Arithmetic functions monotonic at consecutive arguments

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Abstract

For a large class of arithmetic functions f, it is possible to show that, given an arbitrary integer $k \geq 2$, the string of inequalities $f(n+1) < f(n+2) < \cdots < f(n+k)$ holds for infinitely many positive integers n. For other arithmetic functions f, such a property fails to hold even for $k = 3$. We examine arithmetic functions from both classes. In particular, we show that there are only finitely many values of n satisfying $\sigma_2(n-1) < \sigma_2(n) < \sigma_2(n+1)$, where $\sigma_2(n) = \sum_{d|n} d^2$. On the other hand, we prove that for the function $f(n) := \sum_{p|n} p^2$, we do have $f(n-1) < f(n) < f(n+1)$ infinitely often.

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1 Introduction

Computing the average value of an arithmetic functions is often a somewhat easy task, at least compared to the tremendous challenge of fully understanding its local behavior. In any event, it is still possible in some instances to compare the values of an arithmetic function at consecutive values. For instance, we were able to show (see our recent book [4], Proposition 8.9) that, given any integer $k \geq 2$ and letting ϕ stand for the Euler function, $\phi(n+1) < \phi(n+2) < \cdots < \phi(n+k)$ holds for infinitely many positive integers n. The same type of statement can be made for the sum of divisors function $\sigma(n)$. Besides these and other multiplicative functions, similar statements can be made for additive functions. For instance, De Koninck, Friedlander and Luca [3] proved that, given any integer $k \geq 2$ and setting $g(n) = \sum_{p|n} 1$ or $g(n) = \sum_{p^\alpha \| n} \alpha$, then

$$g(n+1) < g(n+2) < \cdots < g(n+k) \qquad \text{holds infinitely often.} \tag{1.1}$$

For other functions g, property (1.1) does not hold, even for $k = 3$. For instance, as we will show in the next section, setting $\sigma_2(n) = \sum_{d|n} d^2$, there are only finitely many values of n satisfying $\sigma_2(n-1) < \sigma_2(n) < \sigma_2(n+1)$.

The fact that $\sigma_2(n)$ is a "large" function does not ensure that for any other "large" function f,

$$f(n-1) < f(n) < f(n+1) \tag{1.2}$$

will hold for only finitely many n's. Indeed, as we will show in Section 3, for the function $f(n) = \sum_{p|n} p^2$, the string of inequalities (1.2) does hold infinitely often.

2 The case of the squared divisors function

Each of the two strings $\sigma_2(n-1) < \sigma_2(n) < \sigma_2(n+1)$ and $\sigma_2(n-1) > \sigma_2(n) > \sigma_2(n+1)$ holds only for finitely many positive integers n. Clearly, this result follows from the following theorem.

Theorem 1. *For all integers $n \geq 8$,*

$$\sigma_2(2n) > \sigma_2(2n \pm 1). \tag{2.1}$$

Proof. Since, for all positive integers m,

$$\frac{\sigma_2(m)}{m^2} = \prod_{p^\alpha \| m} \left(1 + \frac{1}{p^2} + \cdots + \frac{1}{p^{2\alpha}}\right),$$

it follows that

$$\frac{\sigma_2(2n)}{(2n)^2} \geq 1 + \frac{1}{4} = \frac{5}{4} \tag{2.2}$$

and that, if $\zeta(s)$ stands for the Riemann Zeta Function,

$$\frac{\sigma_2(2n \pm 1)}{(2n \pm 1)^2} \leq \prod_{p \geq 3} \left(1 + \frac{1}{p^2} + \cdots\right) = \zeta(2)\left(1 - \frac{1}{2^2}\right) = \frac{3}{4}\zeta(2) = \frac{\pi^2}{8}. \tag{2.3}$$

From (2.2) and (2.3), it follows that

$$\frac{\sigma_2(2n)}{\sigma_2(2n \pm 1)} \geq \frac{(5/4)(2n)^2}{(\pi^2/8)(2n \pm 1)^2},$$

and since this last expression is strictly larger than 1 for all integers $n \geq 76$ and since (2.1) can be checked to be true for $8 \leq n \leq 75$, the proof of Theorem 1 is complete. □

Remark 1. As we mentioned in the Introduction, $\sigma(n-1) < \sigma(n) < \sigma(n+1)$ holds infinitely often, while we just proved that $\sigma_2(n-1) < \sigma_2(n) < \sigma_2(n+1)$ does not. What about the string

$$\sigma_k(n-1) < \sigma_k(n) < \sigma_k(n+1) \tag{2.4}$$

for the more general function $\sigma_k(n) := \sum_{d|n} d^k$? Using the approach used in the proof of Theorem 1, one can prove that the string of inequalities (2.4) also fails to hold infinitely often for all real $k > 1.1905$.

3 The case of the sum of the squared prime factors function

Our goal in this section is to prove the following result.

Theorem 2. *Let* $f(n) := \sum_{p|n} p^2$. *Then,*

$$f(n-1) < f(n) < f(n+1) \qquad \text{holds for infinitely many integers } n. \tag{3.1}$$

Let $P(n)$ stand for the largest prime factor of $n \geq 2$. If one could prove that

$$P(p^2+1) > p \qquad \text{for infinitely many primes } p, \tag{3.2}$$

then Theorem 2 would follow immediately, simply by setting $n = p^2$ in (3.1). However, although (3.2) is most likely true, proving it seems to be a very hard challenge. Observe that Hooley [6] has shown that $P(n^2+1) > n^\theta$, with $\theta = 11/10$, for infinitely many integers n, a result which was later improved by Deshouillers and Iwaniec [2] when they proved that one can take $\theta = 6/5$. In each case, deep analytic results were used to obtain the conclusion.

In order to prove (3.1), we shall use another approach which does not depend on deep analytic results. We start with the following lemma.

Lemma 1. *For each integer* $n \geq 2$, *we have* $P(n)^2 \leq f(n) < 2P(n)^2 \log n$.

Proof. The left inequality is obvious. For the right one, let $\omega(n) = \sum_{p|n} 1$ stand for the number of distinct prime factors of n and note that since $2^{\omega(n)} \leq n$, it follows that $\omega(n) \leq (\log n)/(\log 2) < 2\log n$. Hence,

$$f(n) = \sum_{p|n} p^2 \leq \omega(n)P(n)^2 < 2P(n)^2 \log n.$$

□

The next result comes from a paper of Erdős and Pomerance [5].

Lemma 2. *The inequality*

$$\#\{n \leq x : x^{0.32} < P(n) < x^{0.46} \text{ and } P(n) > P(n-1)\} > 0.0099x$$

holds for all x *sufficiently large.*

The next lemma holds the key to the proof of Theorem 2.

Lemma 3. *There exists a constant* $c_0 > 0$ *such that*

$$\#\{n \leq x : P(n^{2^j}-1) < P(n) < P(n^{2^j}+1) \text{ for some } j = 0, 1, \ldots, 11\} > c_0 x.$$

Remark 2. It is interesting to mention that Balog [1] used a somewhat similar configuration of consecutive integers (in the case $j = 1$) to study the frequency of the reversed pattern, that is $P(n-1) > P(n) > P(n+1)$. More precisely, he proved that the number of positive integers $m \le x$ such that $P(m^2-2) > P(m^2-1) > P(m^2)$ is $\gg x^{1/2}$. However, his proof uses deep analytic results such as the non trivial Bombieri-Vinogradov theorem. Our proof on the other hand is elementary and uses only an averaging argument as well as the distribution of primes in arithmetic progressions with small (fixed) moduli. Our proof uses also some ideas similar to those developed in [5].

Proof. First, we use Lemma 2 to conclude that if we set

$$\mathcal{A}_0 = \{x/10^4 < n < x : x^{0.32} < P(n) < x^{0..46} \text{ and } P(n-1) < P(n)\},$$

then $\#\mathcal{A}_0 > 0.009x$ for $x > x_0$. Let $\varepsilon > 0$ be some small number to be chosen later and set

$$\mathcal{B}_0 = \{n \in \mathcal{A}_0 : P(n+1) > P(n)\}$$

and assume that

$$\#\mathcal{B}_0 < \varepsilon x. \tag{3.3}$$

Then, setting

$$\mathcal{A}_1 = \mathcal{A}_0 \backslash \mathcal{B}_0,$$

we have that $\#\mathcal{A}_1 \ge (0.009-\varepsilon)x$. Observe that the numbers $n \in \mathcal{A}_1$ have the property that $\max\{P(n-1), P(n+1)\} < P(n)$ and therefore that $P(n^2-1) < P(n)$. Now let

$$\mathcal{B}_1 = \{n \in \mathcal{A}_1 : P(n) < P(n^2+1)\}$$

and assume that

$$\#\mathcal{B}_1 < \varepsilon x. \tag{3.4}$$

Further setting

$$\mathcal{A}_2 = \mathcal{A}_1 \backslash \mathcal{B}_1,$$

we have that $\#\mathcal{A}_2 \ge (0.009 - 2\varepsilon)x$. Observe that the numbers $n \in \mathcal{A}_2$ have the property that $\max\{P(n^2-1), P(n^2+1)\} < P(n)$ and therefore that $P(n^4-1) < P(n)$. We continue in this way by inductively creating the sets

$$\mathcal{A}_j = \{x/10^4 < n < x : x^{0.32} < P(n) < x^{0.46} \text{ and } P(n^{2^j} - 1) < P(n)\}$$

for $j = 3, 4, \ldots$, each time assuming that $\#\mathcal{A}_j > (0.009 - j\varepsilon)x$, that is for $j = 0, 1, 2, \ldots$. At each step, we create the set

$$\mathcal{B}_j = \{n \in \mathcal{A}_j : P(n) < P(n^{2^j} + 1)\},$$

with the property that

$$\#\mathcal{B}_j < \varepsilon x. \tag{3.5}$$

Each time, $\mathcal{A}_{j+1} = \mathcal{A}_j \backslash \mathcal{B}_j$ and the process continues. We will show that the above process must necessarily stop for some $j \le 12$, assuming of course that ε is chosen to be sufficiently small. Indeed, suppose that we are at some step $j \ge 2$ and that $\#\mathcal{A}_j \ge (0.009 - j\varepsilon)x$. Then

$$\begin{aligned}\prod_{n\in\mathcal{A}_j}(n^{2^j}-1) &> \zeta(2)^{-1}\prod_{n\in\mathcal{A}_j} n^{2^j} \ge \zeta(2)^{-1}(x/10^4)^{2^j\#\mathcal{A}_j} \\ &\ge \exp\left(2^j(0.009-\varepsilon j)x\log x + O(x)\right). \end{aligned} \tag{3.6}$$

Note the constant implied by the above O–symbol depends on j. On the other hand,

$$\begin{aligned}\prod_{n\in\mathcal{A}_j}(n^{2^j}-1) &= \prod_{n\in\mathcal{A}_j}\left((n-1)(n+1)(n^2+1)(n^4+1)\cdots(n^{2^{j-1}}+1)\right) \\ &= \left(\prod_{n\in\mathcal{A}_j}(n-1)\right)\left(\prod_{n\in\mathcal{A}_j}(n+1)\right)\cdots\prod_{n\in\mathcal{A}_j}(n^{2^{j-1}}+1). \end{aligned} \tag{3.7}$$

Clearly,

$$\prod_{n\in\mathcal{A}_j}(n-1) \le \lfloor x\rfloor! < \exp(x\log x) \tag{3.8}$$

and

$$\prod_{n\in\mathcal{A}_j}(n+1) \le \lfloor x+1\rfloor! \le (x+1)^{x+1} = \exp(x\log x + O(x)). \tag{3.9}$$

Next, let $i \in \{1,\ldots,j-1\}$ and let us examine the product

$$\prod_{n\in\mathcal{A}_j}(n^{2^i}+1).$$

Since $n \in \mathcal{A}_j$, it follows that $P(n^{2^j}-1) < P(n) \le x$, implying that $P(n^{2^i}+1) \le x$. Observe also that if $p \mid n^{2^i}+1$ for some $i \ge 1$, then either $p = 2$ or $p \equiv 1 \pmod{2^{i+1}}$. Denote by $\mathcal{P}_i$ the set of primes $p \le x$ such that $p \equiv 1 \pmod{2^{i+1}}$ and write

$$\prod_{n\in\mathcal{A}_j}(n^{2^i}+1) = 2^{\alpha_{2,i,j}}\prod_{p\in\mathcal{P}_i}p^{\alpha_{p,i,j}} = \exp\left(\sum_{p\in\mathcal{P}_i}\alpha_{p,i,j}\log p + O(\alpha_{2,i,j})\right), \tag{3.10}$$

where $\alpha_{2,i,j}$ and $\alpha_{p,i,j}$ are the exponents of 2 and of p in the factorization of the number appearing in the left–hand side of (3.10). Since $n^{2^i}+1$ is either odd or congruent to 2 modulo 4, it follows that

$$\alpha_{2,i,j} \le \#\mathcal{A}_j \le x. \tag{3.11}$$

Moreover,

$$\alpha_{p,i,j} = \sum_{\substack{n\in\mathcal{A}_j \\ n^{2^i}+1\equiv 0 \pmod{p}}} 1 + \sum_{\substack{n\in\mathcal{A}_j \\ n^{2^i}+1\equiv 0 \pmod{p^2}}} 1 + \cdots.$$

Now $n^{2^i}+1 \equiv 0 \pmod p$ separates the integers $n \le x$ into 2^i progressions modulo p. By Hensel's lifting lemma (concerning roots of a polynomial equation modulo a prime power), these lift to 2^i progressions for n modulo p^s for any $s \ge 1$. Thus, for a fixed $s \ge 1$, we have

$$\sum_{\substack{n\in\mathcal{A}_j \\ n^{2^i}+1\equiv 0 \pmod{p^s}}} 1 \le 2^i\left(\frac{x}{p^s}+1\right).$$

We then split the sum appearing on the right hand side of (3.10) into two sums S_1 and S_2 according to whether $p^s \le x$ or $p^s > x$. Observe that if $p^s \le x$, we have

$$\sum_{\substack{n\in\mathcal{A}_j \\ n^{2^i}+1\equiv 0 \pmod{p^s}}} 1 \le \frac{2^{i+1}x}{p^s}.$$

Hence,

$$\begin{aligned} S_1 &= \sum_{p^s\le x}\log p \sum_{\substack{n\in\mathcal{A}_j \\ n^{2^i}+1\equiv 0 \pmod{p^s}}} 1 \le 2^{i+1}x\sum_{p\in\mathcal{P}_i}\sum_{s\ge 1}\frac{\log p}{p^s} \\ &= 2^{i+1}x\sum_{p\in\mathcal{P}_i}\frac{\log p}{p-1} = 2^{i+1}x\left(\frac{\log x}{2^i}+O(1)\right) = 2x\log x + O(x), \qquad (3.12) \end{aligned}$$

where the constant implied by the above O–symbol depends on j. Assume next that $p^s > x$. Since $p \le x$, it follows that $s \ge 2$. Let s_p be the maximal s such that $p^s \mid n^{2^i}+1$ for some $n \in \mathcal{A}_j$. Then

$$p^{s_p} \le x^{2^i}+1 < x^{2^{i+1}}, \quad \text{so that} \quad s_p < 2^{i+1}\frac{\log x}{\log p}.$$

Thus,

$$\begin{aligned} S_2 &= \sum_{p^s > x}\log p \sum_{\substack{n\in\mathcal{A}_j \\ n^{2^i}+1\equiv 0 \pmod{p^s}}} 1 \\ &\le \sum_{p\le x}(\log p)2^{i+1}s_p \le 2^{2i+2}\log x\sum_{p\le x}1 \\ &= 2^{2i+2}(\log x)\pi(x) = O(x). \qquad (3.13) \end{aligned}$$

Gathering (3.12) and (3.13), we get that

$$\sum_{p\in\mathcal{P}_i}\alpha_{p,i,j}\log p = S_1 + S_2 \le 2x\log x + O(x).$$

Substituting this estimate in relation (3.10), we get, using (3.11),

$$\prod_{n\in\mathcal{A}_j}(n^{2^i}+1)\le \exp(2x\log x+O(x)). \tag{3.14}$$

With equation (3.7), we get

$$\begin{aligned}\prod_{n\in\mathcal{A}_j}(n^{2^j}-1) &\le \exp\left(x\log x(1+1+\underbrace{2+2+\cdots+2}_{j-1\text{ times}})+O(x)\right)\\ &= \exp\left(2jx\log x+O(x)\right).\end{aligned}$$

Combining the above estimate with (3.6), we get that

$$2^j(0.009-j\varepsilon)\le 2j+O(1/\log x).$$

One can check that the largest j such that $2^j 0.009<2j$ is $j=11$. In fact,

$$2\times 11/2^{11}=0.0107422\ldots>0.009 \quad\text{but}\quad 2\times 12/2^{12}=0.00585938\ldots.$$

Thus, taking $\varepsilon=0.001/12=1/12000$, we see that the inequality

$$2^j(0.009-j\varepsilon)<2j+O(1/\log x)$$

is false for $j\ge 12$ and $x>x_0$. Thus, $j\le 11$. This shows that with the constructed sets $\mathcal{B}_0,\ldots,\mathcal{B}_{11}$, one of them must have cardinality at least εx, which is what we wanted to prove with $c_0=\varepsilon=1/12000$. $\square$

We now have the necessary tools to complete the proof of Theorem 2. So, let $j_0\in\{0,1,\ldots,11\}$ be such that

$$\#\mathcal{B}_{j_0}\ge c_0x.$$

The existence of such an integer j_0 is guaranteed by Lemma 2. Assume now that $n\in\mathcal{B}_{j_0}$ but that the inequalities $f(n^{2^{j_0}}-1)<f(n)<f(n^{2^{j_0}}+1)$ do not hold.

First assume that $f(n^{2^j_0}-1)<f(n)$ does not hold. Then, by Lemma 1, we have

$$2^{16}(\log n)P(n^{2^{j_0}}-1)^2>2\log(n^{2^{j_0}}-1)P(n^{2^{j_0}}-1)^2>f(n^{2^{j_0}}-1)\ge P(n)^2,$$

implying that

$$P(n^{2^{j_0}}-1)\ge\frac{P(n)}{2^8\sqrt{\log(x/10^{14})}}>\frac{P(n)}{\log x}\quad\text{for}\quad x>x_0.$$

Now, fix a prime $p\in[x^{0.32},x^{0.46}]$ and a prime $q\in(p/\log x,p)$, and let us count the number of possible integers $n\le x$ such that $P(n)=p$ and $P(n^{2^{j_0}}-1)=q$. Given such an integer n, write it as $n=pm$ for some $m\le x$. Observe that $(pm)^{2^{j_0}}\equiv 1 \pmod q$. For fixed p, the above congruence puts m into at most $2^{j_0}\le 2^{11}$ arithmetic progressions of ratios q. Since $m \le x/p$, it follows that the number of such possibilities for m is

$$\le 2^{11}\left(\frac{x}{pq}+1\right) < \frac{2^{12}x}{pq} \quad \text{because} \quad pq < (x^{0.46})^2 = x^{0.92} < x.$$

We sum up the above inequality over all possible p and q and get that the number of such $n \le x$ is

$$\begin{aligned}
&\le \quad x\ 2^{12} \sum_{x^{0.32}<p<x^{0.46}} \frac{1}{p} \sum_{p/\log x<q<p} \frac{1}{q} \\
&\ll \quad x \sum_{x^{0.32}<p<x^{0.46}} \frac{1}{p}\left(\log\log p - \log\log\left(\frac{p}{\log x}\right) + O\left(\frac{1}{\log p}\right)\right) \\
&\ll \quad x \sum_{x^{0.32}<p<x^{0.46}} \frac{\log\log x}{p\log p} + O\left(\frac{x}{\log x}\right) \\
&\ll \quad \frac{x\log\log x}{\log x} = o(x) \quad \text{as} \quad x \to \infty.
\end{aligned}$$

Thus, for large x, most of the numbers in $n \in \mathcal{A}_{j_0}$ will in fact also satisfy the inequality $f(n^{2^{j_0}}-1) < f(n)$.

It remains to consider the case of the numbers $n \in \mathcal{A}_{j_0}$ for which the inequality $f(n) < f(n^{2^{j_0}}+1)$ fails. But this case can be dealt with in the same way and we may also conclude that the number of such integers n is also $o(x)$ as $x \to \infty$. So, for large x_0, there are at least $(\varepsilon/2)x$ values of $n \le x$ such that $f(n^{2^{j_0}}-1) < f(n^{2^{j_0}}) < f(n^{2^{j_0}}+1)$ holds. This completes the proof of the theorem.

Observe that our approach also gives that for $n > x_0$, the number of $n \le x$ such that $f(n-1) < f(n) < f(n+1)$ is $> c_1 x^{1/2^{11}} > x^{0.0004}$ for some positive constant c_1.

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