theorem in its various forms, in particular as:

$$\pi(x) = (1 + o(1))\frac{x}{\log x} \qquad (x \to \infty), \tag{1.8}$$

$$\prod_{p\leq z} p = \exp\{(1 + o(1))z\} \qquad (z \to \infty), \tag{1.9}$$

$$\sum_{p\leq x} \log p = (1 + o(1))x \qquad (x \to \infty), \tag{1.10}$$

$$p_k = (1 + o(1))k \log k \qquad (k \to \infty), \tag{1.11}$$

where $p_k$ stands for the $k$-th prime number.

## 2 The distribution of the values of $\mathrm{pow}(n)$ and of $\mathrm{sq}(n)$

We examine the distribution of the values of $\mathrm{pow}(n)$ and of $\mathrm{sq}(n)$ separately in the following two theorems.

**Theorem 1.** *For $2 \leq y \leq x$,*

$$\sum_{\substack{n\leq x\\ \mathrm{sq}(n)\leq y}} 1 = 2C_0\frac{\zeta(3/2)}{\zeta(3)}\sqrt{xy} + O\left(x^{1/3}y^{2/3}\log y\right), \tag{2.1}$$

*where*

$$C_0 = \prod_p \left(1 + \frac{1 - 1/p}{p(1 + 1/p^{3/2})}\right)\left(1 - \frac{1}{p}\right) \approx 0.38. \tag{2.2}$$

**Remark 1.** *It is clear that estimate (2.1) has some interest only if the error term is of smaller order than the main term, that is when $y\log^6 y = o(x)$ as $x \to \infty$.*

**Theorem 2.** *For $2 \leq y \leq x$,*

$$\sum_{\substack{n\leq x\\ \mathrm{pow}(n)\leq y}} 1 = x\frac{6}{\pi^2}\sum_{\substack{m\leq y\\ m \text{ powerful}}} \frac{1}{m\prod_{p|m}\left(1 + \frac{1}{p}\right)} + O\left(\sqrt{x}\log y\right). \tag{2.3}$$

**Remark 2.** *Observe that the main term on the right hand side of (2.3) is asymptotic to $x$ as $y \to \infty$. Indeed, this follows from the fact that*

$$\begin{aligned}\frac{6}{\pi^2}\lim_{y\to\infty}\sum_{\substack{m\le y\\ m \ \text{powerful}}}\frac{1}{m\prod_{p|m}\left(1+\frac{1}{p}\right)} &= \prod_p\left(1-\frac{1}{p^2}\right)\prod_p\left(1+\frac{1}{p^2(1+\frac{1}{p})}+\frac{1}{p^3(1+\frac{1}{p})}+\cdots\right)\\ &= \prod_p\left(1-\frac{1}{p^2}\right)\prod_p\left(1+\frac{1}{p^2-1}\right)=1.\end{aligned}$$

In order to prove these two theorems, we first establish some preliminary results.

**Lemma 1.** *Given an arbitrary positive integer $r$,*

$$\#\{n\le x : n \text{ powerful},\ (n,r)=1\} = \frac{\phi(r)}{r}\frac{\zeta(3/2)}{\zeta(3)}\prod_{p|r}\left(1+\frac{1}{p^{3/2}}\right)^{-1}\sqrt{x}+O\left(2^{\omega(r)}x^{1/3}\right), \tag{2.4}$$

*where the constant implied by the Landau symbol is absolute.*

*Proof.* Using the estimate

$$\sum_{\substack{n\le X\\ (n,r)=1}} 1 = \frac{\phi(r)}{r}X+O\left(2^{\omega(r)}\right) \qquad \text{uniform for } r\ge 1, \tag{2.5}$$

we have, using the fact that any powerful number $n$ can be written in a unique way as $n=a^2b^3$ with $\mu^2(b)=1$,

$$\begin{aligned}\sum_{\substack{n\le x\\ n \text{ powerful}\\ (n,r)=1}} 1 &= \sum_{\substack{a^2b^3\le x\\ (a,r)=1\\ (b,r)=1}}\mu^2(b) = \sum_{\substack{b^3\le x\\ (b,r)=1}}\mu^2(b)\sum_{\substack{a^2\le x/b^3\\ (a,r)=1}} 1\\ &= \sum_{\substack{b^3\le x\\ (b,r)=1}}\mu^2(b)\left(\frac{\phi(r)}{r}\frac{x^{1/2}}{b^{3/2}}+O\left(2^{\omega(r)}\right)\right)\\ &= x^{1/2}\frac{\phi(r)}{r}\sum_{\substack{b\le x^{1/3}\\ (b,r)=1}}\frac{\mu^2(b)}{b^{3/2}}+O\left(2^{\omega(r)}x^{1/3}\right). \end{aligned}\tag{2.6}$$

Using the fact that

$$\begin{aligned}\sum_{\substack{b\le x^{1/3}\\ (b,r)=1}}\frac{\mu^2(b)}{b^{3/2}} &= \sum_{\substack{b=1\\ (b,r)=1}}^{\infty}\frac{\mu^2(b)}{b^{3/2}}+O\left(\sum_{b>x^{1/3}}\frac{1}{b^{3/2}}\right)\\ &= \prod_{p\nmid r}\left(1+\frac{1}{p^{3/2}}\right)+O\left(\int_{x^{1/3}}^{\infty}\frac{1}{t^{3/2}}\,dt\right)\end{aligned}$$

$$= \prod_{p \nmid r} \left(1 + \frac{1}{p^{3/2}}\right) + O\left(\frac{1}{x^{1/6}}\right),$$

we obtain from (2.6) that

$$\begin{aligned}
\sum_{\substack{n \le x \\ n \text{ powerful} \\ (n,r)=1}} 1 &= x^{1/2} \frac{\phi(r)}{r} \left( \prod_{p \nmid r} \left(1 + \frac{1}{p^{3/2}}\right) + O\left(\frac{1}{x^{1/6}}\right) \right) + O\left(2^{\omega(r)} x^{1/3}\right) \\
&= x^{1/2} \frac{\phi(r)}{r} \prod_{p \nmid r} \left(1 + \frac{1}{p^{3/2}}\right) + O\left(2^{\omega(r)} x^{1/3}\right) \\
&= x^{1/2} \frac{\phi(r)}{r} \frac{\prod_p \left(1 + \frac{1}{p^{3/2}}\right)}{\prod_{p \mid r} \left(1 + \frac{1}{p^{3/2}}\right)} + O\left(2^{\omega(r)} x^{1/3}\right),
\end{aligned}$$

which proves (2.4). □

**Lemma 2.** *Given an arbitrarily small $\delta > 0$, as $x \to \infty$,*

$$\sum_{r \le x} \frac{\mu^2(r)\phi(r)}{r \prod_{p \mid r} \left(1 + \frac{1}{p^{3/2}}\right)} = C_0 x + O\left(x^{\frac{1}{2}+\delta}\right), \tag{2.7}$$

*where $C_0$ is the constant defined in (2.2).*

*Proof.* For $s > 1$, we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{\mu^2(n)\phi(n)/(n \prod_{p \mid n}(1 + \frac{1}{p^{3/2}}))}{n^s} &= \prod_p \left(1 + \frac{1 - 1/p}{1 + \frac{1}{p^{3/2}}} \frac{1}{p^s}\right) \\
&= \zeta(s) \prod_p \left(1 - \frac{1}{p^s}\right) \prod_p \left(1 + \frac{1 - 1/p}{1 + \frac{1}{p^{3/2}}} \frac{1}{p^s}\right) \\
&= \zeta(s) H(s),
\end{aligned}$$

say. Observing that, for any fixed $s > \frac{1}{2}$, the infinite product $H(s)$ is absolutely convergent, then (2.7) follows by using Wintner's Theorem (see for instance Theorem 6.13 and Problem 6.3 in the book of De Koninck and Luca [4]). □

As an immediate consequence of this lemma, we obtain the following.

**Lemma 3.** *Given an arbitrarily small $\delta > 0$,*

$$\sum_{r \le y} \frac{\mu^2(r)\phi(r)}{r^{3/2} \prod_{p \mid r} \left(1 + \frac{1}{p^{3/2}}\right)} = 2C_0 \sqrt{y} + O\left(y^{\frac{1}{4}+\delta}\right).$$

**Lemma 4.** *Given an arbitrary powerful number $m$,*

$$\sum_{\substack{r\le x/m\\(r,m)=1}} \mu^2(r) = \alpha(m)x + O\left(\sqrt{\frac{x}{m}}\beta(m)\right),$$

*where*

$$\alpha(m) = \frac{6}{\pi^2 m}\prod_{p|m}\left(1+\frac{1}{p}\right)^{-1} \qquad \text{and} \qquad \beta(m) = \prod_{p|m}\left(1-\frac{1}{\sqrt{p}}\right)^{-1}.$$

*Proof.* This is relation (1) in the paper of De Koninck, Kátai and Subbarao [3] with a correction by Cloutier [2]. ☐

We now have the necessary tools to prove Theorems 1 and 2.

*Proof of Theorem 1.* As a consequence of Lemma 1, we have, as $x\to\infty$,

$$\begin{aligned}
\sum_{\substack{n\le x\\ \mathrm{sq}(n)\le y}} 1 &= \sum_{\substack{mr\le x\\ m \text{ powerful}\\ r\le y\\ (m,r)=1}} \mu^2(r) = \sum_{r\le y}\mu^2(r) \sum_{\substack{m\le x/r\\ m\text{ powerful}\\ (m,r)=1}} 1\\
&= \sum_{r\le y}\mu^2(r)\left(\frac{\phi(r)}{r}\frac{\zeta(3/2)}{\zeta(3)}\prod_{p|r}\left(1+\frac{1}{p^{3/2}}\right)^{-1}\sqrt{\frac{x}{r}} + O\left(2^{\omega(r)}\left(\frac{x}{r}\right)^{1/3}\right)\right)\\
&= \sqrt{x}\frac{\zeta(3/2)}{\zeta(3)}\sum_{r\le y}\frac{\mu^2(r)\phi(r)}{r^{3/2}\prod_{p|r}\left(1+\frac{1}{p^{3/2}}\right)} + O\left(x^{1/3}\sum_{r\le y}\frac{2^{\omega(r)}}{r^{1/3}}\right)\\
&= \sqrt{x}\frac{\zeta(3/2)}{\zeta(3)}\sum_{r\le y}\frac{\mu^2(r)\phi(r)}{r^{3/2}\prod_{p|r}\left(1+\frac{1}{p^{3/2}}\right)} + O\left(x^{1/3}y^{2/3}\log y\right),
\end{aligned}$$

where we used the fact that, since $\displaystyle\sum_{r=1}^{\infty}\frac{2^{\omega(r)}}{r^s} = \frac{\zeta^2(s)}{\zeta(2s)}$ say for $s>1$, it follows that

$$\sum_{r\le y} 2^{\omega(r)} = \frac{1}{\zeta(2)}y\log y + O(y), \tag{2.8}$$

implying by partial summation that

$$\sum_{r\le y}\frac{2^{\omega(r)}}{r^{1/3}} \ll y^{2/3}\log y.$$

Using Lemma 3, the proof of Theorem 1 is thus complete. ☐

*Proof of Theorem 2.* Using Lemma 4, we may write

$$\begin{aligned}
\sum_{\substack{n\le x\\ \mathrm{pow}(n)\le y}} 1 &= \sum_{\substack{m\le y\\ m \text{ powerful}}} \sum_{\substack{r\le x/m\\ (r,m)=1}} \mu^2(r) = x \sum_{\substack{m\le y\\ m \text{ powerful}}} \alpha(m) + O\left( \sum_{\substack{m\le y\\ m \text{ powerful}}} \sqrt{\frac{x}{m}}\beta(m)\right) \\
&= x\frac{6}{\pi^2} \sum_{\substack{m\le y\\ m \text{ powerful}}} \frac{1}{m\prod_{p|m}(1+\frac{1}{p})} + O\left(\sqrt{x} \sum_{\substack{m\le y\\ m \text{ powerful}}} \frac{\beta(m)}{\sqrt{m}}\right). \qquad (2.9)
\end{aligned}$$

Using the estimate

$$\sum_{\substack{m\le y\\ m \text{ powerful}}} \frac{\beta(m)}{\sqrt{m}} \le \prod_{p\le y}\left(1+\frac{\beta(p)}{p^{2/2}}+\frac{\beta(p)}{p^{3/2}}+\cdots\right) \le \prod_{p\le y}\left(1+\frac{1}{p}+O\left(\frac{1}{p^{3/2}}\right)\right) \ll \log y$$

in (2.9) completes the proof of Theorem 2. □

# 3 How often is $\mathrm{pow}(n)$ larger than $\mathrm{sq}(n)$

In this section, we show that the set of positive integers $n$ such that $\mathrm{pow}(n) > \mathrm{sq}(n)$ is of zero density. In fact, we show much more.

**Theorem 3.** *As $x$ becomes large,*

$$\#\{n \le x : \mathrm{pow}(n) > \mathrm{sq}(n)\} = D_0 x^{3/4} + O\left(x^{2/3}\log x\right), \qquad (3.1)$$

*where* $D_0 = \frac{4}{3}C_0\frac{\zeta(3/2)}{\zeta(3)} \approx 1.10.$

*Proof.* We clearly have

$$\begin{aligned}
\#\{n \le x : \mathrm{pow}(n) > \mathrm{sq}(n)\} &= \sum_{r\le\sqrt{x}} \mu^2(r) \sum_{\substack{r<m\le x/r\\ m \text{ powerful}\\ (m,r)=1}} 1 \\
&= \sum_{r\le\sqrt{x}} \mu^2(r) \sum_{\substack{m\le x/r\\ m \text{ powerful}\\ (m,r)=1}} 1 - \sum_{r\le\sqrt{x}} \mu^2(r) \sum_{\substack{m\le r\\ m \text{ powerful}\\ (m,r)=1}} 1 \\
&= T_1(x) - T_2(x), \qquad (3.2)
\end{aligned}$$

say. On the one hand, using Lemma 1 and (2.8), we have

$$T_1(x) = \sqrt{x}\frac{\zeta(3/2)}{\zeta(3)} \sum_{r\le\sqrt{x}} \frac{\mu^2(r)\phi(r)}{r^{3/2}\prod_{p|r}(1+1/p^{3/2})} + O(x^{1/3}\log x)$$

$$= \quad 2C_0\frac{\zeta(3/2)}{\zeta(3)}x^{3/4} + O(x^{1/3}\log x). \tag{3.3}$$

On the other hand, using (2.5) and following the same reasoning as in the proof of Lemma 2, we obtain

$$\begin{aligned}
\sum_{\substack{m\le r\\ m \text{ powerful}\\ (m,r)=1}} 1 &= \sum_{\substack{a^2b^3\le r\\ (a,r)=1\\ (b,r)=1}} \mu^2(b) = \sum_{\substack{b^3\le r\\ (b,r)=1}} \mu^2(b) \sum_{\substack{a^2\le r/b^3\\ (a,r)=1}} 1\\
&= \sum_{\substack{b\le r^{1/3}\\ (b,r)=1}} \mu^2(b)\left(\frac{\phi(r)}{r}\frac{r^{1/2}}{b^{3/2}} + O(2^{\omega(r)})\right)\\
&= \frac{\phi(r)}{r^{1/2}}\sum_{\substack{b\le r^{1/3}\\ (b,r)=1}} \frac{\mu^2(b)}{b^{3/2}} + O\left(2^{\omega(r)}r^{1/3}\right)\\
&= \frac{\phi(r)}{r^{1/2}}\frac{\zeta(3/2)}{\zeta(3)}\prod_{p|r}\left(1+\frac{1}{p^{3/2}}\right)^{-1} + O\left(2^{\omega(r)}r^{1/3}\right).
\end{aligned}$$

Using this last estimate, we obtain that

$$\begin{aligned}
T_2(x) &= \frac{\zeta(3/2)}{\zeta(3)}\sum_{r\le\sqrt{x}}\frac{\mu^2(r)\phi(r)}{r^{1/2}\prod_{p|r}\left(1+\frac{1}{p^{3/2}}\right)} + O\left(\sum_{r\le\sqrt{x}} 2^{\omega(r)}r^{1/3}\right)\\
&= \frac{2}{3}C_0\frac{\zeta(3/2)}{\zeta(3)}x^{3/4} + O\left(x^{2/3}\log x\right). \qquad (3.4)
\end{aligned}$$

Using (3.3) and (3.4) in (3.2), estimate (3.1) follows thus completing the proof of Theorem 3. □

# 4 The sums $A_y(x)$ and $B_y(x)$

Recall the definitions of $A_y(x)$ and $B_y(x)$ given in (1.7) as well as the following.

**Definition 1.** *The Dickman function $\rho(u)$ is the continuous function satisfying $\rho(u) = 1$ for $0 \le u \le 1$ and $u\rho'(u) = -\rho(u-1)$ for $u > 1$.*

It follows from this definition that

$$\rho(u) = \rho(k) - \int_k^u \frac{\rho(z-1)}{z}dz. \tag{4.1}$$

**Theorem 4.** *Let $y = x^{1/u}$, with $u \ge 1$ fixed. Then, as $x \to \infty$,*

$$\sum_{\substack{n\le x\\ P(n)\le y}} \operatorname{sq}(n) = (1+o(1))\rho(u)\frac{d_2}{2}x^2 \tag{4.2}$$

*and*

$$\sum_{\substack{n\le x\\ P(n)\le y}} \mathrm{pow}(n) = (1+o(1))\rho(u/2)\frac{d_1}{3}x^{3/2}. \tag{4.3}$$

*Proof.* The proof of Theorem 4 is very similar to the proof of the estimate

$$\Psi(x, x^{1/u}) = (1+o(1))\rho(u)x \qquad (x\to\infty),$$

as performed for instance in the book of Tenenbaum [10]. We start by studying the expression

$$\begin{aligned}\sum_{\substack{n\le x\\ y<P(n)\le x}} \mathrm{sq}(n) &= \sum_{y<p\le x}\sum_{\substack{n\le x\\ P(n)=p}} \mathrm{sq}(n)\\ &= \sum_{y<p\le x}\sum_{\substack{m\le x/p\\ P(m)<p}} \mathrm{sq}(mp) + \sum_{y<p\le x}\sum_{\substack{m\le x/p\\ P(m)=p}} \mathrm{sq}(mp)\\ &= \Sigma_1(x)+\Sigma_2(x), \end{aligned} \tag{4.4}$$

say.

We first show that $\Sigma_2(x)$ is relatively small and can thus be neglected. Writing $m = r\cdot p$, we obtain

$$\Sigma_2(x) = \sum_{y<p\le x}\sum_{\substack{rp\le x/p\\ P(r)\le p}} \mathrm{sq}(rp^2) \le \sum_{y<p\le x}\sum_{r\le x/p^2} \mathrm{sq}(r) \le \sum_{y<p\le x}\sum_{r\le x/p^2} r \ll x^2 \sum_{y<p\le x} p^{-4} \ll \frac{x^2}{y^3}. \tag{4.5}$$

We now turn our attention to the estimation of $\Sigma_1(x)$, which we can write as

$$\Sigma_1(x) = \sum_{y<p\le x} p \sum_{\substack{m\le x/p\\ P(m)<p}} \mathrm{sq}(m).$$

In the range $1\le u\le 2$, the condition $m\le x/p$ implies $P(m)\le x^{1/2}<p$, so we can drop the condition $P(m)<p$. Hence, using equation (1.2), we obtain, as $x\to\infty$,

$$\begin{aligned}\Sigma_1(x) &= \sum_{y<p\le x} p\sum_{m\le x/p} \mathrm{sq}(m) = (1+o(1))\sum_{y<p\le x} p\frac{d_2}{2}\left(\frac{x}{p}\right)^2\\ &= (1+o(1))\frac{d_2}{2}x^2\sum_{y<p\le x}\frac{1}{p} = (1+o(1))\frac{d_2}{2}x^2\int_y^x \frac{1}{t\log t}dt\\ &= (1+o(1))\frac{d_2}{2}(\log u)x^2. \end{aligned} \tag{4.6}$$

Again using (1.2) and after substituting (4.6) and (4.5) in (4.4), we obtain that, in the range $1\le u\le 2$,

$$\sum_{\substack{n\le x\\ P(n)\le y}} \mathrm{sq}(n) = \sum_{n\le x}\mathrm{sq}(n) - \sum_{\substack{n\le x\\ y<P(n)\le x}} \mathrm{sq}(n)$$

$$\begin{aligned} &= (1+o(1))\frac{d_2}{2}(1-\log u)x^2 + O\left(\frac{x^2}{y^3}\right) \\ &= (1+o(1))\frac{d_2}{2}\rho(u)x^2, \end{aligned} \tag{4.7}$$

where we used the fact that $\rho(u) = 1 - \log u$ when $u \in [1,2]$.

In order to extend the range of validity of (4.7) for $u > 2$, we proceed recursively. Let $M \geq 2$ be a positive integer and assume that the asymptotic estimate

$$\sum_{\substack{n\leq x \\ P(n)\leq x^{1/u}}} \mathrm{sq}(n) = (1+o(1))\frac{d_2}{2}\rho(u)x^2 \quad \text{holds for any } u \in [1,M] \text{ as } x \to \infty. \tag{4.8}$$

We will then show that it also holds for $u \in [M, M+1]$.

First observe that, in light of (4.5), we have

$$\begin{aligned} \sum_{\substack{n\leq x \\ x^{1/u}<P(n)\leq x^{1/M}}} \mathrm{sq}(n) &= \sum_{x^{1/u}<p\leq x^{1/M}} \sum_{\substack{m\leq \frac{x}{p} \\ P(m)\leq p}} \mathrm{sq}(mp) \\ &= \sum_{x^{1/u}<p\leq x^{1/M}} p \sum_{\substack{m\leq \frac{x}{p} \\ P(m)<p}} \mathrm{sq}(m) + O\left(x^{2-3/u}\right) \\ &= \sum_{x^{1/u}<p\leq x^{1/M}} p \left( \sum_{\substack{m\leq \frac{x}{p} \\ P(m)\leq p}} \mathrm{sq}(m) - \sum_{\substack{m\leq \frac{x}{p} \\ P(m)=p}} \mathrm{sq}(m) \right) + O\left(x^{2-3/u}\right) \\ &= \sum_{x^{1/u}<p\leq x^{1/M}} p \sum_{\substack{m\leq \frac{x}{p} \\ P(m)\leq p}} \mathrm{sq}(m) + O\left(x^{2-1/u}\right). \end{aligned} \tag{4.9}$$

Using our induction hypothesis (4.8), we obtain, as $x \to \infty$,

$$\begin{aligned} \sum_{x^{1/u}<p\leq x^{1/M}} p \sum_{\substack{m\leq \frac{x}{p} \\ P(m)\leq p}} \mathrm{sq}(m) &= (1+o(1))\frac{d_2}{2} \sum_{x^{1/u}<p\leq x^{1/M}} p \left(\frac{x}{p}\right)^2 \rho\left(\frac{\log x}{\log p} - 1\right) \\ &= (1+o(1))\frac{d_2}{2}x^2 \sum_{x^{1/u}<p\leq x^{1/M}} \frac{1}{p}\rho\left(\frac{\log x}{\log p} - 1\right) \\ &= (1+o(1))\frac{d_2}{2}x^2 \int_{x^{1/u}}^{x^{1/M}} \frac{1}{t\log t}\rho\left(\frac{\log x}{\log t} - 1\right) dt. \end{aligned} \tag{4.10}$$

Setting $z = \log x/\log t$, we obtain $dz = \frac{-\log x}{t(\log t)^2}dt$, so that $\frac{1}{t\log t}dt = -\frac{dz}{z}$ and

$$\frac{d_2}{2}x^2 \int_{x^{1/u}}^{x^{1/M}} \frac{1}{t\log t}\rho\left(\frac{\log x}{\log t} - 1\right) dt = (1+o(1))\frac{d_2}{2}x^2 \int_M^u \frac{\rho(z-1)}{z} dz. \tag{4.11}$$

Gathering (4.9), (4.10) and (4.11), we obtain

$$\sum_{\substack{n\le x\\ x^{1/u}<P(n)\le x^{1/M}}} \mathrm{sq}(n) = (1+o(1))\frac{d_2}{2}x^2\int_M^u \frac{\rho(z-1)}{z}dz. \tag{4.12}$$

From equation (4.12) and our induction hypothesis (4.8), we then obtain

$$\begin{aligned}\sum_{\substack{n\le x\\ P(n)\le x^{1/u}}} \mathrm{sq}(n) &= \sum_{\substack{n\le x\\ P(n)\le x^{1/M}}} \mathrm{sq}(n) - \sum_{\substack{n\le x\\ x^{1/u}<P(n)\le x^{1/M}}} \mathrm{sq}(n)\\ &= (1+o(1))\frac{d_2}{2}x^2\left(\rho(M)-\int_M^u \frac{\rho(z-1)}{z}dz\right),\end{aligned}$$

which, in light of (4.1), can be replaced by

$$\sum_{\substack{n\le x\\ P(n)\le x^{1/u}}} \mathrm{sq}(n) = (1+o(1))\frac{d_2}{2}x^2\rho(u),$$

thus completing the proof of (4.2).

We now turn to the proof of (4.3). We begin by writing

$$\sum_{\substack{n\le x\\ x^{1/u}<P(n)\le x}} \mathrm{pow}(n) = \Sigma_1(x)+\Sigma_2(x)+\Sigma_3(x), \tag{4.13}$$

where $\Sigma_1(x), \Sigma_2(x), \Sigma_3(x)$ stand respectively for the three expressions

$$\sum_{\substack{n\le x\\ x^{1/u}<P(n)\le x\\ P(n)\| n}} \mathrm{pow}(n), \qquad \sum_{\substack{n\le x\\ x^{1/u}<P(n)\le x\\ P(n)^2\| n}} \mathrm{pow}(n), \qquad \sum_{\substack{n\le x\\ x^{1/u}<P(n)\le x\\ P(n)^3| n}} \mathrm{pow}(n).$$

First observe that, in light of (1.1),

$$\begin{aligned}\Sigma_1(x) &= \sum_{x^{1/u}<p\le x}\ \sum_{\substack{m\le \frac{x}{p}\\ P(m)<p}} \mathrm{pow}(mp) = \sum_{x^{1/u}<p\le x}\ \sum_{\substack{m\le \frac{x}{p}\\ P(m)<p}} \mathrm{pow}(m)\\ &\ll \sum_{x^{1/u}<p\le x}\left(\frac{x}{p}\right)^{3/2} \ll x^{3/2}\sum_{t>x^{1/u}}\frac{1}{t^{3/2}} = O\left(x^{3/2-1/2u}\right).\end{aligned}$$

On the other hand, again using (1.1), we have

$$\Sigma_3(x) = \sum_{a\ge 3}\ \sum_{x^{1/u}<p\le x}\ \sum_{\substack{m\le \frac{x}{p^a}\\ P(m)<p}} \mathrm{pow}(mp^a) \le \sum_{a\ge 3}\ \sum_{x^{1/u}<p\le x} p^a \sum_{m\le \frac{x}{p^a}} \mathrm{pow}(m)$$

$$\ll \quad x^{3/2}\sum_{a\geq 3}\sum_{x^{1/u}<p\leq x}\frac{1}{p^{a/2}}\ll\sum_{a\geq 3}x^{\frac{3}{2}-\frac{1}{u}\left(\frac{a}{2}-1\right)}=O\left(x^{3/2-1/2u}\right).$$

It follows from these two estimates that

$$\Sigma_1(x)+\Sigma_3(x)=O\left(x^{3/2-1/2u}\right). \tag{4.14}$$

Now estimating $\Sigma_2(x)$ in the range $1\leq u\leq 2$, we have, since $P(n)^2>x$,

$$\Sigma_2(x)=\sum_{\substack{n\leq x\\ x^{1/u}<P(n)\leq x\\ P(n)^2\|n}}\operatorname{pow}(n)=0. \tag{4.15}$$

Recalling (1.1) and using both (4.14) and (4.15) in (4.13), it follows that, in the range $u\in[1,2]$,

$$\begin{aligned}\sum_{\substack{n\leq x\\ P(n)\leq x^{1/u}}}\operatorname{pow}(n) &= \sum_{n\leq x}\operatorname{pow}(n)-\sum_{\substack{n\leq x\\ x^{1/u}<P(n)\leq x}}\operatorname{pow}(n)\\ &= (1+o(1))\frac{d_1}{3}x^{3/2}-(\Sigma_1(x)+\Sigma_2(x)+\Sigma_3(x))\\ &= (1+o(1))\frac{d_1}{3}\rho(u/2)x^{3/2}, \end{aligned} \tag{4.16}$$

where we used the fact that $\rho(u/2)=1$ for $u\in[1,2]$.

In order to extend the range of validity of (4.16) for $u>2$, we will proceed recursively, namely by assuming that the asymptotic estimate

$$\sum_{\substack{n\leq x\\ P(n)\leq x^{1/v}}}\operatorname{pow}(n)=(1+o(1))\frac{d_1}{3}\rho(v/2)x^{3/2}\quad\text{holds for any } v\in[1,2M] \text{ as } x\to\infty \tag{4.17}$$

and then showing that it also holds for $u\in[2M,2M+2]$. Indeed we have

$$\begin{aligned}\sum_{\substack{n\leq x\\ x^{1/u}<P(n)\leq x^{1/2M}}}\operatorname{pow}(n) &= \sum_{x^{1/u}<p\leq x^{1/2M}}\sum_{\substack{m\leq \frac{x}{p^2}\\ P(m)<p}}\operatorname{pow}(mp^2)+O\left(x^{3/2-1/2u}\right)\\ &= \sum_{x^{1/u}<p\leq x^{1/2M}}p^2\sum_{\substack{m\leq \frac{x}{p^2}\\ P(m)<p}}\operatorname{pow}(m)+O\left(x^{3/2-1/2u}\right)\\ &= \sum_{x^{1/u}<p\leq x^{1/2M}}p^2\sum_{\substack{m\leq \frac{x}{p^2}\\ P(m)\leq p}}\operatorname{pow}(m)-S_1(x)+O\left(x^{3/2-1/2u}\right)\end{aligned} \tag{4.18}$$

where

$$
\begin{aligned}
S_1(x) \quad &:= \quad \sum_{x^{1/u}<p\leq x^{1/2M}} p^2 \sum_{\substack{j\leq \frac{x}{p^3} \\ P(j)\leq p}} \operatorname{pow}(jp) \\
&= \quad \sum_{x^{1/u}<p\leq x^{1/2M}} p^2 \left( \sum_{\substack{j\leq \frac{x}{p^3} \\ P(j)<p}} \operatorname{pow}(j) + \sum_{\substack{j\leq \frac{x}{p^3} \\ P(j)=p}} \operatorname{pow}(jp) \right) \\
&\ll \quad \sum_{x^{1/u}<p\leq x^{1/2M}} p^2 \left( \sum_{j\leq \frac{x}{p^3}} \operatorname{pow}(j) + \sum_{a\geq 1} \sum_{p^a k\leq \frac{x}{p^3}} p^{a+1} \operatorname{pow}(k) \right) \\
&\ll \quad \sum_{x^{1/u}<p\leq x^{1/2M}} p^2 \left( \left(\frac{x}{p^3}\right)^{3/2} + \sum_{a\geq 1} p^{a+1} \left(\frac{x}{p^{a+3}}\right)^{3/2} \right) \\
&\ll \quad x^{3/2-1/u}, \qquad (4.19)
\end{aligned}
$$

where again we used estimate (1.1). Combining (4.18) and (4.19) along with our induction hypothesis (4.17), it follows that

$$
\sum_{\substack{n\leq x \\ x^{1/u}<P(n)\leq x^{1/2M}}} \operatorname{pow}(n) = (1+o(1))\frac{d_1}{3}x^{3/2} \sum_{x^{1/u}<p\leq x^{1/2M}} \frac{1}{p}\rho\left(\frac{\log x}{2\log p} - 1\right),
$$

which we can rewrite in its integral form as

$$
\sum_{\substack{n\leq x \\ x^{1/u}<P(n)\leq x^{1/2M}}} \operatorname{pow}(n) = (1+o(1))\frac{d_1}{3}x^{3/2} \int_{x^{1/u}}^{x^{1/2M}} \frac{1}{t\log t}\rho\left(\frac{\log x}{2\log t} - 1\right) dt. \qquad (4.20)
$$

Setting $z = \frac{\log x}{2\log t}$ in (4.20), we obtain

$$
\sum_{\substack{n\leq x \\ x^{1/u}<P(n)\leq x^{1/2M}}} \operatorname{pow}(n) = (1+o(1))\frac{d_1}{3}x^{3/2} \int_{M}^{u/2} \frac{1}{z}\rho\,(z-1)\,dz.
$$

From this we get

$$
\sum_{\substack{n\leq x \\ P(n)\leq x^{1/u}}} \operatorname{pow}(n) = (1+o(1))\frac{d_1}{3}x^{3/2} \left(\rho(M) - \int_{M}^{u/2} \frac{\rho(z-1)}{z} dz\right),
$$

which, in light of (4.1), yields

$$
\sum_{\substack{n\leq x \\ P(n)\leq x^{1/u}}} \operatorname{pow}(n) = (1+o(1))\frac{d_1}{3}x^{3/2}\rho(u/2),
$$

thus proving (4.3) and completing the proof of Theorem 4. □

# 5 The sizes of $A_y(x)$ and $B_y(x)$ for small $y$

In this section, we establish the size of $A_y(x)$ and $B_y(x)$ when $y = (\log x)^\eta$ for some $\eta > 1$. We prove the following.

**Theorem 5.** *Let $\eta > 1$ be fixed. Then, as $x \to \infty$,*

$$\sum_{\substack{n \le x \\ P(n) \le (\log x)^\eta}} \mathrm{pow}(n) = x^{\frac{3}{2} - \frac{1}{2\eta} + o(1)} \tag{5.1}$$

*and*

$$\sum_{\substack{n \le x \\ P(n) \le (\log x)^\eta}} \mathrm{sq}(n) = x^{2 - \frac{1}{\eta} + o(1)}. \tag{5.2}$$

As we will soon see, Theorem 5 is a consequence of the following lemmas.

**Lemma 5.** *Given a real number $A > 1$,*

$$\Psi(x, \log^A x) = x^{1 - 1/A + o(1)} \qquad (x \to \infty).$$

*Proof.* This is relation (1.14) in the paper of Granville [7]. □

**Lemma 6.** *Let $\eta > 1$ be fixed. Then, as $x \to \infty$,*

$$F_\eta(x) := \#\{n \le x : P(n) \le (\log x)^\eta,\ n \text{ powerful}\} = x^{\frac{1}{2}\left(1 - \frac{1}{\eta}\right) + o(1)}. \tag{5.3}$$

*Proof.* Since all perfect squares are powerful numbers, we have, using Lemma 5,

$$F_\eta(x) \ge \#\{n \le x^{1/2} : P(n) \le (\log x)^\eta\} = x^{\frac{1}{2}\left(1 - \frac{1}{\eta}\right) + o(1)}. \tag{5.4}$$

On the other hand, using Lemma 5 and writing each powerful number $n \le x$ as $n = a^2 b^3$, where $b$ is squarefree, we have for any $\varepsilon > 0$,

$$\begin{aligned}
F_\eta(x) &\le \sum_{\substack{a \le \sqrt{x} \\ P(a) \le (\log x)^\eta}} \#\left\{ b \le \frac{x^{1/3}}{a^{2/3}} : P(b) \le (\log x)^\eta \right\} \\
&\le \sum_{\substack{a \le x^{\frac{1}{2} - \varepsilon} \\ P(a) \le (\log x)^\eta}} \#\left\{ b \le \frac{x^{1/3}}{a^{2/3}} : P(b) \le (\log x)^\eta \right\} + \sum_{\substack{x^{\frac{1}{2} - \varepsilon} < a \le \sqrt{x} \\ P(a) \le (\log x)^\eta}} x^{\frac{2\varepsilon}{3}} \\
&\le \sum_{\substack{a \le x^{\frac{1}{2} - \varepsilon} \\ P(a) \le (\log x)^\eta}} \#\left\{ b \le \frac{x^{1/3}}{a^{2/3}} : P(b) \le (\log x)^\eta \right\} + x^{\frac{2\varepsilon}{3}} \Psi(\sqrt{x}, (\log x)^\eta)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\substack{a \le x^{\frac{1}{2}-\varepsilon} \\ P(a)\le(\log x)^\eta}} \#\left\{ b \le \frac{x^{1/3}}{a^{2/3}} : P(b) \le (\log x)^\eta \right\} + x^{\frac{2\varepsilon}{3}} \cdot x^{\frac{1}{2}(1-\frac{1}{\eta})+o(1)} \\
&= S_1(x) + x^{\frac{1}{2}(1-\frac{1}{\eta})+O(\varepsilon)}, \qquad (5.5)
\end{aligned}
$$

say. Then, using Lemma 5 again, we obtain

$$
\begin{aligned}
S_1(x) &\le \sum_{\substack{a \le x^{\frac{1}{2}-\varepsilon} \\ P(a)\le(\log x)^\eta}} \left(\frac{x}{a^2}\right)^{\frac{1}{3}(1-\frac{1}{\eta})+o(1)} \le x^{\frac{1}{3}(1-\frac{1}{\eta})+o(1)} \sum_{\substack{a \le \sqrt{x} \\ P(a)\le(\log x)^\eta}} a^{-\frac{2}{3}(1-\frac{1}{\eta})} \\
&= x^{\frac{1}{3}(1-\frac{1}{\eta})+o(1)} \int_1^{\sqrt{x}} t^{-\frac{2}{3}(1-\frac{1}{\eta})} d\Psi(t,(\log x)^\eta) \\
&= x^{\frac{1}{3}(1-\frac{1}{\eta})+o(1)} \Bigg( t^{-\frac{2}{3}(1-\frac{1}{\eta})} \Psi(t,(\log x)^\eta) \Big|_1^{\sqrt{x}} \\
&\qquad + \frac{2}{3}\left(1-\frac{1}{\eta}\right) \int_1^{\sqrt{x}} t^{-\frac{2}{3}(1-\frac{1}{\eta})-1} \Psi(t,(\log x)^\eta)\, dt \Bigg) \\
&= x^{\frac{1}{3}(1-\frac{1}{\eta})+o(1)} \left( t^{-\frac{2}{3}(1-\frac{1}{\eta})} \Psi(t,(\log x)^\eta) \Big|_1^{\sqrt{x}} + \frac{2}{3}\left(1-\frac{1}{\eta}\right) \Big( I_1(x) + I_2(x) \Big) \right) \qquad (5.6)
\end{aligned}
$$

say, where

$$
I_1(x) = \int_1^{x^\varepsilon} t^{-\frac{2}{3}(1-\frac{1}{\eta})-1} \Psi(t,(\log x)^\eta)\, dt \quad \text{and} \quad I_2(x) = \int_{x^\varepsilon}^{\sqrt{x}} t^{-\frac{2}{3}(1-\frac{1}{\eta})-1} \Psi(t,(\log x)^\eta)\, dt.
$$

On the one hand,

$$
I_1(x) \le \int_1^{x^\varepsilon} t^{-\frac{2}{3}(1-\frac{1}{\eta})}\, dt \ll (x^\varepsilon)^{-\frac{2}{3}(1-\frac{1}{\eta})} = x^{O(\varepsilon)}. \qquad (5.7)
$$

On the other hand, using Lemma 5,

$$
I_2(x) = \int_{x^\varepsilon}^{\sqrt{x}} t^{-\frac{2}{3}(1-\frac{1}{\eta})-1} \cdot t^{1-\frac{1}{\eta}+o(1)}\, dt \ll \left(\sqrt{x}\right)^{\frac{1}{3}(1-\frac{1}{\eta})+o(1)}. \qquad (5.8)
$$

Then, combining (5.7), (5.8), (5.6) and (5.5) as well as using Lemma 5 one more time, we obtain

$$
\begin{aligned}
F_\eta(x) &\ll x^{\frac{1}{3}(1-\frac{1}{\eta})+o(1)} \left( (\sqrt{x})^{\frac{1}{3}(1-\frac{1}{\eta})+o(1)} + \frac{2}{3}\left(1-\frac{1}{\eta}\right) \left(\sqrt{x}\right)^{\frac{1}{3}(1-\frac{1}{\eta})+o(1)} \right) + x^{\frac{1}{2}(1-\frac{1}{\eta})+O(\varepsilon)} \\
&\ll x^{\frac{1}{3}(1-\frac{1}{\eta})+o(1)} \cdot x^{\frac{1}{6}(1-\frac{1}{\eta})+o(1)} + x^{\frac{1}{2}(1-\frac{1}{\eta})+O(\varepsilon)} \ll x^{\frac{1}{2}(1-\frac{1}{\eta})+O(\varepsilon)},
\end{aligned}
$$

an estimate which in combination with estimate (5.4) completes the proof of Lemma 6. □

**Lemma 7.** *Let $\eta > 1$ be fixed. Then, as $x \to \infty$,*

$$G_\eta(x,y) := \#\{n \le x : \mathrm{pow}(n) > y, P(n) \le (\log x)^\eta\} \ll \left(\frac{x}{y^{1/2}}\right)^{1-\frac{1}{\eta}+o(1)}.$$

*Proof.* Writing each integer $n \le x$ as $n = m \cdot r$, where $m$ and $r$ represent their powerful part and squarefree part, respectively, and using Lemma 5, we get for any $\varepsilon > 0$

$$\begin{aligned}
G_\eta(x,y) &\le \sum_{\substack{y<m\le x\\ m \text{ powerful}\\ P(m)\le(\log x)^\eta}} \sum_{\substack{r\le x/m\\ P(r)\le(\log x)^\eta}} \mu^2(r) \le \sum_{\substack{y<m\le x\\ m \text{ powerful}\\ P(m)\le(\log x)^\eta}} \Psi\left(\frac{x}{m},(\log x)^\eta\right)\\
&= \sum_{\substack{y<m\le x^{1-\varepsilon}\\ m \text{ powerful}\\ P(m)\le(\log x)^\eta}} \Psi\left(\frac{x}{m},(\log x)^\eta\right) + \sum_{\substack{x^{1-\varepsilon}<m\le x\\ m \text{ powerful}\\ P(m)\le(\log x)^\eta}} \Psi\left(\frac{x}{m},(\log x)^\eta\right)\\
&\le x^{1-\frac{1}{\eta}+o(1)} \sum_{\substack{y<m\le x^{1-\varepsilon}\\ m \text{ powerful}\\ P(m)\le(\log x)^\eta}} \frac{1}{m^{1-\frac{1}{\eta}}} + x^\varepsilon \sum_{\substack{x^{1-\varepsilon}<m\le x\\ m \text{ powerful}\\ P(m)\le(\log x)^\eta}} 1.
\end{aligned}$$

Recalling the definition of $F_\eta(x)$ given in (5.3), we then have, using Lemma 6,

$$\begin{aligned}
G_\eta(x,y) &\ll x^{1-\frac{1}{\eta}+o(1)} \int_y^x t^{-1+\frac{1}{\eta}}\, dF_\eta(t) + x^\varepsilon F_\eta(x)\\
&= x^{1-\frac{1}{\eta}+o(1)} \left( t^{-1+\frac{1}{\eta}} F_\eta(t)\Big|_y^x + \left(1-\frac{1}{\eta}\right)\int_y^x t^{-2+\frac{1}{\eta}} F_\eta(t)\, dt\right) + x^{\frac{1}{2}(1-\frac{1}{\eta})+O(\varepsilon)}\\
&= x^{1-\frac{1}{\eta}+o(1)} \left( t^{-\frac{1}{2}(1-\frac{1}{\eta})}\Big|_y^x + \left(1-\frac{1}{\eta}\right)\int_y^x t^{-\frac{3}{2}+\frac{1}{2\eta}}\, dt\right) + x^{\frac{1}{2}(1-\frac{1}{\eta})+O(\varepsilon)}\\
&\le x^{1-\frac{1}{\eta}+o(1)} \left(-y^{-\frac{1}{2}(1-\frac{1}{\eta})} + 2y^{-\frac{1}{2}+\frac{1}{2\eta}}\right) + x^{\frac{1}{2}(1-\frac{1}{\eta})+O(\varepsilon)}\\
&\ll \left(\frac{x}{y^{1/2}}\right)^{1-\frac{1}{\eta}+o(1)},
\end{aligned}$$

thus completing the proof of Lemma 7. □

We now have the necessary tools to prove Theorem 5.

*Proof of Theorem 5.* First observe that, using Lemma 6,

$$\begin{aligned}
\sum_{\substack{n\le x\\ P(n)\le(\log x)^\eta}} \mathrm{pow}(n) &\ge \sum_{\substack{n\le x\\ P(n)\le(\log x)^\eta\\ n \text{ powerful}}} n = \int_1^x t\, dF_\eta(t)\, dt = t\, F_\eta(t)\big|_1^x - \int_1^x F_\eta(t)\, dt\\
&= t^{1+\frac{1}{2}(1-\frac{1}{\eta})+o(1)}\Big|_1^x - \int_1^x t^{\frac{1}{2}(1-\frac{1}{\eta})+o(1)}\, dt \gg x^{\frac{3}{2}-\frac{1}{2\eta}+o(1)}. \qquad (5.9)
\end{aligned}$$

Similarly, using Lemmas 5 and 6, we have

$$\begin{aligned}
\sum_{\substack{n\le x\\ P(n)\le(\log x)^\eta}} \mathrm{pow}(n) &\le \sum_{\substack{m\le x\\ m \text{ powerful}\\ P(m)\le(\log x)^\eta}} m \sum_{\substack{r\le x/m\\ P(r)\le(\log x)^\eta}} \mu^2(r) \le \sum_{\substack{m\le x\\ m \text{ powerful}\\ P(m)\le(\log x)^\eta}} m\Psi\left(\frac{x}{m},(\log x)^\eta\right)\\
&= \sum_{\substack{m\le x^{1-\varepsilon}\\ m \text{ powerful}\\ P(m)\le(\log x)^\eta}} m\left(\frac{x}{m}\right)^{1-\frac{1}{\eta}+o(1)} + \sum_{\substack{x^{1-\varepsilon}<m\le x\\ m \text{ powerful}\\ P(m)\le(\log x)^\eta}} m\Psi\left(\frac{x}{m},(\log x)^\eta\right)\\
&\le x^{1-\frac{1}{\eta}+o(1)} \sum_{\substack{m\le x\\ m \text{ powerful}\\ P(m)\le(\log x)^\eta}} m^{\frac{1}{\eta}} + x^\varepsilon \sum_{\substack{m\le x\\ m \text{ powerful}\\ P(m)\le(\log x)^\eta}} m\\
&= x^{1-\frac{1}{\eta}+o(1)}\int_1^x t^{\frac{1}{\eta}}\, d\,F_\eta(t) + x^\varepsilon\int_1^x t\, dF_\eta(x)\\
&\ll x^{1-\frac{1}{\eta}+o(1)}\cdot x^{\frac{1}{\eta}+\frac{1}{2}-\frac{1}{2\eta}} + x^\varepsilon\cdot x^{\frac{3}{2}-\frac{1}{2\eta}+o(1)} \ll x^{\frac{3}{2}-\frac{1}{2\eta}+O(\varepsilon)}. \qquad (5.10)
\end{aligned}$$

Combining (5.9) and (5.10) and taking $\varepsilon$ arbitrarily small, estimate (5.1) of Theorem 5 follows.

It remains to prove estimate (5.2). First, using Lemma 5, we have

$$\begin{aligned}
\sum_{\substack{n\le x\\ P(n)\le(\log x)^\eta}} \mathrm{sq}(n) &\le \sum_{\substack{n\le x\\ P(n)\le(\log x)^\eta}} n = \int_1^x t d\Psi(t,(\log x)^\eta)\\
&= t\,\Psi(t,(\log x)^\eta)|_1^x - \int_1^x \Psi(t,(\log x)^\eta)\,dt\\
&\le x\cdot x^{1-\frac{1}{\eta}+o(1)} - \int_{x^\varepsilon}^x \Psi(t,(\log x)^\eta)\,dt\\
&\ll x^{2-\frac{1}{\eta}+o(1)}. \qquad (5.11)
\end{aligned}$$

On the other hand, given an arbitrary small number $\varepsilon>0$, we have, using Lemma 7,

$$\begin{aligned}
\sum_{\substack{n\le x\\ P(n)\le(\log x)^\eta}} \mathrm{sq}(n) &= \sum_{\substack{n\le x\\ P(n)\le(\log x)^\eta}} \frac{n}{\mathrm{pow}(n)} \ge x^{-\varepsilon} \sum_{\substack{n\le x\\ P(n)\le(\log x)^\eta\\ \mathrm{pow}(n)\le x^\varepsilon}} n \ge x^{-\varepsilon} \sum_{\substack{x^{1-\varepsilon}<n\le x\\ P(n)\le(\log x)^\eta\\ \mathrm{pow}(n)\le x^\varepsilon}} x^{1-\varepsilon}\\
&= x^{1-2\varepsilon}\left(\Psi(x,(\log x)^\eta) - G_\eta(x,x^\varepsilon) - \Psi(x^{1-\varepsilon},(\log x)^\eta) + G_\eta(x^{1-\varepsilon},x^\varepsilon)\right)\\
&\gg x^{1-2\varepsilon}x^{1-\frac{1}{\eta}+o(1)} = x^{2-\frac{1}{\eta}-2\varepsilon+o(1)}. \qquad (5.12)
\end{aligned}$$

Since $\varepsilon$ can be chosen arbitrarily small, combining (5.11) and (5.12), estimate (5.2) follows, thus completing the proof of Theorem 5. □

## 6 Determining $y$ such that $A_y(x)$ and $B_y(x)$ are of the same order

In this section, the functions

$$g(x) := x \log x - (x-1)\log(x-1)$$

and

$$h(x) := \max_{0<b<1/2} \left( x\log x - (x-b)\log(x-b) - 2b\log b + (1-b)\log(1-b) - (1-2b)\log(1-2b) \right),$$

both defined for real $x > 1$, play an important role.

Our goal is to show that when $y = C\log x$ the expressions $\log(A_y(x)/x)$ and $\log(B_y(x)/x)$ are equal to $(1+o(1))h(C)\dfrac{\log x}{\log\log x}$ and $(1+o(1))g(C)\dfrac{\log x}{\log\log x}$ respectively, so that by choosing $C$ as the solution of $h(C) = g(C)$, both expressions will be asymptotic as $x \to \infty$.

**Theorem 6.** *The estimate*

$$\log(A_y(x)/x) = (1+o(1))\log(B_y(x)/x) \qquad (x\to\infty)$$

*holds for $y = y(x) = 2\log x$.*

The proof of Theorem 6 will follow from a series of lemmas.

**Lemma 8.** *The only solution $x > 1$ to $g(x) = h(x)$ is $x = 2$, in which case $g(2) = h(2) = 2\log 2$.*

*Proof.* Clearly, $g(2) = 2\log 2$. We will prove that $g(2) = h(2)$ and then show that the solution is unique. First observe that

$$\begin{aligned} h(2) - g(2) &= \max_{0<b<1/2} \left(-(2-b)\log(2-b) - 2b\log b + (1-b)\log(1-b) - (1-2b)\log(1-2b)\right) \\ &= \max_{0<b<1/2} f(b), \end{aligned}$$

say. Differentiating $f(b)$ with respect to $b$ yields

$$f'(b) = \log(2-b) - 2\log b - \log(1-b) + 2\log(1-2b).$$

We then have

$$f'(b) = 0 \Longleftrightarrow \frac{(2-b)(1-2b)^2}{b^2(1-b)} = 1 \Longleftrightarrow (3b-2)(b^2-3b+1) = 0,$$

from which we get $b = b_0 := \frac{3-\sqrt{5}}{2}$ as the only solution to $f'(b) = 0$ in $(0, 1/2)$.

It remains to show that $f(b_0) = 0$. This follows directly from the fact that

$$f(b) = bf'(b) - 2\log(2-b) + \log(1-b) - \log(1-2b),$$

which implies that

$$\begin{aligned} f(b_0) = 0 \iff & -2\log(2-b_0) + \log(1-b_0) - \log(1-2b_0) = 0 \\ \iff & \frac{1-b_0}{(2-b_0)^2(1-2b_0)} = 1 \iff (2b_0-3)(b_0^2 - 3b_0 + 1) = 0, \end{aligned}$$

completing the first part of the proof. We also have

$$g'(x) - h'(x) \geq (\log x - \log(x-1)) - \max_{0<b<\frac{1}{2}} (\log x - \log(x-b)) > \log\left(x - \frac{1}{2}\right) - \log(x-1) > 0,$$

so that $g(x)$ is increasing at a faster rate than $h(x)$ and thus proving the solution $x = 2$ to be unique. □

**Lemma 9.** *Given any real number $\kappa > 0$,*

$$\Psi(x, \kappa \log x) = \exp\left\{ g(\kappa+1) \frac{\log x}{\log\log x} \left(1 + O\left(\frac{1}{\log\log x}\right)\right)\right\}. \tag{6.1}$$

*Proof.* This is the first relation on page 271 of the paper of Granville [7]. □

We now introduce the function

$$L(x) := \frac{\log x}{\log\log x}.$$

**Lemma 10.** *Given arbitrary numbers $C > D > 0$ and integer valued functions $a(x)$ and $b(x)$ satisfying $a(x) = (1+o(1))C\,L(x)$ and $b(x) = (1+o(1))D\,L(x)$, then, as $x \to \infty$,*

$$\binom{a(x)}{b(x)} = \exp\left\{(1+o(1))(C\log C - D\log D - (C-D)\log(C-D))L(x)\right\}.$$

*Proof.* Using the well known Stirling formula $n! = (1+o(1))n^n e^{-n}\sqrt{2\pi n}$ as $n \to \infty$, we have, as $x \to \infty$,

$$\begin{aligned} \binom{a(x)}{b(x)} &= \frac{a(x)!}{b(x)!\,(a(x)-b(x))!} \\ &= (1+o(1)) \frac{\sqrt{2\pi a(x)}a(x)^{a(x)}}{\sqrt{2\pi b(x)}b(x)^{b(x)} \cdot \sqrt{2\pi(a(x)-b(x))}\,(a(x)-b(x))^{a(x)-b(x)}} \\ &= (1+o(1)) \frac{1}{\sqrt{2\pi}} \frac{\sqrt{C}}{\sqrt{D}\sqrt{(C-D)L(x)}} \frac{((1+o(1))C)^{a(x)}}{((1+o(1))D)^{b(x)}((1+o(1))(C-D))^{a(x)-b(x)}} \\ &= \exp\left\{(1+o(1))(C\log C - D\log D - (C-D)\log(C-D))L(x)\right\}. \end{aligned}$$

□

**Lemma 11.** *As $x \to \infty$,*

$$\sum_{\substack{n \le x \\ P(n) \le C \log x}} \mathrm{sq}(n) = x e^{(1+o(1)) g(C) L(x)}.$$

*Proof.* To prove this result, we proceed in two steps, establishing upper and lower bounds. We start with the upper bound. Let $\varepsilon > 0$ be an arbitrarily small number. It is clear that

$$\sum_{\substack{n \le x \\ P(n) \le C \log x}} \mathrm{sq}(n) \le x \#\{n \le x : P(n) \le C \log x,\ \mathrm{sq}(n) > x^{1-\varepsilon}\} + x^{1-\varepsilon} \Psi(x, C \log x). \quad (6.2)$$

It follows from Lemma 9 that

$$x^{1-\varepsilon} \Psi(x, C \log x) = o(x). \quad (6.3)$$

Since $n = \mathrm{sq}(n) \,\mathrm{pow}(n)$, we have

$$\#\{n \le x : P(n) \le C \log x,\ \mathrm{sq}(n) > x^{1-\varepsilon}\} \le E_1(x) \cdot E_2(x), \quad (6.4)$$

where

$$\begin{aligned} E_1(x) &:= \#\{m \le x^{\varepsilon} : m \text{ powerful}, P(m) \le C \log x\}, \\ E_2(x) &:= \#\{x^{1-\varepsilon} < m \le x : m \text{ squarefree}, P(m) \le C \log x\}. \end{aligned}$$

Using Lemma 9, we get

$$E_1(x) \le \Psi(x^{\varepsilon}, C \log x) = e^{O(\varepsilon L(x))}. \quad (6.5)$$

To estimate $E_2(x)$, we will need information on the size of $\omega(m)$ for those integers $m$ counted by $E_2(x)$. On the one hand,

$$x^{1-\varepsilon} < \mathrm{sq}(m) = \prod_{p \mid \mathrm{sq}(m)} p \le (C \log x)^{\omega(\mathrm{sq}(m))},$$

which leads to

$$\omega(\mathrm{sq}(m)) \ge (1 - \varepsilon)\,(1 + o\,(1))\, L(x). \quad (6.6)$$

On the other hand, it was proved by Robin [9] that, for $n > e^e$,

$$\omega(n) \le L(n) + 1.45743 \frac{L(n)}{\log \log n}$$

with equality when $n = \prod_{i=1}^{47} p_i$. Hence it follows that for all $n \le x$,

$$\omega(\mathrm{sq}(n)) \le \omega(n) \le (1 + o(1)) L(x) \qquad (x \to \infty). \quad (6.7)$$

Combining equations (6.6) and (6.7), we obtain

$$\omega(\mathrm{sq}(n)) = (1 + O(\varepsilon))\, L(x). \tag{6.8}$$

We can now get an upper bound for $E_2(x)$. Equation (6.8) tells us that there exists $\delta = \delta(\varepsilon) > 0$ tending to 0 with $\varepsilon$ such that

$$\begin{aligned} E_2(x) &\leq \sum_{(1-\delta)L(x)<r<(1+\delta)L(x)} \#\{n \leq x : P(n) \leq C \log x, \omega(n) = r, \mu^2(n) = 1\} \\ &\leq \sum_{(1-\delta)L(x)<r<(1+\delta)L(x)} \binom{\pi(C\log x)}{r} = \exp\{(g(C) + o(1))L(x)\}, \end{aligned} \tag{6.9}$$

where the last equality follows as in the proof of Lemma 10 after observing that, in light of (1.8),

$$\pi(C \log x) = (1 + o(1))\frac{C \log x}{\log\log x} = (1 + o(1))C\, L(x).$$

Using (6.5) and (6.9) in (6.4), and then taking into account (6.3) and (6.2), we have proven the upper bound implied in Lemma 11.

It remains to prove the lower bound. Let us first choose a squarefree number $t < x$ such that $\varepsilon \log x < p(t) < P(t) \leq C \log x$ and also satisfying

$$\omega(t) = \ell_1(x) := \left\lfloor \left(1 - \frac{\varepsilon}{2}\right) L(x) \right\rfloor.$$

For such an integer $t$, using (1.9) and then (1.11), we have

$$t = \prod_{p|t} p \geq \prod_{p \leq p_{\omega(t)}} p \geq e^{\omega(t)\log(\omega(t))(1+o(1))} \geq x^{1-\frac{\varepsilon}{2}+o(1)} \tag{6.10}$$

and

$$t \leq (C \log x)^{\omega(t)} \leq x^{1-\frac{\varepsilon}{2}+o(1)}, \quad \text{so that } t = x^{1-\frac{\varepsilon}{2}+o(1)}. \tag{6.11}$$

Now define $p_t$ as the largest prime number such that

$$t \cdot \prod_{p \leq p_t} p \leq x. \tag{6.12}$$

Note that such a prime $p_t$ exists because of (6.11). It follows from (6.12), using (1.10), that, as $x$ (and thus $t$) tends to infinity,

$$\begin{aligned} &\log t + \sum_{p \leq p_t} \log p \leq \log x, \\ &\log t + (1 + o(1))p_t \leq \log x, \\ &(1 + o(1))p_t \leq \log x - (1 - \varepsilon/2 + o(1)) \log x = (\varepsilon/2 + o(1)) \log x, \end{aligned}$$

where we used (6.10), thus implying that, if $x$ is sufficiently large,

$$p_t \le \left(\frac{\varepsilon}{2} + o(1)\right) \log x < \varepsilon \log x.$$

This allows us to choose an integer $s_t$ defined by $s_t := \prod_{p \le p_t} p$ such that, in light of (6.12), $n = s_t \cdot t \le x$. Observe that we also have

$$\frac{x}{\log x} \le n = s_t \cdot t \tag{6.13}$$

because if we had $s_t \cdot t < \frac{x}{\log x}$, then multiplying both side by $\varepsilon \log x$ would give us $\varepsilon \log x \cdot s_t \cdot t < \varepsilon x < x$, which would contradict the definition of $p_t$. Using (6.13), we therefore have

$$\begin{aligned} \sum_{\substack{n \le x \\ P(n) \le C \log x}} \mathrm{sq}(n) &\ge \frac{x}{\log x} \# \left\{ t \le x : \mu^2(t) = 1, \varepsilon \log x < p(t) < P(t) \le C \log x, \omega(t) = \ell_1(x) \right\} \\ &\ge \frac{x}{\log x} \binom{\pi(C \log x) - \pi(\varepsilon \log x)}{\ell_1(x)}. \end{aligned} \tag{6.14}$$

Finally, observing that as a consequence of (1.8),

$$\pi(C \log x) - \pi(\varepsilon \log x) = (1 + o(1)) C L(x)$$

and that $\ell_1(x) = (1 + o(1)) L(x)$, it follows from (6.14) and Lemma 10 that

$$\sum_{\substack{n \le x \\ P(n) \le C \log x}} \mathrm{sq}(n) \ge x e^{(g(C) + o(1)) L(x)},$$

thus completing the proof of the lower bound and thereby that of Lemma 11. □

**Lemma 12.** *As $x \to \infty$, we have*

$$\sum_{\substack{n \le x \\ P(n) \le C \log x}} \mathrm{pow}(n) = x e^{(1 + o(1)) h(C) L(x)}. \tag{6.15}$$

*Proof.* We begin by proving the implied lower bound by using the trivial inequality

$$\sum_{\substack{n \le x \\ P(n) \le C \log x}} \mathrm{pow}(n) \ge \frac{x}{2} \# \left\{ \frac{x}{2} < n \le x : P(n) \le C \log x, n \text{ powerful} \right\}. \tag{6.16}$$

In this last set, if we focus only on those $n = 2^a \cdot m$ with $m$ odd and $a \ge 2$, we may write

$$\# \left\{ \frac{x}{2} < n \le x : P(n) \le C \log x, n \text{ powerful} \right\}$$

$$\geq \#\left\{m \leq \frac{x}{8} : P(m) \leq C\log x, m \text{ odd, powerful}\right\}.$$

Letting $\varepsilon > 0$ be a small number, we further focus our attention on those integers $m \leq x/8$ such that $\Omega(m) = \ell_2(x) := \lfloor (1-\varepsilon)L(x) \rfloor$, implying that we may write

$$\begin{aligned} &\#\left\{m \leq \frac{x}{8} : P(m) \leq C\log x, m \text{ odd and powerful}\right\} \\ &\quad \geq \#\{m \leq \frac{x}{8} : P(m) \leq C\log x, \Omega(m) = \ell_2(x), m \text{ odd and powerful}\}. \end{aligned} \tag{6.17}$$

Note that the two conditions $\Omega(m) = \ell_2(x)$ and $P(m) \leq C\log x$ imply that

$$m \leq (C\log x)^{\ell_2(x)} \leq x^{(1-\varepsilon)(1+o(1))},$$

thus allowing us to replace $m \leq \frac{x}{8}$ by $m \geq 1$ in the last set appearing in (6.17).

We now write each integer $m$ in that set as $m = s^2 \cdot t$ where $s = \prod_{p|m} p$ and $t = m/s^2$. Observe that for each such integer $m$, we have

$$\Omega(t) = \Omega(m) - \Omega(s^2) = \Omega(m) - 2\omega(s) = \Omega(m) - 2b\Omega(m) = (1-2b)\ell_2(x)$$

for some $b \in (0, 1/2)$. It follows from these observations that

$$\begin{aligned} &\#\{m \geq 1 : P(m) \leq C\log x, \Omega(m) = \ell_2(x), m \text{ odd and powerful}\} \\ &\quad \geq \max_{0<b<1/2} \sum_{\substack{s\geq 1 \\ \mu^2(s)=1 \\ P(s)\leq C\log x \\ \omega(s)=b\ell_2(x) \\ s \text{ odd}}} \#\{t \geq 1 : \Omega(t) = (1-2b)\ell_2(x), p|t \Rightarrow p|s\}. \end{aligned} \tag{6.18}$$

Since $\omega(s) = b\ell_2(x)$, we obtain

$$\begin{aligned} \#\{t \geq 1 : \Omega(t) = (1-2b)\ell_2(x), p|t \Rightarrow p|s\} &= \binom{\Omega(t) + \omega(s) - 1}{\Omega(t)} \\ &= \binom{(1-2b)\ell_2(x) + b\ell_2(x) - 1}{(1-2b)\ell_2(x)} \\ &= \binom{(1-b)\ell_2(x) - 1}{(1-2b)\ell_2(x)} \end{aligned} \tag{6.19}$$

and we also have

$$\#\{s \geq 1 : \mu^2(s) = 1, P(s) \leq C\log x, \omega(s) = b\ell_2(x), s \text{ odd}\} = \binom{\pi(C\log x) - 1}{b\ell_2(x)}. \tag{6.20}$$

From estimates (6.18), (6.19) and (6.20), we obtain

$$\#\{m \geq 1 : P(m) \leq C\log x, \Omega(m) = \ell_2(x), m \text{ odd and powerful }\}$$

$$\geq \max_{0<b<1/2} \binom{(1-b)\ell_2(x)-1}{(1-2b)\ell_2(x)} \binom{\pi(C\log x)-1}{b\ell_2(x)}$$
$$= \exp\left((1+o(1))h(C)L(x)\right), \tag{6.21}$$

where the last equality follows from Lemma 10. The proof of the lower bound in (6.15) then follows from (6.16) and (6.21).

We now prove the upper bound implied in Lemma 12. Let $\varepsilon > 0$ be small. We start with the trivial inequality

$$\sum_{\substack{n\leq x\\ P(n)\leq C\log x}} \mathrm{pow}(n) \leq x\#\{n \leq x : \mathrm{pow}(n) > x^{1-\varepsilon}, P(n) \leq C\log x\} + x^{1-\varepsilon}\Psi\left(x, C\log x\right). \tag{6.22}$$

First observe that, using Lemma 9 and provided that $x$ is large enough, we have

$$x^{1-\varepsilon}\Psi\left(x, C\log x\right) \leq x^{1-\varepsilon}x^{\varepsilon/2} = x^{1-\varepsilon/2}. \tag{6.23}$$

On the other hand, writing each integer $n \in \{n \leq x : \mathrm{pow}(n) > x^{1-\varepsilon}, P(n) \leq C\log x\}$ as $n = \mathrm{pow}(n)\cdot \mathrm{sq}(n) = m\cdot r$, we have

$$\#\{n \leq x : \mathrm{pow}(n) > x^{1-\varepsilon}, P(n) \leq C\log x\} \leq A(x)\times B(x), \tag{6.24}$$

where

$$A(x) := \#\{r \leq x^{\varepsilon}, \mu^2(r) = 1, P(r) \leq C\log x\}$$

and

$$B(x) := \#\{m \leq x : m \text{ powerful}, P(m) \leq C\log x\}.$$

Using Lemma 9, we obtain

$$A(x) \leq \Psi(x^{\varepsilon}, C\log x) = \exp\left(O(\varepsilon)L(x)\right). \tag{6.25}$$

Concerning $B(x)$, we have

$$B(x) \leq \#B_1(x)\times \#B_2(x), \tag{6.26}$$

where

$$\begin{aligned} B_1(x) &:= \{m \leq x : m \text{ powerful}, P(m) \leq \varepsilon\log x\},\\ B_2(x) &:= \{m \leq x : m \text{ powerful}, \varepsilon\log x < p(m) \leq P(m) \leq C\log x\}. \end{aligned}$$

On the one hand, using Lemma 9, we have

$$\#B_1(x) \leq \Psi(x, \varepsilon\log x) \leq \left(\frac{1}{\varepsilon}\right)^{\pi(\varepsilon\log x)} = \exp\left(O(\varepsilon)\frac{\log x}{\log\log x}\right). \tag{6.27}$$

On the other hand, for each $m \in B_2(x)$, we have $(\varepsilon\log x)^{\Omega(m)} < m \leq x$, and therefore, for some small $\delta > 0$,

$$\Omega(m) \leq \frac{\log x}{\log\log x + \log\varepsilon} = (1+\delta)L(x),$$

provided $x$ is large enough. It follows that

$$\begin{aligned}\#B_2(x) &\leq \#\{m \leq x : P(m) \leq C\log x, \Omega(m) < (1+\delta)L(x)\} \\ &\leq (1+\delta)L(x) \times \#\{m \leq x : P(m) \leq C\log x, \Omega(m) = \lfloor (1+\delta)L(x)\rfloor\} \\ &\leq L(x)^2 \\ &\quad \times \max_{0<b<1/2} \#\{m \leq x : P(m) \leq C\log x, \Omega(m) = \lfloor (1+\delta)L(x)\rfloor, \omega(m) = \lfloor bL(x)\rfloor\}.\end{aligned}$$

As we did in the proof of the lower bound, namely when establishing (6.21), it follows from this last series of inequalities that

$$\begin{aligned}&\max_{0<b<1/2} \#\{n \leq x : P(n) \leq C\log x, \Omega(n) = \lfloor (1+\delta)L(x)\rfloor, \omega(n) = \lfloor bL(x)\rfloor\} \\ &\quad = \exp\left((1+o(1))h(C)L(x)\right). \qquad (6.28)\end{aligned}$$

Gathering estimates (6.26) through (6.28), we obtain that

$$B(x) \leq \exp\left((1+O(\varepsilon))h(C)L(x)\right). \qquad (6.29)$$

Collecting estimates (6.22) through (6.25) and also (6.29), the proof of the upper bound implied in Lemma 12 then follows, thus completing the proof of Lemma 12. □

*Proof of Theorem 6.* The proof of Theorem 6 is an immediate consequence of Lemmas 11 and 12. □

**Remark 3.** *While Lemma 12 is valid only if $y = C\log x$ with $C \geq 1$, the domain of validity of Lemma 11 can be extended to $y = C\log x$ for any $C > 0$.*

**Remark 4.** *As is shown in [2], considering for instance only powers of 2, for any $y > 2$, we have $A_y(x) > x/2$.*

**Remark 5.** *For any $\varepsilon > 0$, if $C < 1-\varepsilon$ and $y \leq C\log x$, then*

$$B_y(x) < \left(\prod_{p<y} p\right)\Psi(x,y) < e^{C\log x + o(1)} = x^{C+o(1)}.$$

# 7 Final remarks

Collecting our results from sections 4, 5 and 6, it is interesting to observe that the behaviors of $A_y(x)$ and $B_y(x)$ vary greatly depending on the relative size of $y = y(x)$. Indeed, we obtained asymptotic estimates for $A_y(x)$ and $B_y(x)$ when $y = x^{1/u}$, as $x \to \infty$, namely

$$A_y(x) = (1+o(1))\rho(u/2)\frac{d_1}{3}x^{3/2} \quad \text{and} \quad B_y(x) = (1+o(1))\rho(u)\frac{d_2}{2}x^2.$$

In other ranges, we obtained less precise but nevertheless interesting estimates. In particular, when $y = (\log x)^\eta$ for some $\eta > 1$, we obtained asymptotic formulas for $\log(A_y(x))/\log x$ and $\log(B_y(x))/\log x$, that is,

$$A_y(x) = x^{\frac{3}{2}-\frac{1}{2\eta}+o(1)} \quad \text{and} \quad B_y(x) = x^{2-\frac{1}{\eta}+o(1)} \qquad (x \to \infty).$$

For even smaller values of $y$, that is when $y = y(x) = 2\log x$, we obtained

$$\log(A_y(x)/x) = (1 + o(1))\log(B_y(x)/x) \qquad (x \to \infty),$$

since we proved that both $A_y(x)$ and $B_y(x)$ are equal to $x \exp\left\{(1 + o(1))2\log 2 \frac{\log x}{\log\log x}\right\}$ as $x \to \infty$.

Gathering our results, we obtain that given any $\varepsilon > 0$, if $y < (2-\varepsilon)\log x$, then $B_y(x) > A_y(x)$. On the other hand, although we did not prove it, we believe that $A_y(x) > B_y(x)$ when $y > (2+\varepsilon)\log x$. While this is the case in every range investigated in the paper, a rigorous proof would require studying $A_y(x)$ and $B_y(x)$ in the ranges not covered by the present work, an interesting challenge.

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