The uniform distribution mod 1 of sequences involving the largest prime factor function

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Dedicated to Professor Antanas Laurincikas on his 65-th anniversary

Abstract

Let $P(n)$ stand for the largest prime factor of n and let f be a real valued function satisfying certain conditions. We prove that the sequence $(f(P(n)))_{n\geq 2}$ is uniformly distributed modulo 1. We also show an analogous result if $P(n)$ is replaced by $P_k(n)$, the k-th largest prime factor of n.

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1 Introduction

Given a real number x, let $\{x\}$ denote its fractional part, that is $\{x\} = x - \lfloor x \rfloor$. A sequence of real numbers $(x_n)_{n\geq 1}$ is said to be *uniformly distributed modulo 1* if, given any real numbers $0 < a < b < 1$, then $\lim_{N\to\infty} \frac{1}{b-a}\#\{n \leq N : a < \{x_n\} < b\} = 1$.

For instance, given an irrational number $\alpha > 0$, one can show that the sequence $(\alpha n)_{n\geq 1}$ is uniformly distributed modulo 1. Various other examples of such sequences are given in the book of Kuipers and Niederreiter [3].

On the other hand, it is known that if $2 = p_1 < p_2 < \cdots$ stands for the sequence of primes, then the sequence $(\log p_n)_{n\geq 1}$ is not uniformly distributed modulo 1.

In this paper, we prove that if $P(n)$ stands for the largest prime factor of n and if f is a real valued function satisfying certain conditions, then the sequence $(f(P(n)))_{n\geq 2}$ is uniformly distributed modulo 1. We also prove that if $P_k(n)$ stands for the k-largest prime factor of n and if τ stands for an arbitrary non zero real number, then the sequence $(\tau \log P_k(n))_{n\geq 4}$ is uniformly distributed modulo 1.

2 Main results

Theorem 1. *Let $g : [1,\infty) \to \mathbb{R}$ be a differentiable function and let $f : [0,\infty) \to \mathbb{R}$ be defined by $f(u) = g(\log u)$. Assume that the function $vg'(v)$ is increasing and tends*

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to infinity. For $x \geq 2$, let $R(x) := \pi(x) - li(x)$ be the error term in the Prime Number Theorem and further assume that, for any given real number $d > 0$,

$$\lim_{y\to\infty} \int_y^{y^{1+d}} \frac{|R(u)|}{u} |f'(u)| \, du = 0. \tag{2.1}$$

Then the sequence $(f(P(n)))_{n\geq 1}$ is uniformly distributed modulo 1.

Remark 1. *Observe that it is clear that (2.1) holds for $f(u) := \tau \log u$, where $\tau \neq 0$ is some fixed constant, thus implying that, as a consequence of Theorem 1, the sequence $(\tau \log P(n))_{n\geq 1}$ is uniformly distributed modulo 1.*

For a positive integer n which is not a prime, we let $P_2(n) = P(n/P(n))$ and more generally, given an arbitrary integer $k \geq 2$ and an integer n with at least k prime divisors counting their multiplicity, let $P_k(n) = P_{k-1}(n/P(n))$.

Theorem 2. *Given an arbitrary real number $\tau \neq 0$ and an arbitrary integer $k \geq 1$, the sequence $(\tau \log P_k(n))_{n\geq 2}$ is uniformly distributed modulo 1.*

3 Notation

We will use the standard notation $e(y) := \exp\{2\pi i y\}$.

For $2 \leq y \leq x$, let

$$\Psi(x, y) = \#\{n \leq x : P(n) \leq y\}.$$

Let $\rho(u)$ stand for the Dickman function, that is the unique continuous solution of the differential equation $u\rho'(u) + \rho(u-1) = 0$ with initial condition $\rho(u) = 1$ for $0 \leq u \leq 1$.

The letter c, with or without subscript, always denotes a positive constant, but not necessarily the same at each occurrence.

4 Preliminary results

The following two results are well known and their proofs can be found in the book of Tenenbaum [4].

Lemma 1. *For all $2 \leq y \leq x$, letting $u = \log x / \log y$,*

$$\Psi(x, y) \ll x \exp\{-u/2\}.$$

Lemma 2. *Given a fixed $\varepsilon > 0$, then uniformly for $u = \log x/\log y \leq 1/\varepsilon$,*

$$\Psi(x, y) = (1 + o(1))\rho(u)x \qquad (x \to \infty).$$

Finally, the following result, a proof of which can be found in the book of De Koninck and Luca [2], has the advantage of being true for all $2 \leq y \leq x$.

Lemma 3. *Uniformly for $2 \le y \le x$,*

$$\Psi(x,y) = x\rho(u) + O\left(\frac{x}{\log y}\right).$$

Lemma 4. *Let $(f(p))_{p\in\wp}$ be a sequence of real numbers which is such that, for any given real number $d > 0$ and integer $k \ge 1$,*

$$\lim_{y\to\infty} \sum_{y<p<y^{1+d}} \frac{e(kf(p))}{p} = 0. \tag{4.1}$$

Then the sequence $(f(P(n)))_{n\ge 1}$ is uniformly distributed mod 1.

Proof. We will be using the well known result of H. Weyl [5] which asserts that a sequence $(x_n)_{n\ge1}$ is uniformly distributed mod 1 if and only if

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e(kx_n) = 0$$

for each positive integer k.

Let k be an arbitrary positive integer and let $\varepsilon > 0$ be a small number. We then write

$$\begin{aligned} S(N) &= \frac{1}{N}\sum_{n\le N} e(kf(P(n))) \\ &= \frac{1}{N}\sum_{\substack{n\le N\\ P(n)\le N^\varepsilon}} e(kf(P(n))) + \frac{1}{N}\sum_{\substack{n\le N\\ N^\varepsilon<P(n)\le N}} e(kf(P(n))) \\ &= S_1(N) + S_2(N), \end{aligned} \tag{4.2}$$

say. Hence, in light of Weyl's criterion, it will be sufficient to show that

$$\limsup_{N\to\infty} |S(N)| = 0. \tag{4.3}$$

Now, it follows from Lemma 1 that

$$S_1(N) \le \frac{1}{N}\Psi(N, N^\varepsilon) \le \exp\left\{-\frac{1}{2\varepsilon}\right\}. \tag{4.4}$$

On the other hand, using Lemma 3, we have

$$\begin{aligned} S_2(N) &= \frac{1}{N}\sum_{N^\varepsilon<p\le N} e(kf(p))\psi\left(\frac{N}{p}, p\right) \\ &= \sum_{N^\varepsilon<p\le N}\left(\frac{e(kf(p))}{p}\rho\left(\frac{\log N/p}{\log p}\right) + O\left(\frac{1}{p\log p}\right)\right) \end{aligned}$$

$$= \sum_{N^\varepsilon < p \le N} \frac{e(kf(p))}{p} \rho\left(\frac{\log N/p}{\log p}\right) + O\left(\frac{1}{\varepsilon \log N}\right)$$

$$= S_3(N) + O\left(\frac{1}{\varepsilon \log N}\right), \tag{4.5}$$

say.

It remains to show that

$$|S_3(N)| < c\,\varepsilon \qquad \text{if } N \text{ is large enough.} \tag{4.6}$$

To do so, we proceed as follows. Since $\rho(u)$ is uniformly continuous in the closed interval $[1, 1/\varepsilon - 1]$, there exists a suitable constant $\Delta > 0$ (depending only on ε) such that $|\rho(u_1) - \rho(u_2)| \le \varepsilon$ for all $u_1, u_2 \in [1, 1/\varepsilon - 1]$ satisfying $|u_1 - u_2| \le \Delta$. Let us now consider the sequence $\ell_0, \ell_1, \ldots, \ell_{j_0}$ defined by $\ell_0 = 1$ and $\ell_j = 1 + j\Delta$ for each $j = 1, \ldots, j_0$, where j_0 is the smallest integer such that $j_0\Delta \ge \varepsilon$. Further set $\mathcal{J}_j = [\ell_j, \ell_{j+1})$ and then split the sum $S_3(N)$ as follows.

$$S_3(N) = \sum_{\sqrt{N} < p \le N} \frac{e(kf(p))}{p} + \sum_{j=0}^{j_0 - 1} T_j, \tag{4.7}$$

where

$$T_j = \sum_{\frac{\log N}{\log p} \in \mathcal{J}_j} \frac{e(kf(p))}{p} \rho\left(\frac{\log(N/p)}{\log p}\right) \qquad j = 0, 1, \ldots, j_0 - 1.$$

Since in each T_j, we have

$$\left|\rho\left(\frac{\log(N/p)}{\log p}\right) - \rho(\ell_j - 1)\right| \le \varepsilon,$$

it follows, recalling that $\rho(\ell_j - 1) = \rho(j\Delta)$,

$$|T_j| \le \rho(j\Delta) \left|\sum_{\frac{\log N}{\log p} \in \mathcal{J}_j} \frac{e(kf(p))}{p}\right| + \varepsilon \sum_{\frac{\log N}{\log p} \in \mathcal{J}_j} \frac{1}{p}. \tag{4.8}$$

Now observe that

$$\frac{\log N}{\log p} \in \mathcal{J}_j \Longleftrightarrow \ell_j \le \frac{\log N}{\log p} < \ell_{j+1} \Longleftrightarrow p \in (N^{1/\ell_{j+1}}, N^{1/\ell_j}].$$

We then have

$$|S_3(N)| \le \left|\sum_{\sqrt{N} < p \le N} \frac{e(kf(p))}{p}\right| + \sum_{j=0}^{j_0 - 1} \rho(j\Delta) \left|\sum_{\frac{\log N}{\log p} \in \mathcal{J}_j} \frac{e(kf(p))}{p}\right|$$

$$+\varepsilon\sum_{j=0}^{j_0-1}\sum_{\frac{\log N}{\log p}\in\mathcal{J}_j}\frac{1}{p}. \tag{4.9}$$

Again, observe that since in $S_3(N)$, we sum only over those primes p for which $\frac{\log N}{\log p}\in[2,1/\varepsilon]$, it follows that the number of intervals $\mathcal{J}_j$'s necessary to cover the interval $[2,1/\varepsilon]$ is finite. This is why we can allow j to vary only from 0 to j_0-1.

It follows from the condition (4.1) that the first two terms on the right hand side of (4.9) tend to 0 as $N\to\infty$, implying that

$$\begin{aligned}\limsup_{N\to\infty}|S_3(N)| &\leq \varepsilon\limsup_{N\to\infty}\sum_{\frac{\log N}{\log p}\in[2,1/\varepsilon]}\frac{1}{p}\\ &= \varepsilon\limsup_{N\to\infty}\sum_{N^\varepsilon\leq p\leq N}\frac{1}{p}\\ &< c\varepsilon\log\frac{1}{\varepsilon}. \end{aligned}\tag{4.10}$$

Hence, using estimates (4.4), (4.5) and (4.10) in (4.2), relation (4.3) follows immediately, thus completing the proof of Lemma 4. □

Lemma 5. *Given a fixed positive number $\varepsilon<1$,*

$$\#\{n\leq N: P(n)>N^{1-\varepsilon}\}\leq c\varepsilon N.$$

Proof. This follows from the fact

$$\sum_{\substack{n=mp\leq N\\ N^{1-\varepsilon}<p\leq N}}1\leq N\sum_{N^{1-\varepsilon}<p\leq N}\frac{1}{p}\leq c\varepsilon N.$$

□

Lemma 6. *For any fixed number $\lambda\neq 0$, consider the function*

$$\kappa(y,z):=\sum_{y<p<z}\frac{e(\lambda\log p)}{p}.$$

Then,

$$\sup_{z>y}|\kappa(y,z)|\to 0 \qquad \textit{as } y\to\infty. \tag{4.11}$$

Proof. This result follows essentially from the fact that $\sum_p\frac{1}{p^{i\lambda}}$ is convergent for all $\lambda\neq 0$. □

5 Proof of Theorem 1

With $\pi(u) = \int_2^u \frac{dt}{\log t} + R(u)$, we get that

$$d\,\pi(u) = \frac{du}{\log u} + d\,R(u).$$

Hence

$$\begin{aligned}
\sum_{y<p<y^{1+d}} \frac{e(kf(p))}{p} &= \int_y^{y^{1+d}} \frac{e(kf(u))}{u}\, d\,\pi(u) \\
&= \int_y^{y^{1+d}} \frac{e(kf(u))}{u\log u}\, du + \int_y^{y^{1+d}} \frac{e(kf(u))}{u}\, d\,R(u) \\
&= \mathcal{J}_1(y) + \mathcal{J}_2(y), \qquad (5.1)
\end{aligned}$$

say. With the change of variable $v = \log u$, so that $f(u) = f(e^v) = g(v) = g(\log u)$, we get

$$\mathcal{J}_1(y) = \int_{\log y}^{(1+d)\log y} \frac{e(kg(v))}{v}\, dv. \qquad (5.2)$$

Since

$$(e(kg(v)))' = e(kg(v)) \cdot 2\pi i k \cdot g'(v),$$

it follows from (5.2) that

$$\begin{aligned}
\mathcal{J}_1(y) &= \int_{\log y}^{(1+d)\log y} (e(kg(v)))' \frac{dv}{2\pi i k v g'(v)} \\
&= e(kg(v)) \frac{1}{2\pi i k v g'(v)}\Bigg|_{\log y}^{(1+d)\log y} - \frac{1}{2\pi i k}\int_{\log y}^{(1+d)\log y} e(kg(v)) \left(\frac{1}{vg'(v)}\right)' dv \\
&= \mathcal{J}_1^{(1)}(y) - \mathcal{J}_1^{(2)}(y), \qquad (5.3)
\end{aligned}$$

say.

First of all, since by hypothesis, $vg'(v) \to \infty$ as $v \to \infty$, it is clear that

$$\left|\mathcal{J}_1^{(1)}(y)\right| \to 0 \qquad \text{as } y \to \infty. \qquad (5.4)$$

On the other hand, since by hypothesis, $vg'(v) \to \infty$ monotonically as $v \to \infty$, it follows that $\left(\frac{1}{vg'(v)}\right)'$ is negative for all real $v \geq 1$ and therefore that

$$\left|\mathcal{J}_1^{(2)}(y)\right| \leq \frac{1}{2\pi k}\int_{\log y}^{(1+d)\log y} \left(-\frac{1}{vg'(v)}\right)' dv = \frac{1}{2\pi k}\, \frac{-1}{vg'(v)}\Bigg|_{\log y}^{(1+d)\log y} \to 0, \qquad (5.5)$$

as $y \to \infty$.

On the other hand,

$$\begin{aligned}
\mathcal{J}_2(y) &= \int_y^{y^{1+d}} \frac{e(kf(u))}{u}\, d\, R(u) \\
&= \frac{R(u)e(kf(u))}{u}\bigg|_y^{y^{1+d}} - \int_y^{y^{1+d}} R(u) \left(\frac{e(kf(u))}{u}\right)' du \\
&= \mathcal{J}_2^{(1)}(y) - \mathcal{J}_2^{(2)}(y), \qquad (5.6)
\end{aligned}$$

say.

Now, on the one part, we clearly have

$$\left|\mathcal{J}_2^{(1)}(y)\right| \to 0 \qquad \text{as } y \to \infty. \qquad (5.7)$$

while on the other hand, since

$$\mathcal{J}_2^{(2)}(y) = \int_y^{y^{1+d}} R(u) \frac{e(kf(u)) \cdot 2\pi ik \cdot uf'(u) - e(kf(u))}{u^2}\, du,$$

it follows that

$$\left|\mathcal{J}_2^{(2)}(y)\right| \le 2\pi k \int_y^{y^{1+d}} \frac{|R(u)|}{u} |f'(u)|\, du + \int_y^{y^{1+d}} \frac{|R(u)|}{u^2}\, du \to 0, \qquad (5.8)$$

as $y \to \infty$, where we used (2.1).

Combining estimates (5.4), (5.5), (5.7) and (5.8), then, in light of (5.3) and (5.6), it follows from (5.1) that

$$\sum_{y<p<y^{1+d}} \frac{e(kf(p))}{p} \to 0 \qquad \text{as } y \to \infty.$$

Thus, using Lemma 4, the theorem is proved.

6 Proof of Theorem 2

We will only consider the case $k = 2$, the general case being similar.

Now, given real numbers $0 < \alpha < \beta \le 1$, let us introduce the two expressions

$$\omega_{\alpha,\beta}(n) = \omega_{\alpha,\beta,N}(n) = \sum_{\substack{p|n \\ N^\alpha<p<N^\beta}} 1, \qquad E_{\alpha,\beta}(N) = \sum_{N^\alpha<p<N^\beta} \frac{1}{p}.$$

Since

$$E_{\alpha,\beta}(N) = \log\frac{\beta}{\alpha} + o(1) \qquad \text{as } N \to \infty,$$

it follows from the Turán-Kubilius inequality that

$$\sum_{n\le N}(\omega_{\alpha,\beta}(n)-E_{\alpha,\beta}(N))^2 \le cNE_{\alpha,\beta}(N) < c_1 N\log\frac{\beta}{\alpha}. \tag{6.1}$$

Let $\varepsilon>0$ be an arbitrary small number. Let $\delta\in(0,1)$, to be determined later. Since $\omega_{\delta,1}(n)=0$, 1 or 2 when $P_3(n)<N^\delta$, we have that $\dfrac{(\omega_{\delta,1}(n)-E_{\delta,1}(N))^2}{(E_{\delta,1}(N)-2)^2}\ge 1$, thus implying, using (6.1), that

$$\begin{aligned}
\#\{n\le N : P_3(n)<N^\delta\} &\le \frac{1}{(E_{\delta,1}(N)-2)^2}\sum_{n\le N}(\omega_{\delta,1}(n)-E_{\delta,1}(N))^2\\
&\le c_1 N\frac{1}{E_{\delta,1}(N)} \qquad \text{provided } E_{\delta,1}(N)\ge 4,\\
&\le c_1\frac{N}{\log(1/\delta)}\\
&\le c_2\varepsilon N, \qquad (6.2)
\end{aligned}$$

provided we choose $\delta=\exp\{-1/\varepsilon\}$.

Let δ_1,δ_2 be two small positive numbers, to be determined later. We will now count the number $T(N)$ of positive integers $n\le N$ for which there exist two prime numbers p,q satisfying $pq|n$ and $N^{\delta_1}<p\le q<p^{1+\delta_2}$. It is clear that

$$\begin{aligned}
T(N) &\le N\sum_{N^{\delta_1}<p\le q<p^{1+\delta_2}}\frac{1}{pq}\le N\sum_{N^{\delta_1}<p\le N}\frac{1}{p}\sum_{p\le q<p^{1+\delta_2}}\frac{1}{q}\\
&\le cN\cdot\log\frac{1}{\delta_1}\cdot\log(1+\delta_2)\\
&\le c_1\varepsilon N, \qquad (6.3)
\end{aligned}$$

provided we choose δ_1 and δ_2 so that $\delta_2\log\dfrac{1}{\delta_1}\le\varepsilon$; for instance, the choice $\delta_1=\delta$ and $\delta_2=\varepsilon^2$ is appropriate.

Note that our main goal will be to prove that

$$A(N):=\sum_{n\le N}e(\ell\log P_2(n))=o(N)$$

for every non zero real number ℓ. Indeed, choosing $\ell=k\tau$, with $0\ne k\in\mathbb{Z}$, we will then be guaranteed by Weyl's criterion that $(\tau\log P_2(n))_{n\ge1}$ is uniformly distributed modulo 1, as claimed.

So, let us write each number $n\le N$ as $n=\nu p_2p_1$, where $P(\nu)<p_2<p_1$. Choose $\varepsilon,\delta_1,\delta_2$ as above. We first drop those positive integers $n\le N$ for which

$$p_1>N^{1-\varepsilon} \quad\text{or}\quad P(\nu)<N^\delta \quad\text{or}\quad p_2\le p_1<p_2^{1+\delta_2}.$$

Indeed, we can do this in light of Lemma 5, (6.2) and (6.3).

We have thus established that

$$A(N) = A_1(N) + O(\varepsilon N),$$

where $A_1(N)$ counts the number of positive integers $n \le N$ counted by $A(N)$ but that were not dropped in the above process.

With the above notation, we may therefore write

$$|A_1(N)| \ll \left| \sum_{p_1 p_2 \le N^{1-\delta_1}} e(\ell \log p_2) \Psi\left(\frac{N}{p_1 p_2}, p_2\right) \right| = |A_2(N)| + |A_3(N)|, \tag{6.4}$$

where in $A_2(N)$, the summation runs over those primes p_1, p_2 such that $\frac{N}{p_1 p_2} < p_2$, while in $A_3(N)$ it runs over those such that $\frac{N}{p_1 p_2} \ge p_2$.

Observe that for those p_1, p_2 running in $A_2(N)$, we have

$$\Psi\left(\frac{N}{p_1 p_2}, p_2\right) = \left\lfloor \frac{N}{p_1 p_2} \right\rfloor = \frac{N}{p_1 p_2} + O(1),$$

thus implying that

$$\begin{aligned} A_2(N) &= N \sum_{\substack{p_1 p_2 \le N^{1-\delta_1} \\ p_1 p_2^2 > N}} \frac{e(\ell \log p_2)}{p_1 p_2} + O\left(\sum_{p_1 p_2 \le N} 1 \right) \\ &= N \sum_{\substack{p_1 p_2 \le N^{1-\delta_1} \\ p_1 p_2^2 > N}} \frac{e(\ell \log p_2)}{p_1 p_2} + O\left(\frac{N \log\log N}{\log N} \right). \end{aligned} \tag{6.5}$$

We have

$$\begin{aligned} N \sum_{p_1 p_2^2 > N} \frac{e(\ell \log p_2)}{p_1 p_2} &= N \sum_{p_1 > N^{\delta_1}} \frac{1}{p_1} \sum_{\sqrt{\frac{N}{p_1}} \le p_2 < p_1} \frac{e(\ell \log p_2)}{p_2} \\ &= N \sum_{N^{\delta_1} < p_1 < N^{1-\delta_1}} \frac{1}{p_1} \kappa(\sqrt{N/p_1}, p_1). \end{aligned} \tag{6.6}$$

In light of Lemma 6, we have that $|\kappa(\sqrt{N/p_1}, p_1)| < \varepsilon$ if N is sufficiently large and therefore that it follows from (6.5) and (6.6) that

$$|A_2(N)| \ll \varepsilon N \sum_{N^{\delta_1} < p_1 < N^{1-\delta_1}} \frac{1}{p_1} < \varepsilon N \log \frac{1}{\delta_1}. \tag{6.7}$$

On the other hand, using Lemma 2, we have, as $N \to \infty$,

$$A_3(N) = \sum_{\substack{p_2 < p_1,\ p_1 p_2 \le N^{1-\delta_1} \\ \frac{N}{p_1 p_2} \ge p_2}} e(\ell \log p_2) \Psi\left(\frac{N}{p_1 p_2}, p_2\right)$$

$$= \quad (1+o(1)) \sum_{\substack{p_2<p_1,\ p_1p_2\le N^{1-\delta_1}\\ \frac{N}{p_1p_2}\ge p_2}} e(\ell\log p_2)\frac{N}{p_1p_2}\rho\left(\frac{\log(N/p_1p_2)}{\log p_2}\right)$$

$$= \sum_{\substack{p_2<p_1,\ p_1p_2\le N^{1-\delta_1}\\ \frac{N}{p_1p_2}\ge p_2}} e(\ell\log p_2)\frac{N}{p_1p_2}\rho\left(\frac{\log(N/p_1p_2)}{\log p_2}\right)$$

$$+o\left(\sum_{\substack{p_2<p_1,\ p_1p_2\le N^{1-\delta_1}\\ \frac{N}{p_1p_2}\ge p_2}} \frac{N}{p_1p_2}\right)$$

$$= B_1(N)+o(B_2(N)), \tag{6.8}$$

say. It is clear that

$$B_2(N) < N \sum_{N^{\delta_1}<p_1<N^{1-\delta_1}} \frac{1}{p_1}\cdot \sum_{N^{\delta_1}<p_2<N^{1-\delta_1}} \frac{1}{p_2} \ll N\left(\log\frac{1}{\delta_1}\right)^2. \tag{6.9}$$

To estimate $B_1(N)$, let us denote the interval $\left[N^{\delta_1}, \min\left(\sqrt{\frac{N}{p_1}}, p_1\right)\right]$ by $I(p_1)$ and then, using the same approach as that used in Theorem 1, we subdivide each interval $I(p_1)$ into subintervals $\mathcal{J}_j - 1 = [\ell_j - 1, \ell_{j+1}-1)$ depending if

$$\ell_j - 1 \le \nu_{p_1,p_2} := \frac{\log(N/p_1p_2)}{\log p_2} = \frac{\log(N/p_1)}{\log p_2} - 1 \le \ell_{j+1}-1.$$

This is useful because the function $\rho(\nu_{p_1,p_2})$ varies very little in each interval $\mathcal{J}_j - 1 = [\ell_j - 1, \ell_{j+1}-1)$, in the sense that

$$\left|\rho(\nu_{p_1,p_2}) - \rho(\nu_{p_1,p_2^*})\right| < \varepsilon_1 \text{ for all } \nu_{p_1,p_2}, \nu_{p_1,p_2^*} \in \mathcal{J}_j - 1.$$

Taking into account the fact that the number of such intervals $\mathcal{J}_j$ is bounded for each p_1 by a constant $c(p_1)$ and moreover that we can conclude to the existence of a universal constant $C>0$ such that $\sup_{p_1} c(p_1) \le C$, it follows from (6.8) and since $|\rho(\nu_{p_1,p_2}) - \rho(j\Delta)| \le \varepsilon_1$, that

$$|B_1(N)| \ll \varepsilon_1 \cdot N \sum_{p_1>N^{\delta_1}} \frac{1}{p_1}\sum_{j=0}^{j_0-1}\rho(j\Delta)\left|\sum_{\nu_{p_1,p_2}\in\mathcal{J}_j-1}\frac{e(\ell\log p_2)}{p_2}\right|. \tag{6.10}$$

Hence, it follows from (6.10), (6.9) and (6.8) that

$$|A_3(N)| \ll o\left(N\left(\log\frac{1}{\delta_1}\right)^2\right) + \varepsilon_1\kappa_0(N)N\sum_{N^{\delta_1}<p_1<N^{1-\delta_1}}\frac{1}{p_1}\sum_j \rho(j\Delta), \tag{6.11}$$

where

$$\kappa_0(N) := \sup_{j,p_1} \left| \sum_{\nu_{p_1,p_2} \in -1+\mathcal{J}_j} \frac{e(\ell \log p_2)}{p_2} \right|$$

tends to 0 uniformly in p_1 and in j as $N \to \infty$. Hence, (6.11) yields

$$\limsup_{N\to\infty} \frac{|A_3(N)|}{N} \leq \varepsilon_1 \left(\log \frac{1}{\delta_1}\right)^2.$$

Since ε_1 can be chosen arbitrarily small, we conclude that

$$\limsup_{N\to\infty} \frac{|A_3(N)|}{N} = 0. \tag{6.12}$$

Gathering estimates (6.7) and (6.12) in (6.4), the proof of Theorem 2 (in the case $k = 2$) is complete.

7 Example

Given a real number $\alpha > 0$, set $f(u) = (\log u)^\alpha$. Setting $v = \log u$, we get $g(v) = v^\alpha$, so that $vg'(v) = \alpha v^\alpha \to \infty$ as $v \to \infty$. It remains to check that condition (2.1) is satisfied. On the one hand,

$$f'(u) = \alpha(\log u)^{\alpha-1}\frac{1}{u}, \tag{7.1}$$

while on the other hand, it is known since de la Vallée-Poussin (see [1]) that, for some constant $C > 0$,

$$R(u) \ll u \exp\{-C\sqrt{\log u}\}. \tag{7.2}$$

Using (7.1) and (7.2), we get, by setting $v = \log u$,

$$\int_y^{y^{1+d}} \frac{|R(u)|}{u}|f'(u)|\,du \leq \alpha \int_y^{y^{1+d}} \frac{(\log u)^{\alpha-1}}{u} e^{-C\sqrt{\log u}}\,du = \alpha \int_{\log y}^{(1+d)\log y} \frac{v^{\alpha-1}}{e^{C\sqrt{v}}}\,dv. \tag{7.3}$$

Since this last quantity clearly tends to 0 as $y \to \infty$, condition (2.1) of Theorem 1 is satisfied, thereby implying that, if $f(m) = (\log m)^\alpha$, then the sequence $f(P(n))_{n\geq 1}$ is uniformly distributed mod 1.

References

[1] Charles-Jean de la Vallée-Poussin, *Sur la fonction $\zeta(s)$ de Riemann et le nombre de nombres premiers inférieurs à une limite donnée*, Mém. couronnées et autres mém. publ. l'Acad. roy. Sci. Lettres Beaux-Arts Belgique **59** (1899-1900), No.1, 74 pp.

[2] J.M. De Koninck and F. Luca, *Analytic Number Theory: Exploring the Anatomy of Integers*, Graduate Studies in Mathematics, Vol. 134, American Mathematical Society, Providence, Rhode Island, 2012.

[3] L. Kuipers and H. Niederreiter, *Uniform Distribution of Sequences*, John Wiley & Sons, New York, 1974.

[4] G. Tenenbaum, *Introduction à la théorie analytique des nombres*, Collection Échelles, Belin, 2008.

[5] H. Weyl, *Über die Gleichverteilung von Zahlen mod Eins*, Math Ann. **77** (1916), 313-352.

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