Constructing normal numbers using residues of selective prime factors of integers

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Dedicated to Professor András Benczur on the occasion of his seventieth anniversary

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Abstract

Given an integer $N \geq 1$, for each integer $n \in J_N := [e^N, e^{N+1})$, let $q_N(n)$ be the smallest prime factor of n which is larger than N; if no such prime factor exists, set $q_N(n) = 1$. Fix an integer $Q \geq 3$ and consider the function $f(n) = f_Q(n)$ defined by $f(n) = \ell$ if $n \equiv \ell \pmod{Q}$ with $(\ell, Q) = 1$ and by $f(n) = \Lambda$ otherwise, where Λ stands for the empty word. Then consider the sequence $(\kappa(n))_{n\geq 1} = (\kappa_Q(n))_{n\geq 1}$ defined by $\kappa(n) = f(q_N(n))$ if $n \in J_N$ with $q_N(n) > 1$ and by $\kappa(n) = \Lambda$ if $n \in J_N$ with $q_N(n) = 1$. Then, for each integer $N \geq 1$, consider the concanetation of the numbers $\kappa(1), \kappa(2), \ldots$, that is define $\theta_N := \mathrm{Concat}(\kappa(n) : n \in J_N)$. Then, set $\alpha_Q := \mathrm{Concat}(\theta_N : N = 1, 2, 3, \ldots)$. Finally, let $B_Q = \{\ell_1, \ell_2, \ldots, \ell_{\varphi(Q)}\}$ be the set of reduced residues modulo Q, where φ stands for the Euler function. We show that α_Q is a normal sequence over B_Q.

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1 Introduction

In previous papers ([1], [2], [3]), we showed how one could construct normal numbers by concatenating the digits of the numbers $P(2)$, $P(3)$, $P(4), \ldots$, where $P(n)$ stands for the largest prime factor of n, then similarly by using the k-th largest prime factor instead of the largest prime factor and finally by doing the same replacing $P(n)$ by $p(n)$, the smallest prime factor of n.

Here, we consider a different approach which uses the residue modulo an integer $Q \geq 3$ of the smallest element of a particular set of prime factors of an integer n. But first, we need to set the table.

For a given integer $Q \geq 3$, let $A_Q := \{0, 1, \ldots, Q-1\}$. Given an integer $t \geq 1$, an expression of the form $i_1 i_2 \ldots i_t$, where each $i_j \in A_Q$, is called a *finite word* of length

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t. The symbol Λ will denote the *empty word*. We let A_Q^t stand for the set of all words of length t. An infinite sequence of digits $a_1a_2\ldots$, where each $a_i \in A_Q$, is called an *infinite word*.

An infinite sequence $a_1a_2\ldots$ of base Q digits is called a *normal sequence* over A_Q if any preassigned sequence of k digits occurs at the expected frequency of $1/Q^k$.

Given a fixed integer $Q \geq 3$, let

$$f_Q(n) := \begin{cases} \Lambda & \text{if} \quad (n,Q) \neq 1, \\ \ell & \text{if} \quad n \equiv \ell \pmod Q, \quad (\ell, Q) = 1. \end{cases} \tag{1.1}$$

Write $p_1 < p_2 < \cdots$ for the sequence of consecutive primes, and consider the infinite word

$$\xi_Q = f_Q(p_1)f_Q(p_2)f_Q(p_3)\ldots$$

Let

$$B_Q = \{\ell_1, \ell_2, \ldots, \ell_{\varphi(Q)}\}$$

be the set of reduced residues modulo Q, where φ stands for the Euler totient function.

In an earlier paper [4], we conjectured that the word ξ_Q is a normal sequence over B_Q in the sense that given any integer $k \geq 1$ and any word $\beta = r_1 \ldots r_k \in B_Q^k$, and further setting

$$\xi_Q^{(N)} = f_Q(p_1)f_Q(p_2)\ldots f_Q(p_N) \quad \text{for each} \quad N \in \mathbb{N}$$

and

$$M_N(\xi_Q|\beta) := \#\{(\gamma_1,\gamma_2)|\xi_Q^{(N)} = \gamma_1\beta\gamma_2\},$$

we have

$$\lim_{N\to\infty} \frac{M_N(\xi_Q|\beta)}{N} = \frac{1}{\varphi(Q)^k}.$$

In this paper, we consider a somewhat similar but more simple problem, namely by using the residue of the smallest prime factor of n (modulo Q) which is larger than a certain quantity, and this time we obtain an effective result.

2 Main result

Given an integer $N \geq 1$, for each integer $n \in J_N := [x_N, x_{N+1}) := [e^N, e^{N+1})$, let $q_N(n)$ be the smallest prime factor of n which is larger than N; if no such prime factor exists, set $q_N(n) = 1$. Fix an integer $Q \geq 3$ and consider the function $f(n) = f_Q(n)$ defined by (1.1). Then consider the sequence $(\kappa(n))_{n\geq 1} = (\kappa_Q(n))_{n\geq 1}$ defined by $\kappa(n) = f(q_N(n))$ if $n \in J_N$ with $q_N(n) > 1$ and by $\kappa(n) = \Lambda$ if $n \in J_N$ with $q_N(n) = 1$. Then, for each integer $N \geq 1$, consider the concatenation of $\kappa(1), \kappa(2), \kappa(3), \ldots$, that is define

$$\theta_N := \text{Concat}(\kappa(n) : n \in J_N).$$

Then, settting

$$\alpha_Q := \text{Concat}(\theta_N : N = 1, 2, 3, \ldots),$$

we will prove the following result.

Theorem 1. *The sequence α_Q is a normal sequence over B_Q.*

3 Proof of the main result

We first introduce the notation $\lambda_N = \log\log N$. Moreover, from here one, the letters p and π, with or without subscript, always stand for primes. Finally, let $\wp$ stand for the set of all primes.

Fix an arbitrary large integer N and consider the interval $J := [x, x+y] \subseteq J_N$. Let $p_1, p_2, \ldots, p_k$ be k distinct primes belonging to the interval $(N, N^{\lambda_N}]$. Then, set

$$S_J(p_1, p_2, \ldots, p_k) := \#\{n \in J : q_N(n+j) = p_j \text{ for } j = 1, 2, \ldots, k\}.$$

We know by the Chinese Remainder Theorem that the system of congruences (*) $n + j \equiv 0 \pmod{p_j}$, $j = 1, 2, \ldots, k$, has a unique solution $n_0 < p_1 p_2 \cdots p_k$ and that any solution $n \in J$ of (*) is of the form

$$n = n_0 + m p_1 p_2 \cdots p_k \quad \text{for some non negative integer } m.$$

Let us now reorder the primes $p_1, p_2, \ldots, p_k$ as

$$p_{i_1} < p_{i_2} < \cdots < p_{i_k}.$$

If $\pi \in \wp$ and $N < \pi < p_{i_1}$, it is clear that we will have $(n+j, \pi) = 1$ for each $j \in \{1, 2, \ldots, k\}$. Similarly, if $\pi \in \wp$ and $p_{i_1} < \pi < p_{i_2}$, then $(n+j, \pi) = 1$ for each $j \in \{1, 2, \ldots, k\} \setminus \{i_1\}$, and so on. Let us now introduce the function $\rho : \wp \cap (N, p_{i_k}] \to \{0, 1, 2, \ldots, k\}$ defined by

$$\rho(\pi) = \begin{cases} k & \text{if } N < \pi < p_{i_1}, \\ k-1 & \text{if } p_{i_1} < \pi < p_{i_2}, \\ \vdots & \qquad \vdots \\ 1 & \text{if } p_{i_{k-1}} < \pi < p_{i_k}, \\ 0 & \text{if } \pi \in \{p_1, p_2, \ldots, p_k\}. \end{cases}$$

By using the Eratosthenian sieve (see for instance the book of Halberstam and Richert [5]), we easily obtain that, as $y \to \infty$,

$$S_J(p_1, \ldots, p_k) = (1 + o(1)) \frac{y}{p_1 \cdots p_k} \prod_{N < \pi < p_{i_k}} \left(1 - \frac{\rho(\pi)}{\pi}\right). \tag{3.1}$$

Setting $U := \prod_{N<\pi<p_{i_k}} \left(1 - \frac{\rho(\pi)}{\pi}\right)$, one can see that, as $N \to \infty$,

$$\begin{aligned} \log U &= k \log\log N - k \log\log p_{i_1} - (k-1)\log\log p_{i_2} + (k-1)\log\log p_{i_1} \\ &\qquad - \cdots - \log\log p_{i_k} + \log\log p_{i_{k-1}} + o(1) \\ &= k \log\log N - \log\log p_{i_1} - \cdots - \log\log p_{i_k} + o(1), \end{aligned}$$

implying that

$$U = (1+o(1)) \prod_{j=1}^{k} \frac{\log N}{\log p_j} \qquad (N \to \infty). \tag{3.2}$$

Hence, in light of (3.2), relation (3.1) can be replaced by

$$S_J(p_1, \ldots, p_k) = (1+o(1)) \frac{y}{p_1 \cdots p_k} \prod_{j=1}^{k} \frac{\log N}{\log p_j} \qquad (y \to \infty). \tag{3.3}$$

Now let $r_1, \ldots, r_k$ be an arbitrary collection of reduced residues modulo Q and let us define

$$B_y(r_1, \ldots, r_k) := \sum_{\substack{p_j \equiv r_j \pmod Q \\ N < p_j \le N^{\lambda_N} \\ j=1,\ldots,k}} S_J(p_1, \ldots, p_k).$$

From the Prime Number Theorem in arithmetic progressions, we have that

$$\sum_{\substack{u \le p \le u + u/(\log u)^{10} \\ p \equiv \ell \pmod Q}} \frac{1}{p \log p} = (1+o(1)) \frac{1}{\varphi(Q)} \sum_{u \le p \le u + u/(\log u)^{10}} \frac{1}{p \log p} \quad (u \to \infty). \tag{3.4}$$

On the other hand, it is clear that, from the Prime Number Theorem,

$$\sum_{N < p \le N^{\lambda_N}} \frac{1}{p \log p} = (1+o(1)) \int_N^{N^{\lambda_N}} \frac{du}{u \log^2 u} = \frac{1+o(1)}{\log N} \qquad (N \to \infty). \tag{3.5}$$

Combining (3.3), (3.5), and (3.4), it follows that, as $y \to \infty$,

$$\begin{aligned} B_y(p_1, \ldots, p_k) &= (1+o(1)) y \sum_{\substack{p_j \equiv r_j \pmod Q \\ N < p_j < N^{\lambda_N} \\ j=1,\ldots,k}} \prod_{j=1}^{k} \frac{\log N}{p_j \log p_j} \\ &= (1+o(1)) \frac{y}{\varphi(Q)^k}. \end{aligned} \tag{3.6}$$

Observe also that

$$\frac{1}{x_N} \#\{n \in J_N : q_N(n) > N^{\lambda_N}\} \to 0 \quad \text{as } x_N \to \infty. \tag{3.7}$$

Indeed, it is clear that if $q_N(n) > N^{\lambda_N}$, then $\left(n, \prod_{N<\pi<N^{\lambda_N}} \pi\right) = 1$. Therefore, for some positive absolute constant C, we have

$$\#\{n \in J_N : q_N(n) > N^{\lambda_N}\} \le C x_N \prod_{N<\pi\le N^{\lambda_N}} \left(1 - \frac{1}{\pi}\right) \le C \frac{x_N}{\lambda_N},$$

which proves (3.7).

We now examine the first M digits of α_Q, say $\alpha_Q^{(M)}$. Let N be such that $x_N \le M < x_{N+1}$ and set $x := x_N$, $y := M - x_N$ and $J_0 = [x, x+y]$.

It follows from (3.6) that, as $y \to \infty$,

(3.8)

$$\#\{n \in J_0 : q_N(n+j) \equiv r_j \pmod{Q} \text{ for } j = 1, \ldots, k\} = (1+o(1)) \frac{y}{\varphi(Q)^k} + O\left(\frac{x_N}{\lambda_N}\right),$$

where the error term comes from (3.7) and accounts for those integers $n \in J_N$ for which $q_N(n) > N^{\lambda_N}$. Running the same procedure for each positive integer $H < N$, each time choosing $J_H = [x_H, x_{H+1})$, we then obtain a formula similar to the one in (3.8).

Gathering the resulting relations allows us to obtain that, for $X = x + y$,

$$\begin{aligned}
&\lim_{X\to\infty} \frac{1}{X} \# \{n \le X : q_N(n+j) \equiv r_j \pmod{Q} \text{ for } j = 1, 2, \ldots, k\} \\
&= \lim_{X\to\infty} \frac{1}{X} \Big(\sum_{H=1}^{N-1} \# \{n \in J_H : q_N(n+j) \equiv r_j \pmod{Q} \text{ for } j = 1, 2, \ldots, k\} \\
&\qquad + \#\{n \in J_0 : q_N(n+j) \equiv r_j \pmod{Q} \text{ for } j = 1, \ldots, k\}\Big) \\
&\qquad = \frac{1}{\varphi(Q)^k},
\end{aligned}$$

thus completing the proof of Theorem 1.

4 Final remarks

Let $\Omega(n) := \sum_{p^\alpha \| n} \alpha$ stand for the number of prime factors of n counting their multiplicity. Fix an integer $Q \ge 3$ and consider the function $u_Q(m) = \ell$, where ℓ is the unique non negative number $\le Q - 1$ such that $m \equiv \ell \pmod{Q}$. Now consider the infinite sequence

$$\xi_Q = \text{Concat}\left(u_Q(\Omega(n)) : n \in \mathbb{N}\right).$$

We conjecture that ξ_Q is a normal sequence over $\{0, 1, \ldots, Q-1\}$.

Moreover, let $\widetilde{\wp} \subset \wp$ be a subset of primes such that $\sum_{p\in\widetilde{\wp}} 1/p = +\infty$ and consider the function $\Omega_{\widetilde{\wp}}(n) := \sum_{\substack{p^\alpha \| n \\ p\in\widetilde{\wp}}} \alpha$. We conjecture that

$$\xi_Q(\widetilde{\wp}) := \text{Concat}\,(u_Q(\Omega_{\widetilde{\wp}}(n)) : n \in \mathbb{N})$$

is also a normal sequence over $\{0, 1, \ldots, Q-1\}$.

Finally, observe that we can also construct normal numbers by first choosing a monotonically growing sequence $(w_N)_{N\geq 1}$ such that $w_N > N$ for each positive integer N and such that $(\log w_N)/N \to 0$ as $N \to \infty$, and then defining $q_N(n)$ as the smallest prime factor of n larger than w_N if $n \in J_N$, setting $q_N(n) = 1$ otherwise. The proof follows along the same lines as the one of our main result.

References

[1] J.M. De Koninck and I. Kátai, *On a problem on normal numbers raised by Igor Shparlinski*, Bulletin of the Australian Mathematical Society **84** (2011), 337–349.

[2] J.M. De Koninck and I. Kátai, *Construction of normal numbers using the distribution of the k-th largest prime factor*, Journal of the Australian Mathematical Society **88** (2013), 158-168.

[3] J.M. De Koninck and I. Kátai, *Normal numbers generated using the smallest prime factor function*, preprint.

[4] J.M. De Koninck and I. Kátai, *Prime-like sequences leading to the construction of normal numbers*, Functiones et Approximatio (to appear).

[5] H.H. Halberstam and H.E. Richert, *Sieve Methods*, Academic Press, London, 1974.

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