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# NORMAL NUMBERS CREATED FROM PRIMES AND POLYNOMIALS

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ABSTRACT. Given an integer $D \geq 2$, a *D-normal number*, or simply a *normal number*, is a real number whose $D$-ary expansion is such that any preassigned sequence, of length $k \geq 1$, of base $D$ digits from this expansion, occurs at the expected frequency, namely $1/D^k$. We construct large families of normal numbers using primes and polynomials.

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## 1. Introduction

The concept of a normal number goes back to 1909: it was first introduced by Émile Borel [1]. Given an integer $D \geq 2$, a *D-normal number*, or simply a *normal number*, is a real number whose $D$-ary expansion is such that any preassigned sequence, of length $k \geq 1$, of base $D$ digits from this expansion, occurs at the expected frequency, namely $1/D^k$. Equivalently, given a positive real number $\eta < 1$ whose expansion is $\eta = 0.a_1a_2\ldots$, where each $a_j \in \{0, 1, \ldots, D-1\}$, that is, $\eta = \sum_{j=1}^{\infty} \frac{a_j}{D^j}$, we say that $\eta$ is a normal number if the sequence $\{D^m\eta\}$, $m = 1, 2, \ldots$ (here $\{y\}$ stands for the fractional part of $y$), is uniformly distributed in the interval $[0, 1[$. Clearly, both definitions are equivalent, because the sequence $\{D^m\eta\}$, $m = 1, 2, \ldots$, is uniformly distributed in $[0, 1[$ if and only if for every integer $k \geq 1$ and $b_1 \ldots b_k \in \{0, 1, \ldots, D-1\}^k$, we have

$$\lim_{N\to\infty} \frac{1}{N}\#\{j < N : a_{j+1} \ldots a_{j+k} = b_1 \ldots b_k\} = \frac{1}{D^k}.$$

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Identifying if a given real number is a normal number is unresolved. For instance, classical numbers such as $\pi$, $e$, $\sqrt{2}$, $\log 2$ as well as the famous Apéry constant $\zeta(3)$, have not yet been proven to be normal numbers, although numerical evidence clearly indicates that they are. It is interesting to observe that Borel [1] showed one century ago that almost all numbers are normal.

Even constructing specific normal numbers is no small challenge.

Several authors studied the problem of constructing normal numbers. One of the first was Champernowne [2] who, in 1933, was able to prove that the number made up of the concatenation of the natural numbers, namely the number

$$0.1234567891011121314151617181920 21\ldots,$$

is normal in base 10. In 1946, Copeland and Erdős [3] showed that the same is true if one replaces the sequence of natural numbers by the sequence of primes, namely for the number

$$0.23571113171923293137\ldots$$

In the same paper, they conjectured that if $f(x)$ is any nonconstant polynomial whose values at $x = 1, 2, 3, \ldots$ are positive integers, then the number $0.f(1)f(2)f(3)\ldots$, where $f(n)$ is written in base 10, is a normal number. In 1952, Davenport and Erdős [4] proved this conjecture.

In 1997, Nakai and Shiokawa [13] showed that if $f(x)$ is any nonconstant polynomial taking only positive integral values for positive integral arguments, then the number $0.f(2)f(3)f(5)f(7)\ldots f(p)\ldots$, where $p$ runs through the prime numbers, is normal. In 2008, Madritsch, Thuswaldner and Tichy [12] extended the results of Nakai and Shiokawa by showing that, if $f$ is an entire function of logarithmic order, then the numbers

$$0.[f(1)]_D[f(2)]_D[f(3)]_D\ldots \quad \text{and} \quad 0.[f(2)]_D[f(3)]_D[f(5)]_D[f(7)]_D\ldots,$$

where $[f(n)]_D$ stands for the base $D$ expansion of the integer part of $f(n)$, are normal.

In this paper, we use a totally different approach to create large families of normal numbers. Indeed, in order to compare the above approaches with our new approach, let us introduce the notion of *typical number* (which is not a mathematical concept!). Let us say that a large positive integer $n$ is *typical* if, given an arbitrary positive integer $k$, one can establish that the number of occurrences of the various subsequences of consecutive digits of length $k$ in the $D$-ary expansion of $n$ are essentially the same. In other words, $n$ is typical if, given arbitrary positive integers $\beta_1, \beta_2, \ldots, \beta_s$, every one of them made up of $k$ digits, they each occur at the same frequency in the $D$-ary expansion of $n$. Then, all the results mentioned above are based on the fact that $f(n)$, $f(p)$, and so on, are typical integers for almost all $n$.

In 1995 (see [6]), we observed that one can map the set of positive integers $n$ into the set of $D$-ary integers by using the multiplicative structure of the positive integers $n$. Indeed, we proved that if we subdivide the set of primes $\wp$ into $D$ distinct subsets $\wp_j$, $j = 0, 1, \ldots, D-1$, of essentially the same size, and if $p_1 < \cdots < p_r$ are the prime divisors of $n$ with $p_j \in \wp_{\ell_j}$ for certain $\ell_j \in \{0, 1, \ldots, D-1\}$, then, for almost all $n$, the corresponding number $\ell_1 \ldots \ell_r$ is typical. Using this result, we recently constructed (see [5]) large families of normal numbers.

In this paper, we further expand on this approach but this time using the prime factorization of the values taken by primitive irreducible polynomials defined on the set of positive integers.

## 2. Notation

Given a set of positive integers $B$, we let $\mathcal{N}(B)$ stand for the multiplicative semigroup generated by $B$.

Let $D \geq 2$ be a fixed integer. Given an integer $t \geq 1$, an expression of the form $i_1 i_2 \ldots i_t$, where each $i_j$ is one of the numbers $0, 1, \ldots, D-1$, is called a *word* of length $t$. Given a word $\alpha$, we shall write $\lambda(\alpha) = t$ to indicate that $\alpha$ is a word of length $t$. We shall also use the symbol $\Lambda$ to denote the *empty word* and write $\lambda(\Lambda) = 0$.

Let $E = E_D = \{0, 1, 2, \ldots, D-1\}$. Then, $E^k$ will stand for the set of words of length $k$ over $E$, while $E^* = E_D^*$ will stand for the set of finite words over $E_D$, including the empty word $\Lambda$. The operation on $E_D^*$ is the concatenation $\alpha\beta$ for $\alpha, \beta \in E_D^*$. It is clear that $\lambda(\alpha\beta) = \lambda(\alpha) + \lambda(\beta)$. Also, we will say that $\alpha$ is a prefix of a word $\gamma$ if for some $\delta$, we have $\gamma = \alpha\delta$.

Given $n \in \mathbb{N}$, we shall write its $D$-ary expansion as

$$n = \varepsilon_0(n) + \varepsilon_1(n) D + \varepsilon_2(n) D^2 + \ldots + \varepsilon_t(n) D^t,$$

where $\varepsilon_i(n) \in E$ for $0 \leq i \leq t$ and $\varepsilon_t(n) \neq 0$. To this representation, we associate the word $\overline{n} = \varepsilon_0(n)\varepsilon_1(n) \ldots \varepsilon_t(n) \in E^{t+1}$.

Let $k$ be a fixed positive integer. For each word $\beta = b_1 \ldots b_k \in E^k$, we let $\nu_\beta(\overline{n})$ stand for the number of occurrences of $\beta$ in the $D$-ary expansion of the positive integer $n$, that is, the number of times that $\varepsilon_j(n) \ldots \varepsilon_{j+k-1}(n) = \beta$ as $j$ varies from 0 to $t - (k-1)$.

Given $\eta_\infty = \varepsilon_1\varepsilon_2\varepsilon_3\ldots$, where each $\varepsilon_i$ is an element of $E_D$ and, for each positive integer $N$, we let $\eta_N = \varepsilon_1\varepsilon_2\ldots\varepsilon_N$. Moreover, for each $\beta = \delta_1\ldots\delta_k \in E_D^k$ and integer $N \geq 2$, let $M(N,\beta)$ stand for the number of occurrences of $\beta$ as a subsequence of consecutive digits of $\eta_N$, that is

$$M(N,\beta) = \#\{(\alpha,\gamma) : \eta_N = \alpha\beta\gamma,\ \alpha,\gamma \in E_D^*\}.$$

We will say that $\eta_\infty$ is a *normal sequence* if

$$\lim_{N\to\infty} \frac{M(N,\beta)}{N} = \frac{1}{D^{\lambda(\beta)}} \quad \text{for all } \beta \in E_D^*. \tag{2.1}$$

Let $\xi < 1$ be a positive real number whose $D$-ary expansion is

$$\xi = 0.\varepsilon_1\varepsilon_2\varepsilon_3\ldots$$

and, for each integer $N \geq 1$, set

$$\xi_N = 0.\varepsilon_1\varepsilon_2\ldots\varepsilon_N.$$

With $\beta$ and $M(N,\beta)$ as above, we will say that $\xi$ is normal if (2.1) holds.

Let $Q_1, Q_2, \ldots, Q_h \in \mathbb{Z}[x]$ be distinct irreducible primitive monic polynomials each of degree no larger than 3. Recall that a polynomial with integer coefficients is said to be *primitive* if the greatest common divisor of its coefficients is 1. For each $\nu = 0,1,2,\ldots,D-1$, let $c_1^{(\nu)}, c_2^{(\nu)}, \ldots, c_h^{(\nu)}$ be distinct integers, $F_\nu(x) = \prod_{j=1}^h Q_j(x + c_j^{(\nu)})$, with $F_\nu(0) \neq 0$ for each $\nu$. Moreover, assume that the integers $c_i^{(\nu)}$ are chosen in such a way that $F_\nu(x)$ are squarefree polynomials and $\gcd(F_\nu(x), F_\mu(x)) = 1$ when $\nu \neq \mu$.

Let $\wp_0$ be the set of prime numbers $p$ for which there exist $\mu \neq \nu$ and $m \in \mathbb{N}$ such that $p|\gcd(F_\nu(m), F_\mu(m))$. It follows from Lemma 1 below that $\wp_0$ is a finite set. Now let

$$U(n) = F_0(n)F_1(n)\cdots F_{D-1}(n) = \vartheta\, q_1^{a_1} q_2^{a_2}\cdots q_r^{a_r},$$

where $\vartheta \in \mathcal{N}(\wp_0)$ and $q_1 < q_2 < \cdots < q_r$ are primes not belonging to $\mathcal{N}(\wp_0)$ with positive integers $a_i$. Then, let $h_n$ be defined on the prime powers $q^a$ of $U(n)$ by

$$h_n(q^a) = h_n(q) = \begin{cases} \Lambda & \text{if } q|\vartheta, \\ \ell & \text{if } q|F_\ell(n),\ q \notin \wp_0 \end{cases}$$

and further define $\alpha_n$ as

$$\alpha_n = h_n(q_1^{a_1})h_n(q_2^{a_2})\ldots h_n(q_r^{a_r}),$$

where on the right hand side we omit $\Lambda$ when $h_n(q_i^{a_i}) = \Lambda$ for some $i$. Finally, we let $\eta$ be the real number whose $D$-ary expansion is

$$\eta = 0.\alpha_1\alpha_2\alpha_3\ldots \tag{2.2}$$

As a simple example, take $h = 1$, $Q_1(x) = x$, $F_\nu(x) = x + \nu$ for $\nu = 0, 1, \ldots, D-1$, in which case we have $\wp_0 = \{p : p \le D-1\}$. Then,

$$U(n) = n(n+1)\cdots(n+D-1) = e(n) q_1^{a_1} \cdots q_r^{a_r},$$

where $e(n) := \prod_{\substack{q^\alpha \| U(n) \\ q \le D-1}} q^\alpha$, so that

$$h_n(q^a) = h_n(q) = \begin{cases} \Lambda & \text{if } q | e(n), \\ \ell & \text{if } q | n+\ell, \ q \notin \wp_0 \end{cases}$$

and

$$\alpha_n = h_n(q_1^{a_1}) h_n(q_2^{a_2}) \ldots h_n(q_r^{a_r}),$$

thus giving rise to the number

$$\eta = 0.\alpha_1\alpha_2\alpha_3\ldots$$

In the particular case $D = 5$, we get $U(n) = n(n+1)(n+2)(n+3)(n+4)$ so that $\wp_0 = \{2, 3\}$ and

$$h_n(q^a) = h_n(q) = \begin{cases} \Lambda & \text{if } q \in \{2, 3\}, \\ \ell & \text{if } q | n+\ell, \ q \ge 5, \ \text{where } \ell \in \{0, 1, 2, 3, 4\}. \end{cases}$$

In this case, one can check that

$$\eta = 0.\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\ldots = 0.43241302\ldots$$

Given an integer $n \ge 2$, we let $\omega(n)$ stand for the number of distinct prime divisors of $n$ and set $\omega(1) = 0$. We let $\varphi$ stand for the Euler function. Also, given a real number $x \ge 2$, we let $\pi(x)$ stand for the number of primes $p \le x$, while for coprime integers $k, \ell$, we let $\pi(x; k, \ell)$ stand for the number of prime numbers $p \le x$ such that $p \equiv \ell \pmod{k}$. For each real number $x \ge 2$, we set $\mathrm{li}(x) := \int_2^x \frac{dt}{\log t}$, a function often called the logarithmic integral. From here on, the letters $p$ and $q$, with or without subscript, always denote primes, while the letter $c$ always denotes a positive constant, but not necessarily the same at each occurrence. Finally, it will be convenient at times to write $x_1$ instead of $\log x$, and $x_2$ for $\log x_1$.

## 3. Main results

**Theorem 1.** *The number $\eta$ defined by (2.2) is a normal number.*

**Theorem 2.** *With the notations of Section 2 and assuming that deg$(Q_j) \leq 2$ for $j = 1, 2, \ldots, h$, then the number*

$$\xi = 0.\alpha_2\alpha_3\alpha_5 \ldots \alpha_p \ldots$$

*(where the above subscripts run over primes $p$) is a normal number.*

## 4. Preliminary lemmas

**Lemma 1.** *Given $F_1, F_2 \in \mathbb{Z}[x]$, which are relatively prime, then the congruences*

$$F_1(m) \equiv 0 \pmod a \quad \textit{and} \quad F_2(m) \equiv 0 \pmod a$$

*have common roots for at most finitely many $a$'s.*

Proof. A proof of this result was established by Tanaka [14]. □

**Lemma 2.** *Let $F$ be an arbitrary primitive polynomial with integer coefficients and of degree $\nu$. Let $D$ be the discriminant of $F$ and assume that $D \neq 0$. Let $\rho(m)$ be the number of solutions $n$ of $F(n) \equiv 0 \pmod m$. Then $\rho$ is a multiplicative function whose values on the prime powers $p^\alpha$ satisfy*

$$\rho(p^\alpha) \quad \begin{cases} = \rho(p) & \text{if } p \not| D, \\ \leq 2D^2 & \text{if } p | D. \end{cases}$$

*Moreover, there exists a positive constant $c = c(F)$ such that $\rho(p^\alpha) \leq c$ for all prime powers $p^\alpha$.*

Proof. This assertion is well known. □

**Lemma 3.** *If $g \in \mathbb{Q}[x]$ is an irreducible polynomial and $\rho(m)$ stands for the number of residue classes mod $m$ for which $g(n) \equiv 0 \pmod m$, then*

*(i)* $\displaystyle\sum_{p \leq x} \rho(p) = \frac{x}{\log x} + O\left(\frac{x}{\log^2 x}\right)$;

*(ii)* $\displaystyle\sum_{p \leq x} \frac{\rho(p)}{p} = \log\log x + c + O\left(\frac{1}{\log x}\right)$.

Proof. This result is due to Landau [11]. □

**Lemma 4.** (Brun-Titchmarsh Inequality) *There exists a positive constant $c$ such that*

$$\pi(x;k,\ell) < c\frac{x}{\varphi(k)\log(x/k)} \qquad \text{for all } k < x.$$

Proof. For a proof, see the book of Halberstam and Richert [8]. □

**Lemma 5.** (Bombieri-Vinogradov Theorem) *Given any fixed number $A > 0$, there exists a number $B = B(A) > 0$ such that*

$$\sum_{k \le \sqrt{x}/(\log^B x)} \max_{(k,\ell)=1} \max_{y \le x} \left| \pi(y;k,\ell) - \frac{li(y)}{\varphi(k)} \right| = O\left(\frac{x}{\log^A x}\right).$$

*Moreover, an appropriate choice for $B(A)$ is $2A+6$.*

Proof. For a proof, see the book of Iwaniec and Kowalski [10]. □

**Lemma 6.** *Let $F$ be a squarefree polynomial with integer coefficients and of positive degree such that the degree of each of its irreducible factors is of degree no larger than 3. Let $Y(x)$ be a function which tends to $+\infty$ as $x \to +\infty$. Then*

$$\lim_{x\to\infty} \frac{1}{x} \#\{n \le x : p^2 | F(n) \text{ for some } p > Y(x)\} = 0.$$

Proof. For a proof, see the book of Hooley [9] (pp. 62-69). □

**Lemma 7.** *Let $F$ and $Y$ be as in Lemma 6. Assume that each of the irreducible factors of $F$ is of degree no larger than 2 and that $F(0) \neq 0$. Then*

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \#\{p \le x : q^2 | F(p) \text{ for some } q > Y(x)\} = 0.$$

Proof. For a proof, see the book of Hooley [9] (pp. 69-72). □

**Lemma 8.** *Let $f(n)$ be a real valued non negative arithmetic function. Let $a_n$, $n = 1, \ldots, N$, be a sequence of integers. Let $r$ be a positive real number, and let $p_1 < p_2 < \cdots < p_s \le r$ be prime numbers. Set $Q = p_1 \cdots p_s$. If $d|Q$, then let*

$$\sum_{\substack{n=1 \\ a_n \equiv 0 \pmod d}}^{N} f(n) = \kappa(d)X + R(N,d), \tag{4.1}$$

*where $X$ and $R$ are real numbers, $X \ge 0$, and $\kappa(d_1 d_2) = \kappa(d_1)\kappa(d_2)$ whenever $d_1$ and $d_2$ are co-prime divisors of $Q$.*

*Assume that for each prime $p$, $0 \le \kappa(p) < 1$. Setting*

$$I(N,Q) := \sum_{\substack{n=1 \\ (a_n,Q)=1}}^{N} f(n),$$

*then the estimate*

$$I(N,Q) = \{1 + 2\theta_1 H\} X \prod_{p|Q} (1-\kappa(p)) + 2\theta_2 \sum_{\substack{d|Q \\ d \le z^3}} 3^{\omega(d)} |R(N,d)|$$

*holds uniformly for $r \ge 2$, $\max(\log r, S) \le \frac{1}{8}\log z$, where $|\theta_1| \le 1$, $|\theta_2| \le 1$, and*

$$H = \exp\left(-\frac{\log z}{\log r}\left\{\log\left(\frac{\log z}{S}\right) - \log\log\left(\frac{\log z}{S}\right) - \frac{2S}{\log z}\right\}\right)$$

*and*

$$S = \sum_{p|Q} \frac{\kappa(p)}{1-\kappa(p)} \log p.$$

*When these conditions are satisfied, there exists an absolute positive constant $c$ such that $2H \le c < 1$.*

P r o o f. This result is Lemma 2.1 in the book of Elliott [7]. □

**Lemma 9.** *There exists a positive constant $c = c(h,D)$ such that*

$$\frac{1}{x}\sum_{n \le x} |\omega(U(n)) - hD\log\log x|^2 \le c\, \log\log x, \tag{4.2}$$

$$\frac{1}{x} \sum_{\substack{n \le x \\ \omega(U(n)) > hDx_2 + cx_2^{3/4}}} \omega(U(n)) \ll \sqrt{\log\log x} \tag{4.3}$$

*and*

$$\frac{1}{x} \sum_{\substack{n \le x \\ \omega(U(n)) < hDx_2 - cx_2^{3/4}}} \omega(U(n)) \ll \sqrt{\log\log x}. \tag{4.4}$$

P r o o f. First observe that

$$\omega(U(n)) = \sum_{j=0}^{D-1} \omega(F_j(n)) + O(1), \tag{4.5}$$

where the term $O(1)$ accounts for the possible common prime divisors of $F_\nu(n)$ and $F_\mu(n)$, which as we saw are in finite number.

From the Turán-Kubilius inequality,

$$\frac{1}{x}\sum_{n\le x}\left(\omega(F_\nu(n))-\sum_{p\le x}\frac{\rho_{F_\nu}(p)}{p}\right)^2 < c\left(1+\sum_{p\le x}\frac{\rho_{F_\nu}(p)}{p}\right). \tag{4.6}$$

On the other hand, it follows from Lemma 3 (ii) that

$$\sum_{p\le x}\frac{\rho_{F_\nu}(p)}{p}=\sum_{j=0}^{h-1}\sum_{p\le x}\frac{\rho_{Q_j(x+c_j^{(\nu)})}(p)}{p}=h\log\log x+O(1). \tag{4.7}$$

Combining (4.5), (4.6) and (4.7), inequality (4.2) follows.
Setting

$$\Sigma_A:=\sum_{\substack{n\le x\\ \omega(U(n))>hDx_2+cx_2^{3/4}}}1$$

and using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\sum_{\substack{n\le x\\ \omega(U(n))>hDx_2+cx_2^{3/4}}}\omega(U(n)) &= \sum_{\substack{n\le x\\ \omega(U(n))>hDx_2+cx_2^{3/4}}}(\omega(U(n))-hDx_2) + hDx_2\sum_{\substack{n\le x\\ \omega(U(n))>hDx_2+cx_2^{3/4}}}1\\ &\le \Sigma_A^{1/2}\times\left(\sum_{\substack{n\le x\\ \omega(U(n))>hDx_2+cx_2^{3/4}}}|\omega(U(n))-hDx_2|^2\right)^{1/2}+hDx_2\,\Sigma_A. \end{aligned}\tag{4.8}$$

Now, it follows from (4.2) that

$$\Sigma_A\le\frac{x}{\sqrt{x_2}}. \tag{4.9}$$

Hence, in light of (4.2) and (4.9), estimate (4.8) yields

$$\sum_{\substack{n\le x\\ \omega(U(n))>hDx_2+cx_2^{3/4}}}\omega(U(n))\ll\Sigma_A^{1/2}\sqrt{x}\cdot x_2^{1/2}+x_2\Sigma_A\ll x\,x_2^{1/4}+x\,x_2^{1/2}\ll x\,x_2^{1/2},$$

thereby completing the proof of inequality (4.3). Clearly, (4.4) can be obtained in a similar way.

□

**Lemma 10.** *There exists a positive constant $c = c(h, D)$ such that*

$$\frac{1}{li(x)} \sum_{p \le x} |\omega(U(p)) - hD \log\log x|^2 \le c \log\log x, \tag{4.10}$$

$$\frac{1}{li(x)} \sum_{\substack{p \le x \\ \omega(U(p)) > hDx_2 + cx_2^{3/4}}} \omega(U(p)) \ll \sqrt{\log\log x} \tag{4.11}$$

*and*

$$\frac{1}{li(x)} \sum_{\substack{p \le x \\ \omega(U(p)) < hDx_2 - cx_2^{3/4}}} \omega(U(p)) \ll \sqrt{\log\log x}. \tag{4.12}$$

P r o o f. The proof follows essentially by observing that

$$\max_{p \le x} \sum_{\substack{q \mid U(p) \\ q > x^{1/4}}} 1 = O(1)$$

and then using Lemma 5 and the Turán-Kubilius inequality. □

Let $\varepsilon_x = 1/\sqrt{x_2}$, $Y_x = \exp\{x_1^{\varepsilon_x}\}$ and $Z_x = \exp\{x_1^{1-\varepsilon_x}\}$. Also, let

$$\wp_1 = \{p : p \le Y_x, p \notin \wp_0\}, \quad \wp_2 = \{p : Y_x < p < Z_x\}, \quad \wp_3 = \{p : p \ge Z_x\}.$$

Finally, for each $j = 0, 1, 2, 3$, set $\omega_j(n) = \sum_{\substack{p \mid n \\ p \in \wp_j}} 1$.

**Lemma 11.** *With the above notation, we have*

$$\sum_{n \le x} \omega_1(U(n)) \ll x \sum_{p \le Y_x} \frac{1}{p} \ll x\varepsilon_x x_2 = x\sqrt{x_2}, \tag{4.13}$$

$$\sum_{n \le x} \omega_3(U(n)) \ll x \sum_{Z_x \le p < x^{1/4}} \frac{1}{p} + O(x) \ll x\sqrt{x_2}, \tag{4.14}$$

$$\sum_{p \le x} \omega_1(U(p)) \ll li(x)\sqrt{x_2}, \tag{4.15}$$

$$\sum_{p \le x} \omega_3(U(p)) \ll li(x)\sqrt{x_2}. \tag{4.16}$$

P r o o f. Estimates (4.13) and (4.14) are straightforward. The other two estimates follow using Lemma 4. □

Let us write each positive integer $n$ as $n = A(n)B(n)C(n)$, where $A(n) \in \mathcal{N}(\wp_0 \cup \wp_1)$, $B(n) \in \mathcal{N}(\wp_2)$ and $C(n) \in \mathcal{N}(\wp_3)$. Observe that $\rho_{F_\nu}(n) = \rho_{F_\mu}(n)$ for every $\nu$, $\mu$.

**Lemma 12.** *Let $m_0, m_1, \dots, m_{D-1}$ be squarefree coprime numbers belonging to $\mathcal{N}(\wp_2)$, with $M = m_0 m_1 \cdots m_{D-1} \le \sqrt{x}$. Let $T(x|m_0, m_1, \dots, m_{D-1})$ be the number of those integers $n \le x$ for which $B(F_j(n)) = m_j$ for $j = 0, 1, \dots, D-1$. Then,*

$$\left| T(x|m_0, m_1, \dots, m_{D-1}) - \frac{x\rho(M)\varphi(M)}{M^2} \prod_{p\in\wp_2} \left(1 - \frac{D\rho_F(p)}{p}\right) K(M) \right|$$
$$\ll \frac{x\rho(M)}{M} \exp\{-x_1^{\varepsilon_x}\}, \tag{4.17}$$

*where*

$$K(M) = \prod_{p|M} \left(1 - \frac{D\rho_F(p)}{p}\right)^{-1}.$$

**Remark 1.** *Observe that $K(M) = 1 + o(1)$.*

P r o o f. Observe that $M$ is squarefree. For convenience, let $\rho = \rho_{F_\nu}$. It is clear that the congruences

$$B(F_j(m)) \equiv 0 \pmod{m_j} \qquad (j = 0, 1, \dots, D-1)$$

hold for $n \equiv \ell_i \pmod{M}$, $i = 1, 2, \dots, \rho(M)$.

Let us now consider $\ell = \ell_i$ for a fixed $i \in [1, \rho(M)]$ and define

$$\begin{aligned} \varphi_j(k) &= \frac{F_j(\ell + kM)}{m_j} \qquad (j = 0, 1, \dots, D-1), \\ \Phi(k) &= \varphi_0(k)\varphi_1(k)\cdots\varphi_{D-1}(k). \end{aligned} \tag{4.18}$$

Finally, let $Q = \prod_{p\in\wp_2} p$.

We now apply Lemma 8 with $f(k) = 1$, $a_k = \Phi(k)$ and $X = x/M$, and obtain an estimate for each corresponding $I_i(X, Q)$ for the particular choice $\ell = \ell_i$. With this set up, we have

$$T(x|m_0, m_1, \dots, m_{D-1}) = \sum_{i=1}^{\rho(M)} I_i(X, Q). \tag{4.19}$$

Observe that $\eta(p^\alpha) = \eta(p) = 0$ if $p \in \wp_1$. On the other hand, for $p \in \wp_2 \cup \wp_3$, we have $\rho_{\varphi_j}(p^\alpha) = \rho_{\varphi_j}(p)$ and also that if $p|m_j$, then $\rho_{\varphi_j}(p) = 1$ and $\rho_{\varphi_\ell}(p) = 0$ for $\ell \ne j$, while on the other hand if $(p, M) = 1$, then $\rho_{\varphi_j}(p) = \rho(p)$ for $j = 0, 1, \dots, D-1$.

Now we denote by $\eta(M)$ the number of those $k$ mod $M$ such that $\Phi(k) \equiv 0 \pmod{M}$. Then one can easily show that

$$\eta(p^\alpha) = \eta(p) = \begin{cases} 0 & \text{if } p \in \wp_1, \\ \rho_{\varphi_j}(p) = 1 & \text{if } p | m_j, \\ \rho(p) & \text{if } p \in \wp_2 \cup \wp_3,\ (p, M) = 1. \end{cases} \tag{4.20}$$

It is also clear that the error term in (4.1) satisfies

$$|R(x,d)| \leq D\rho(d). \tag{4.21}$$

It follows from Lemma 8 that

$$I_i(X,Q) = (1 + O(H))\frac{x}{M}\prod_{p|Q}\left(1 - \frac{\eta(p)}{p}\right) + O\left(\sum_{\substack{d|Q \\ d \leq z^3}} 3^{\omega(d)}|R(X,d)|\right). \tag{4.22}$$

Using the notation of Lemma 8, we have

$$S = \sum_{p|Q} \frac{\eta(p)}{p - \eta(p)} \log p,$$

and one can show that there exist two positive constants $c_1 < c_2$ such that

$$c_1 < \frac{S}{(\log x)^{\varepsilon_x}} < c_2. \tag{4.23}$$

Moreover, we have that $\log r = (\log x)^{\varepsilon_x}$. So, we choose $\log z = (\log x)^{\delta_x}$, with $0 < \varepsilon_x < \delta_x$, where $\delta_x$ is a function which tends to 0 as $x \to \infty$ and which will be determined later.

We can prove that for $z \geq 2$,

$$\sum_{\substack{d|Q \\ d \leq z^3}} 3^{\omega(d)}\eta(d) \leq cz^3(\log z)^K, \tag{4.24}$$

for a suitable large constant $K$. Indeed,

$$\sum_{d \leq Y} 3^{\omega(d)}\eta(d)|\mu(d)|\log d \leq \sum_{pu \leq Y} 3^{\omega(pu)}(\log p)\eta(p)\eta(u)|\mu(u)|$$

$$\leq 3\sum_{u \leq Y} 3^{\omega(u)}\eta(u)|\mu(u)| \sum_{p \leq Y/u} \eta(p)\log p. \tag{4.25}$$

Since $\displaystyle\sum_{p\le Y/u}\eta(p)\log p\le c\frac{Y}{u}$, (4.25) becomes

$$\begin{aligned}\sum_{d\le Y}3^{\omega(d)}\eta(d)|\mu(d)|\log d &\le cY\sum_{u\le Y}\frac{3^{\omega(u)}\eta(u)}{u}|\mu(u)|\\ &\le cY\prod_{p\le Y}\left(1+\frac{3\eta(p)}{p}\right)\le cY\exp\left\{3\sum_{p\le Y}\frac{\eta(p)}{p}\right\}\\ &\le cY\exp(3h\log\log Y)=cY(\log Y)^{3h}. \qquad (4.26)\end{aligned}$$

Let us write

$$\sum_{d\le Y}3^{\omega(d)}\eta(d)|\mu(d)|=\sum_{d\le\sqrt{Y}}+\sum_{\sqrt{Y}<d\le Y}=S_1+S_2, \qquad (4.27)$$

say. Clearly we have

$$S_1\ll\sqrt{Y}\cdot Y^{\varepsilon}, \qquad (4.28)$$

where $\varepsilon>0$ can be taken arbitrarily small. On the other hand, in light of (4.26), we have

$$S_2\le\frac{2}{\log Y}\cdot cY(\log Y)^{3h}\ll Y(\log Y)^{3h-1}. \qquad (4.29)$$

Setting $Y=z^3$ and using (4.28) and (4.29) in (4.27) proves (4.24).

Coming back to our choice of $z$ and to the size of $S$ given by (4.23), we have

$$\frac{\log z}{(\log x)^{\varepsilon_x}}=x_1^{\delta_x-\varepsilon_x},\qquad \frac{\log z}{S}\approx x_1^{\delta_x-\varepsilon_x}.$$

Therefore, by choosing $\delta_x=2\varepsilon_x$, we obtain

$$H\le C\exp\left\{-\frac{1}{2}(\delta_x-\varepsilon_x)x_2\cdot x_1^{\delta_x-\varepsilon_x})\right\},$$

that is,

$$H\le C\exp\{-\varepsilon_x\cdot x_2\cdot x_1^{\varepsilon_x}\}. \qquad (4.30)$$

Moreover,

$$\begin{aligned}\prod_{p|Q}\left(1-\frac{\eta(p)}{p}\right) &= \prod_{\substack{p\in\wp_2\\(p,M)=1}}\left(1-\frac{D\rho_F(p)}{p}\right)\prod_{p|M}\left(1-\frac{1}{p}\right)\\ &= \frac{\varphi(M)}{M}K(M)\prod_{p\in\wp_2}\left(1-\frac{D\rho_F(p)}{p}\right). \qquad (4.31)\end{aligned}$$

Using (4.24), (4.30) and (4.31) in (4.22), and then using this in (4.19), we obtain that inequality (4.17) follows immediately, thus completing the proof of Lemma 12.

□

As a corollary of Lemma 12, we have the following.

**Lemma 13.** *Let* $\varepsilon_x = 1/\sqrt{x_2}$. *Let* $M$ *squarefree,* $M \in \mathcal{N}(\wp_2)$, $M \le \sqrt{x}$. *Now assume that* $m_0, m_1, \ldots, m_{D-1}$ *and* $m'_0, m'_1, \ldots, m'_{D-1}$ *are so chosen that*

$$M = m_0 m_1 \cdots m_{D-1} = m'_0 m'_1 \cdots m'_{D-1}.$$

*Then,*

$$\left| T(x|m_0, m_1, \ldots, m_{D-1}) - T(x|m'_0, m'_1, \ldots, m'_{D-1}) \right| \ll \frac{x\rho(M)}{M} \exp\{-x_1^{\varepsilon_x}\},$$

*while*

$$T(x|m_0, m_1, \ldots, m_{D-1}) \gg \frac{x\rho(M)\varphi(M)}{M^2} \exp\{-c\sqrt{x_2}\}.$$

We want to estimate the number of primes $p \le x$ such that

$$B(F_\nu(p)) = m_\nu \qquad (\nu = 0, 1, 2, \ldots, D-1).$$

Let

$$M = m_0 m_1 \cdots m_{D-1}, \qquad M \in \mathcal{N}(\wp_2) \text{ and } \mu^2(M) = 1.$$

Let the solutions of

$$F_\nu(n) \equiv 0 \pmod{m_\nu} \qquad (\nu = 0, 1, \ldots, D-1)$$

modulo $M$ be $\ell_1, \ldots, \ell_{\rho(M)}$. It is clear that $(\ell_i, M) = 1$ for each $i$. Indeed, assume that $p^*|(\ell_i, M)$. Then, $p^*|\ell_i + kM$ for some $k$, while $p^*|m_\nu$ for some $\nu$, thereby implying that $0 \equiv F_\nu(\ell_i + km) \equiv F_\nu(0) \pmod{p^*}$, which in turn yields $p^*|F_\nu(0)$, which is not possible since $F_\nu(0) \ne 0$ and $p^* \ge Y_x$.

We now fix an integer $\ell \in \{\ell_1, \ldots, \ell_{\rho(M)}\}$, consider the function $\Phi(k)$ defined in (4.18) and define the function

$$f_\ell(k) = \begin{cases} 1 & \text{if } \ell + kM \in \wp, \\ 0 & \text{otherwise.} \end{cases}$$

We shall now estimate $\displaystyle\sum_{\substack{kM+\ell \le x \\ (\Phi(k), Q)=1}} f_\ell(k)$.

First we consider the sum $\sum_{\substack{kM+\ell\le x\\ \Phi(k)\equiv 0 \pmod d}} f_\ell(k)$. Assume that $\Phi(k)\equiv 0 \pmod d$ holds for $k=k_1,\ldots,k_{\rho_\Phi}$. For a fixed $k=k^*$, we need to count the number of primes $p=\ell+k^*M+\nu dM\le x$. We will now make use of Lemma 6. Letting

$$\Delta(x,H)=\sup_{(\ell,H)=1}\left|\pi(x;H,\ell)-\frac{\mathrm{li}(x)}{\varphi(H)}\right|,$$

we have

$$\sum_{j=1}^{\rho_\Phi}\pi(x;dM,\ell+k_jM)=\frac{\mathrm{li}(x)}{\varphi(dM)}\sum_{j=1}^{\rho_\Phi}\sum_{(\ell+k_jM,d)=1}1+O\left(\Delta(x,dM)\right).$$

Let $\kappa(d):=\sum_{\substack{j=1\\(\ell+k_jM,d)=1}}^{\rho_\Phi}\frac{1}{\varphi(dM)}$ and set

$$V:=\sum_{d|Q}\mu(d)\kappa(d)=\sum_{j=1}^{\rho_\Phi}\sum_{\substack{d|Q\\(\ell+k_jM,d)=1}}\frac{\mu(d)}{\varphi(dM)}.$$

Moreover, let $d=d_1d_2$, where $d_1|M$, $(d_2,M)=1$. Since $(M,\ell+k_jM)=1$, then $(d_1,\ell+k_jM)=1$ holds. Therefore,

$$\begin{aligned}
V &= \sum_{j=1}^{\rho_\Phi}\frac{1}{\varphi(M)}\cdot\sum_{d_1|M}\frac{\mu(d_1)}{d_1}\cdot\sum_{\substack{d_2|Q\\(d_2,(\ell+k_jM)M)=1}}\frac{\mu(d_2)}{\varphi(d_2)}\\
&= \sum_{j=1}^{\rho_\Phi}\frac{1}{M}\sum_{\substack{d_2|Q\\(d_2,(\ell+k_jM)M)=1}}\frac{\mu(d_2)}{\varphi(d_2)}\\
&= \sum_{j=1}^{\rho_\Phi}\frac{1}{M}\prod_{p|Q}\left(1-\frac{1}{p-1}\right)\cdot R_{\ell,j},
\end{aligned}$$

where $R_{\ell,j}=\prod_{\substack{p|Q\\p|(\ell+k_jM)M}}\left(1-\frac{1}{p-1}\right)^{-1}$.

Since $R_{\ell,j}\ge 1$ and

$$\log R_{\ell,j}\le 2\sum_{\substack{p|Q\\p|(\ell+k_jM)M}}\frac{1}{p}\le\frac{\log M+\log(\ell+k_jM)}{Y_x\log Y_x}\le\frac{c}{e^{(\log x)^{\varepsilon_x}}\cdot(\log x)^{\varepsilon_x}},$$

it follows that

$$R_{\ell,j} = 1 + O\left(e^{-(\log x)^{\varepsilon_x}}\right).$$

On the other hand,

$$\begin{aligned}
\sum_{\substack{d\mid Q\\ d\le z^3}} \Delta(x,d)\cdot 3^{\omega(d)} &\le \sum_{\substack{d\mid Q\\ \omega(d)>Ax_2}} c\frac{3^{\omega(d)}\mathrm{li}(x)}{\varphi(d)} + \sum_{\substack{d\mid Q\\ \omega(d)\le Ax_2}} c3^{Ax_2}\,|\Delta(x,d)| \\
&\le c\,\mathrm{li}(x)\sum_{\substack{d\mid Q\\ \omega(d)>Ax_2}} \frac{3^{\omega(d)}}{\varphi(d)} + c3^{Ax_2}\sum_{d\le\sqrt{x}}|\Delta(x,d)| \\
&\le c\,\mathrm{li}(x)3^{-Ax_2}\prod_{p\mid Q}\left(1+\frac{3^2}{p-1}\right) + c\frac{3^{Ax_2}\mathrm{li}(x)}{\log^B x} \\
&\le \frac{c\,\mathrm{li}(x)}{(\log x)^C}, \qquad (4.32)
\end{aligned}$$

where $C$ is an arbitrary constant. Indeed, let $A = C$. Since $z^3 \le x^{1/4}$, we may apply Lemma 6 for an arbitrary constant $B$, thereby yielding (4.32).

Hence, with

$$M = m_0 m_1 \cdots m_{D-1} \in \mathcal{N}(\wp_2), \qquad \mu^2(M) = 1$$

and

$$K(x\mid m_0, m_1, \ldots, m_{D-1}) = \#\{p \le x : B(F_\nu(p)) = m_\nu, \quad \nu = 0,1,\ldots,D-1\},$$

we have

$$\left| K(x\mid m_0, m_1,\ldots,m_{D-1}) - \frac{x}{M}\prod_{p\mid Q}\left(1-\frac{1}{p-1}\right)\sum_{\ell,j} R_{\ell,j}\right|$$

$$\ll \mathrm{li}(x)\frac{\rho(M)}{M}(\log x)^{-B},$$

where $B$ is an arbitrary constant.

This setup will now facilitate the proofs of our theorems.

## 5. Proof of Theorem 1

Recall that given a word $\beta = b_1 b_2 \ldots b_k \in E_D^k$, $\nu_\beta(\delta)$ stands for the number of occurrences of $\beta$ in $\delta$, that is the number of solutions $\tau_1, \tau_2 \in E_D^*$ such that

$\delta = \tau_1\beta\tau_2$. Note that it is clear that, given any $\gamma_1, \gamma_2 \in E_D^k$,

$$\nu_\beta(\gamma_1) + \nu_\beta(\gamma_2) \le \nu_\beta(\gamma_1\gamma_2) \le \nu_\beta(\gamma_1) + \nu_\beta(\gamma_2) + k.$$

Let $N$ be a large integer and let $\theta_N$ be the prefix of length $N$ of the infinite sequence $\alpha_1\alpha_2\ldots$. Moreover, let $x$ be the largest integer for which

$$\lambda(\alpha_1 \ldots \alpha_x) \le N < \lambda(\alpha_1 \ldots \alpha_x\alpha_{x+1}).$$

Since $\lambda(\alpha_{x+1}) \le \omega(U(x+1)) \le c\log x$, we have

$$N + O(\log x) = \sum_{n\le x} \lambda(\alpha_n) = \sum_{n\le x} (\omega(U(n)) + O(1)) = O(x) + hDx\log\log x.$$

Therefore,

$$x = \frac{N}{hD\log\log N} + O\left(\frac{N}{(\log\log N)^2}\right).$$

Let $\theta_N = \alpha_1 \ldots \alpha_x$. For each $n \in [1, x]$, let $\alpha_n = \gamma_n\kappa_n\delta_n$, where $\gamma_n$ is the word composed from $h_n(q)$ where $q$ runs over those prime divisors of $U(n)$ which belong to the set $\wp_1$ and similarly $\delta_n$ is composed from those $h_n(q)$ where $q$ runs over the prime divisors of $U(n)$ which belong to $\wp_3$.

We have $\lambda(\gamma_n) \le \omega_1(U(n))$ and $\lambda(\delta_n) \le \omega_3(U(n))$, so that by (4.13) and (4.14), we obtain that

$$\sum_{n\le x} \lambda(\gamma_n) \ll x\sqrt{x_2} \qquad \text{and} \qquad \sum_{n\le x} \lambda(\delta_n) \ll x\sqrt{x_2},$$

thereby implying that

$$\nu_\beta(\theta_N) = \sum_{n=1}^{x} \nu_\beta(\kappa_n) + O(x\sqrt{x_2}). \tag{5.1}$$

Using estimates (4.3) and (4.4) of Lemma 9, it follows from (5.1) that

$$\nu_\beta(\theta_N) = \sum_{\substack{n=1\\ n\in\mathcal{J}}}^{x} \nu_\beta(\kappa_n) + O(x\sqrt{x_2}), \tag{5.2}$$

where

$$\mathcal{J} := \{n : |\omega(U(n)) - hDx_2| \le cx_2^{3/4}\}.$$

Now, let

$$\mathcal{J}' := \{n \in \mathcal{J} : q^2|U(n) \text{ for } q \in \wp_2\}.$$

We claim that we can drop from the sum in (5.2) those $n \in \mathcal{J}'$, since one can show by Lemma 6 that

$$\sum_{\substack{n\le x\\ n\in\mathcal{J}'}} \nu_\beta(\kappa_n) = o(x\log\log x) \qquad (x\to\infty).$$

For the remaining integers $n \le x$, $n \in \mathcal{J} \setminus \mathcal{J}'$, we have

$$B(F_\nu(n)) = m_\nu \qquad (\nu = 0, 1, \ldots, D-1),$$

with $M = m_0 m_1 \cdots m_{D-1}$, $M$ squarefree, $|\omega(M) - hD \log\log x| \le C x_2^{3/4}$. We then have

$$M \le Z_x^{2hDx_2} \le x^{\varepsilon_x},$$

say.

Now, let $M \in \mathcal{N}(\wp_2)$, squarefree, $M \le x^{\varepsilon_x}$, $M = q_1 \cdots q_S$ for primes $q_1 < \ldots < q_S$, $|S - hDx_2| \le c x_2^{3/4}$.

With $M = m_0 m_1 \cdots m_{D-1}$ being any representation, we have by Lemma 12,

$$\begin{aligned} T(x|m_0, m_1, \ldots, m_{D-1}) &= x \frac{\rho(M)\varphi(M)}{M} \prod_{p \in \wp_2} \left(1 - \frac{D\rho_F(p)}{p}\right) \cdot K(M) \\ &\quad + O\left(x \frac{\rho(M)}{M} \exp\{-x^{\varepsilon_x}\}\right). \end{aligned}$$

For a fixed $M$, consider all $m_0, m_1, \ldots, m_{D-1}$ for which $M = m_0 m_1 \cdots m_{D-1}$. Let $\tau_D(M)$ be the number of solutions of $M = m_0 m_1 \cdots m_{D-1}$. It is clear that $\tau_D$ is a multiplicative function and that $\tau_D(p) = p$. If $m_0, m_1, \ldots, m_{D-1}$ run over all the possible choices, then the corresponding $\beta_n$'s run over all the possible words of length $S$ in $E_D^k$. Indeed, let $\varepsilon_1 \ldots \varepsilon_S \in E_D^S$ and let $m_j = \prod_{\varepsilon_\ell = j} q_\ell$ $(j = 0, 1, \ldots, S-1)$. We then have

$$\begin{aligned} \nu_\beta(\theta_N) &= x \sum_{\substack{M \le x^{\varepsilon x} \\ M \text{ squarefree } \in \mathcal{N}(\wp_2) \\ |\omega(M) - hDx_2| \le c x_2^{3/4}}} \frac{\rho(M)\varphi(M)}{M^2} K(M) \prod_{p \in \wp_2} \left(1 - \frac{D\rho_F(p)}{p}\right) \sum_{\rho \in E_D^S} \nu_\beta(\rho) \\ &\quad + O\left(\sum_{M \le x^{\varepsilon x}} x \frac{\rho(M)\omega(M)\tau_D(M)}{M} \exp\{-x_1^{\varepsilon_x}\}\right) + O\left(x \cdot x_2^{3/4}\right). \end{aligned}$$

Let $\Sigma_0$ be the second error term above.

It is easy to see that

$$\sum_{\rho \in E_D^S} \nu_\beta(\rho) = (s - k + 1) D^{s-k}.$$

Hence, it follows that

$$\begin{aligned} \Sigma_0 &\ll x \exp\{-x_1^{\varepsilon_x}\} x_2 \prod_{p \in \wp_2} \left(1 + \frac{\rho(p)\tau_D(p)}{p}\right) \\ &\ll x \exp\{-x_1^{\varepsilon_x}\} x_2 \cdot (\log x)^\kappa \end{aligned}$$

$$\ll \quad x.$$

Now, let $\beta_1, \beta_2$ be arbitrary distinct words belonging to $E_D^k$. Then,

$$|\nu_{\beta_1}(\theta_N) - \nu_{\beta_2}(\theta_N)| \ll x \cdot x_2^{3/4}.$$

Since

$$\sum_{\beta \in E_D^k} \nu_\beta(\theta_N) = N + O(\log N)$$

and since $x \approx N/(\log\log N)$, it follows that

$$\left|\nu_\beta(\theta_N) - \frac{N}{D^k}\right| \le \frac{1}{D^k} \sum_{\beta_1 \in E_D^k} |\nu_\beta(\theta_N) - \nu_{\beta_1}(\theta_N)| + O\left(\frac{N}{(\log\log N)^{1/4}}\right),$$

thus establishing that

$$\limsup_{N\to\infty} \frac{\nu_\beta(\theta_N)}{N} = \frac{1}{D^k}$$

and thereby completing the proof of Theorem 1.

## 6. The proof of Theorem 2

The proof of Theorem 2 can be obtained along the same lines as that of Theorem 1, if one uses Lemma 13 instead of Lemma 12, along with Lemmas 7 and 10, as well as inequalities (4.13) and (4.14) of Lemma 11.

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