Distribution of consecutive digits in the q-ary expansions of some subsequences of integers II

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Abstract

We study the normality of the distribution of consecutive digits in the q-ary expansion of integers belonging to particular subsequences of the positive integers.

1 Introduction

For each integer $q \geq 2$, let $A_q = \{0, 1, \ldots, q-1\}$. The q-ary expansion of a positive integer n is the representation

$$n = \sum_{j=0}^{\infty} a_j(n) q^j, \qquad \text{where each } a_j(n) \in A_q, \tag{1.1}$$

observing that the above sum is clearly finite, since $a_j(n) = 0$ as soon as $q^j > n$.

Let $P \in \mathbb{Z}[x]$ be an arbitrary polynomial of degree r with a positive leading coefficient.

Let k be a positive integer. We write A_q^k to denote the set $\underbrace{A_q \times \cdots \times A_q}_{k}$. A typical element of A_q^k will be denoted by $\underline{b} = (b_0, b_1, \ldots, b_{k-1})$ with each $b_\nu \in A_q$.

Let $F = F_k : A_q^k \to \mathbb{R}$ be such that $F(0, 0, \ldots, 0) = 0$.

Moreover, for positive integers n represented as in (1.1), we consider the functions $\phi_0^k(n), \phi_1^k(n), \ldots$ given by

$$\phi_j^k(n) = (a_j(n), \ldots, a_{j+k-1}(n)).$$

With these notations, we further introduce the sequences $\{\alpha_n\}_{n\geq 1}$ and $\{\beta_n\}_{n\geq 1}$ defined by

$$\alpha_n = \sum_{j=0}^{\infty} F_k(\phi_j^k(P(n))), \qquad \beta_n = \sum_{j=0}^{\infty} F_k(\phi_j^k(n)).$$

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Let also $M = \frac{1}{q^k} \sum_{\underline{b} \in A_q^k} F_k(\underline{b})$ and, for each $h = 0, \dots, k-1$,

$$\sigma_h^2 = \frac{1}{q^{k+h}} \sum_{(b_0,\dots,b_{k+h-1}) \in A_q^{k+h}} \left(F_k(b_0,\dots,b_{k-1}) - M\right)\left(F_k(b_h,\dots,b_{h+k-1}) - M\right).$$

Also, set

$$\sigma^2 = \sigma_0^2 + 2\sum_{h=1}^{k-1} \sigma_h^2. \tag{1.2}$$

Finally, for convenience, from here on, assume that x is a large number and let $N = \left\lfloor \frac{\log x}{\log q} \right\rfloor$.

In 1996, Bassily and Kátai [2] proved the following two theorems.

Theorem A. *Assume that $\sigma \neq 0$. Then, for every real number y,*

$$\lim_{x\to\infty} \frac{1}{x} \#\{n \le x : \frac{\alpha_n - MNr}{\sigma\sqrt{Nr}} < y\} = \Phi(y), \tag{1.3}$$

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \#\{p \le x : \frac{\alpha_p - MNr}{\sigma\sqrt{Nr}} < y\} = \Phi(y), \tag{1.4}$$

where Φ stands for the Gaussian Law.

Moreover, let $[\alpha, \beta) \subset [0,1]$, $\chi : [0,1) \to \mathbb{R}$ be the periodic modulo 1 function defined by

$$\chi(t) = \chi_{[\alpha,\beta)}(t) := \begin{cases} 1 - (\beta - \alpha) & \text{if } t \in [\alpha, \beta), \\ -(\beta - \alpha) & \text{if } t \in [0,1) \setminus [\alpha, \beta). \end{cases}$$

Theorem B. *Assume that*

$$\sigma^2 := \int_0^1 \chi(t)^2\, dt + 2\sum_{k=1}^{\infty} \int_0^1 \chi(t)\chi(q^k t)\, dt \neq 0.$$

Then, for every real number y,

$$\lim_{x\to\infty} \frac{1}{x} \#\{n \le x : \sum_{j=1}^{Nr} \chi\left(\frac{P(n)}{q^j}\right) < y\sigma\sqrt{Nr}\} = \Phi(y),$$

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \#\{p \le x : \sum_{j=1}^{Nr} \chi\left(\frac{P(p)}{q^j}\right) < y\sigma\sqrt{Nr}\} = \Phi(y).$$

The proof of Theorem A is based essentially on Lemmas 1 and 2 below. By using Theorems A and B, we can estimate the moments of $\dfrac{\alpha_n - MNr}{\sigma\sqrt{Nr}}$ and of $\dfrac{\alpha_p - MNr}{\sigma\sqrt{Nr}}$ and compare them with the moments of $P(n) = n$. Since estimate (4.1) is known to hold in the particular case $P(n) = n$, the Frechet-Shohat Theorem implies the more general estimates (4.1) and (4.2).

In this paper we will show that Theorems A and B are still true when the sums run over some subsets of integers defined in Section 4.

2 Notations

As usual, $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ will stand for the set of positive integers, of integers and real numbers, respectively. As is customary, p will always denote a prime number.

Throughout this paper, we let λ stand for the Lebesgue measure.

We denote by $\mathcal{M}_1$ the set of all complex valued multiplicative functions g satisfying $|g(n)| \leq 1$ for all $n \in \mathbb{N}$.

For each $y \in \mathbb{R}$, let $e(y) := \exp\{2\pi y i\}$.

Given a set $B = \{x_1, \ldots, x_M\}$ of real numbers, the discrepancy of B modulo 1, noted $\mathrm{discr}(B)$ is defined by

$$\mathrm{discr}(B) = \sup_{[\alpha,\beta)\subseteq[0,1]} \left| \frac{1}{M} \sum_{\substack{n=1 \\ \{x_n\}\in[\alpha,\beta)}}^{M} 1 - (\beta - \alpha) \right|.$$

Throughout this paper, the letters c and C always denote positive constants, but not necessarily the same at each occurrence.

3 Preliminary results

Lemma 1. *(Erdős-Turán) If D_M stands for the discrepancy of the real numbers $x_1, \ldots, x_M$ modulo 1, then, there exists a positive constant c such that*

$$D_M \leq c \left(\sum_{h=1}^{K} \frac{|\Psi_h|}{h} + \frac{M}{K} \right)$$

for any positive integer K, where $\Psi_m := \sum_{\ell=1}^{M} e(m x_\ell)$ for each positive integer m.

Proof. This result is due to Erdős and Turán [3]. □

Now, given a real number $\xi \in (0, 1)$, let

$$U = U_\xi := [1 - \xi, 1] \cup \bigcup_{b=1}^{q-1} \left[\frac{b}{q} - \xi, \frac{b}{q} + \xi \right] \cup [0, \xi].$$

Lemma 2. *Given $P \in \mathbb{Z}[x]$. Assume that $P(x)$ has positive degree and that its leading coefficient is positive. For each $j \in \mathbb{N}$, set*

$$E_j(x) := \#\left\{p \le x : \left\{\frac{P(p)}{q^{j+1}}\right\} \in U\right\} \quad \text{and} \quad F_j(x) := \#\left\{n \le x : \left\{\frac{P(n)}{q^{j+1}}\right\} \in U\right\}.$$

Further let $\varepsilon > 0$ be fixed, $N^\varepsilon < j < rN - N^\varepsilon$, η an arbitrary positive constant. Then, uniformly in j, we have

$$E_j(x) \ll \xi\pi(x) + \frac{x}{\log^\eta x} \quad \text{and} \quad F_j(x) \ll \xi x + \frac{x}{\log^\eta x}.$$

Proof. This is Lemma 4 in Bassily and Kátai [1]. □

For an arbitrary sequence of positive integers $\ell_1 < \cdots < \ell_h$ and given $b_1, \ldots, b_h \in A_q$, let

$$\text{(3.1)} \quad \Sigma_1 = \mathcal{N}\left(x \,\middle|\, \begin{matrix} \ell_1, \ldots, \ell_h \\ b_1, \ldots, b_h \end{matrix}\right) = \#\{n \le x : a_{\ell_j}(P(n)) = b_j, j = 1, \ldots, h\},$$

$$\text{(3.2)} \quad \Sigma_2 = \Pi\left(x \,\middle|\, \begin{matrix} \ell_1, \ldots, \ell_h \\ b_1, \ldots, b_h \end{matrix}\right) = \#\{p \le x : a_{\ell_j}(P(p)) = b_j, j = 1, \ldots, h\}.$$

Lemma 3. *Assume that*

$$\text{(3.3)} \qquad N^{1/3} \le \ell_1 < \ell_2 < \cdots < \ell_h \le rN - N^{1/3}$$

and let η be an arbitrary positive constant. Then,

$$\Sigma_1 = \frac{x}{q^h} + o\left(\frac{x}{\log^\eta x}\right) \quad \text{and} \quad \Sigma_2 = \frac{\pi(x)}{q^h} + o\left(\frac{x}{\log^\eta x}\right) \qquad (x \to \infty)$$

hold uniformly for all choices of $\ell_1, \ldots, \ell_h$ satisfying (3.3) and $b_j \in A_q$. The implicit constants in the $o(\ldots)$ terms may depend on P, h and η.

Proof. This is Lemma 1 in Bassily and Kátai [2]. □

Lemma 4. *Let c and η be arbitrary constants. Let x be large. Then, for every choice of $h \le c \log\log x$, of $\ell_1, \ldots, \ell_h$ satisfying condition (3.3) and of $(b_1, \ldots, b_h) \in A_q^h$, we have*

$$\Sigma_1 = \frac{x}{q^h} + o\left(\frac{xh}{\log^\eta x}\right) \quad \text{and} \quad \Sigma_2 = \frac{\pi(x)}{q^h} + o\left(\frac{xh}{\log^\eta x}\right) \qquad (x \to \infty),$$

where the implicit constants in the $o(\ldots)$ terms may depend on P, c and η.

Proof. This is Lemma 2 in Bassily and Kátai [2]. □

Remark. The proofs of Lemmas 2, 3 and 4 depend mainly on results of I.M. Vinogradov [6] and L.K. Hua [4].

4 Main results

Let $J_1, \dots, J_k \subseteq [0,1)$ be a finite number of intervals. Let $P_1(x), \dots, P_k(x)$ be real valued polynomials each of positive degree. Consider the linear combinations

$$Q_{m_1,\dots,m_k}(x) = m_1 P_1(x) + \cdots + m_k P_k(x),$$

where $m_1, \dots, m_k \in \mathbb{Z}$, and assume that $Q_{m_1,\dots,m_k}(x) - Q_{m_1,\dots,m_k}(0)$ has an irrational coefficient for every $m_1, \dots, m_k \in \mathbb{Z}$ except when $m_1 = \cdots = m_k = 0$. Moreover, let $S = \{n \in \mathbb{N} : \{P_\ell(n)\} \in J_\ell \text{ for } \ell = 1, \dots, k\}$.

Then, the second author [5] proved the following result.

Theorem C. *With S, $J_1, \dots, J_k$ and $P_1, \dots, P_k$ as above,*

$$\sup_{g \in \mathcal{M}_1} \left| \frac{1}{x} \sum_{\substack{n \le x \\ n \in S}} g(n) - \frac{\lambda(J_1) \cdots \lambda(J_k)}{x} \sum_{n \le x} g(n) \right| \to 0 \quad \text{as } x \to \infty.$$

As a corollary to this result, the second author showed that, if $u(n)$ is an additive function for which there exist two functions $A(x)$ and $B(x)$ such that

$$F(y) = \lim_{x \to \infty} \frac{1}{x} \#\{n \le x : \frac{u(n) - A(x)}{B(x)} < y\}$$

exists for all $y \in \mathbb{R}$ and represents a distribution function, then, with S, $J_1, \dots, J_k$ and $P_1, \dots, P_k$ as above,

$$\lim_{x \to \infty} \frac{1}{\lambda(J_1) \cdots \lambda(J_k)} \frac{1}{x} \# \left\{ n \le x : n \in S,\ \frac{u(n) - A(x)}{B(x)} < y \right\} = F(y)$$

for every continuity point y of F.

Theorem 1. *As $x \to \infty$,*

$$\Pi_S(x) = \#\{p \le x : p \in S\} = (1 + o(1)) \lambda(J_1) \cdots \lambda(J_k) \pi(x).$$

Let $R \in \mathbb{R}[x]$ be a polynomial of degree $r > 0$ be such that $R(x) \to \infty$ as $x \to \infty$. Write the q-ary expansion of each integer n as

$$n = \sum_{\nu=0}^{t} \varepsilon_\nu(n) q^\nu, \qquad \text{with each } \varepsilon_\nu(n) \in A_q,\ \varepsilon_t(n) \ne 0.$$

Consider the word in A_q^{t+1} defined as

$$\overline{n} = \varepsilon_0(n) \varepsilon_1(n) \dots \varepsilon_t(n).$$

Now, define

$$N_S\left(x\left|\begin{array}{c}\ell_1,\dots,\ell_t\\ a_1,\dots,a_t\end{array}\right.\right) = \#\{n\le x : n\in S,\ \varepsilon_{\ell_j}(R(n))=a_j,\ j=1,\dots,t\},$$
$$\Pi_S\left(x\left|\begin{array}{c}\ell_1,\dots,\ell_t\\ a_1,\dots,a_t\end{array}\right.\right) = \#\{p\le x : p\in S,\ \varepsilon_{\ell_j}(R(p))=a_j,\ j=1,\dots,t\}.$$

Then, we have the following.

Theorem 2. *Let* $N=\left\lfloor\dfrac{\log x}{\log q}\right\rfloor$. *Let* $t\in\mathbb{N}$ *be fixed and let also* $0<\tau\le\frac{1}{2}$ *be fixed. Then, given any* $a_1,\dots,a_t\in A_q$,

$$\sup_{N^\tau\le\ell_1<\cdots<\ell_t<rN-N^\tau}\left|\frac{q^tN_S\left(x\left|\begin{array}{c}\ell_1,\dots,\ell_t\\ a_1,\dots,a_t\end{array}\right.\right)}{N_S(x)}-1\right|\le\delta(x)$$

and

$$\sup_{N^\tau\le\ell_1<\cdots<\ell_t<rN-N^\tau}\left|\frac{q^t\Pi_S\left(x\left|\begin{array}{c}\ell_1,\dots,\ell_t\\ a_1,\dots,a_t\end{array}\right.\right)}{\Pi_S(x)}-1\right|\le\delta(x),$$

where $\delta(x)\to 0$ *as* $x\to\infty$ *and where* $N_S(x)=\#\{n\le x : n\in S\}$ *and* $\Pi_S(x)=\#\{p\le x : p\in S\}$

Let f be a q-additive function and set

$$m_k=\frac{1}{q}\sum_{b\in A_q}f(bq^k)\quad\text{and}\quad\sigma_k^2=\frac{1}{q}\sum_{b\in A_q}f^2(bq^k)-m_k^2.$$

Let also

$$M(x)=\sum_{k=0}^{N}m_k\quad\text{and}\quad D^2(x)=\sum_{k=0}^{N}\sigma_k^2.$$

Assume that $R\in\mathbb{Z}[x]$, $R(x)\to\infty$ as $x\to\infty$ and that r is the degree of R.

Theorem 3. *Let* f *be a* q*-additive function and assume that* $|f(bq^j)|\le C$ *for all* $b\in A_q$ *and all integers* $j\ge 0$. *Assume also that* $\dfrac{D(x)}{\log^{1/3}x}\to\infty$ *as* $x\to\infty$. *Then,*

$$\lim_{x\to\infty}\frac{1}{N_S(x)}\#\{n\le x : n\in S,\ \frac{f(R(n))-M(x^r)}{D(x^r)}<y\} = \Phi(y),$$
$$\lim_{x\to\infty}\frac{1}{\Pi_S(x)}\#\{p\le x : p\in S,\ \frac{f(R(p))-M(x^r)}{D(x^r)}<y\} = \Phi(y).$$

Theorem 4. *Let σ be as in (1.2) and assume that $\sigma \neq 0$. Then, for every real number y,*

$$\lim_{x\to\infty} \frac{1}{N_S(x)} \#\{n \le x : n \in S,\ \frac{\alpha_n - MNr}{\sigma\sqrt{Nr}} < y\} = \Phi(y), \tag{4.1}$$

$$\lim_{x\to\infty} \frac{1}{\Pi_S(x)} \#\{p \le x : p \in S,\ \frac{\alpha_p - MNr}{\sigma\sqrt{Nr}} < y\} = \Phi(y). \tag{4.2}$$

Theorem 5. *Assume that*

$$\sigma^2 = \int_0^1 \chi(t)^2\,dt + 2\sum_{k=1}^{\infty} \int_0^1 \chi(t)\chi(q^k t)\,dt \neq 0.$$

Then, for every real number y,

$$\lim_{x\to\infty} \frac{1}{N_S(x)} \#\{n \le x : n \in S,\ \sum_{j=1}^{Nr} \chi\left(\frac{P(n)}{q^j}\right) < y\sigma\sqrt{Nr}\} = \Phi(y),$$

$$\lim_{x\to\infty} \frac{1}{\Pi_S(x)} \#\{p \le x : p \in S,\ \sum_{j=1}^{Nr} \chi\left(\frac{P(p)}{q^j}\right) < y\sigma\sqrt{Nr}\} = \Phi(y).$$

5 Proof of Theorem 1

Let

$$f_h(x) = \begin{cases} 1 & \text{if } x \in J_h, \\ 0 & \text{if } x \in [0,1) \setminus J_h \end{cases}$$

and extend f_h to the whole set of real numbers by setting $f_h(x+\nu) = f_h(x)$ for all $\nu \in \mathbb{Z}$.

Let Δ be a fixed positive small number. Then, for each $x \in \mathbb{R}$, let

$$g_h(x) = \frac{1}{2\Delta} \int_{-\Delta}^{\Delta} f_h(x+y)\,dy.$$

Then, we write the Fourier series associated with $f_h(x)$ and $g_h(x)$ as

$$f_h(x) = \sum_{m=-\infty}^{\infty} c_m^{(h)} e(mx) \qquad \text{and} \qquad g_h(x) = \sum_{m=-\infty}^{\infty} d_m^{(h)} e(mx),$$

where the constants $c_m^{(h)}$ and $d_m^{(h)}$ satisfy

$$\left|c_m^{(h)}\right| \le \frac{K_h}{|m|} \text{ for } m \neq 0, \quad \text{and } c_0^{(h)} = \lambda(J_h)$$

and

$$\left|d_m^{(h)}\right| \le K_h \min\left(\frac{1}{|m|}, \frac{1}{\Delta m^2}\right) \text{ for } m \neq 0, \quad \text{and } d_0^{(h)} = \lambda(J_h),$$

where K_h is some positive constant depending only on h.

With these notations and conditions, define the two arithmetic functions

$$\begin{aligned}\sigma(n) &= f_1(P_1(n))\cdots f_k(P_k(n)),\\ \kappa(n) &= g_1(P_1(n))\cdots g_k(P_k(n)),\end{aligned}$$

so that

$$\sigma(n)=\begin{cases}1 & \text{if } n\in S,\\ 0 & \text{otherwise,}\end{cases}$$

while $0\le\kappa(n)\le 1$.

Now if $\kappa(n)\neq\sigma(n)$, then for some $j\in\{1,\ldots,k\}$, we have

$$g_j(P_j(n))\neq f_j(P_j(n)). \tag{5.1}$$

In such a case, write $J_j=U_1\cup\cdots\cup U_m$, namely the union of finite disjoint intervals $U_h=[\alpha_h,\beta_h)$, with $\alpha_1<\beta_1<\alpha_2<\beta_2<\cdots<\alpha_m<\beta_m$. Now, if (5.1) does indeed hold, then

(5.2)

$$\{P_j(n)\}\in[\alpha_1-\Delta,\alpha_1+\Delta]\cup[\beta_1-\Delta,\beta_1+\Delta]\cup\cdots\cup[\alpha_m-\Delta,\alpha_m+\Delta]\cup[\beta_m-\Delta,\beta_m+\Delta].$$

But the number of positive integers $n\le x$ satisfying (5.2) is less than

$$2mx\Delta+2mx\cdot\operatorname{discr}(\{P_j(n)\}_{1\le n\le\lfloor x\rfloor}).$$

But, in light of Lemma 1, we have that

$$\operatorname{discr}(\{P_j(n)\}_{1\le n\le\lfloor x\rfloor})\to 0\qquad\text{as } x\to\infty.$$

Similarly,

$$\operatorname{discr}(\{P_j(p)\}_{1\le n\le\lfloor x\rfloor})\to 0\qquad\text{as } x\to\infty.$$

Combining these results, we get that

$$\begin{aligned}\#\{n\le x:\sigma(n)\neq\kappa(n)\} &\le c\Delta x+o(x)\qquad(x\to\infty),\\ \#\{p\le x:\sigma(p)\neq\kappa(p)\} &\le (c\Delta+o(1))\pi(x)\qquad(x\to\infty).\end{aligned}$$

Now using the same argument as the one used in [5] to prove the theorem in that paper, but proceeding only with $g(n)=1$, we get that

$$\begin{aligned}\Pi_S(x) &= \sum_{p\le x}\kappa(p)+O(\Delta\pi(x))\\ &= \sum_{m_1,\ldots,m_k} d^{(1)}_{m_1}\cdots d^{(k)}_{m_k}\sum_{p\le x} e(Q_{m_1,\ldots,m_k}(p))+O(\Delta\pi(x)),\end{aligned}$$

where $d^{(1)}_0\cdots d^{(k)}_0=\lambda(J_1)\cdots\lambda(J_k)$ and

$$\sum_{m_1,\ldots,m_k}\left|d^{(1)}_{m_1}\cdots d^{(k)}_{m_k}\right|<\infty.$$

Then, the proof of Theorem 1 is complete by observing that

$$\frac{1}{\pi(x)} \sum_{p \le x} e(P(p)) \to 0 \qquad \text{as } x \to \infty,$$

which is a well-known result of I.M. Vinogradov.

6 Proof of Theorem 2

For $x \in [0,1)$, let

$$\phi_0(x) = \begin{cases} 1 & \text{if } 0 \le x < 1/q, \\ 0 & \text{if } 1/q \le x \le 1. \end{cases}$$

and extend the definition of $\phi_0(x)$ to all non negative real numbers x using the relation $\phi_0(x+n) = \phi_0(x)$ for all $n \in \mathbb{N}$. Moreover, set

$$\phi_b(x) = \phi_0(x - b/q) \qquad \text{for } b \in A_q$$

and, given a fixed positive small number Δ, let

$$h_b(x) = \frac{1}{2\Delta} \int_{-\Delta}^{\Delta} \phi_b(x+z)\, dz = \sum_{m=-\infty}^{\infty} u_m^{(b)} e(mx).$$

By a simple computation, we easily obtain that $u_0^{(b)} = 1/q$ and that $u_m^{(b)} = 0$ if $m \equiv 0 \pmod q$ and $m \neq 0$, while

$$\left| u_m^{(b)} \right| \le \min\left(\frac{1}{\pi m}, \frac{1}{\Delta \pi m^2} \right).$$

Now, let

$$\begin{aligned} \rho(n) &:= \prod_{j=1}^{t} \phi_{a_j}\left(\frac{R(n)}{q^{\ell_j+1}} \right), \\ \tau(n) &:= \prod_{j=1}^{t} h_{a_j}\left(\frac{R(n)}{q^{\ell_j+1}} \right). \end{aligned}$$

With this definition, it is clear that $\rho(n) = 1$ or 0, depending on wether $\varepsilon_{\ell_j}(n) = a_j$ for $j = 1, \ldots, t$ or not. Moreover, $0 \le \tau(n) \le 1$ and

$$\#\{n \le x : \rho(n) \neq \tau(n)\} \le c\Delta x + x \operatorname{discr}(\{R(1), \ldots, R(\lfloor x \rfloor)\}).$$

Hence, it follows from this that

$$\limsup_{x \to \infty} \frac{1}{x} \#\{n \le x : \rho(n) \neq \tau(n)\} \le c\Delta.$$

Similarly, using Lemmas 1 and 2, we get that

$$\limsup_{x\to\infty} \frac{1}{x}\#\{p \le x : \rho(p) \ne \tau(p)\} \le c\Delta.$$

Hence, we obtain that

$$N_S\left(x \,\middle|\, \begin{matrix} \ell_1, \dots, \ell_t \\ a_1, \dots, a_t \end{matrix}\right) = \sum_{n\le x} \sigma(n)\rho(n) = \sum_{n\le x} \kappa(n)\tau(n) + O(\Delta x) + o(x) \quad (x \to \infty).$$

We now repeat the argument used in [1], namely letting

$$V = \left(\frac{1}{q^{\ell_1+1}}, \dots, \frac{1}{q^{\ell_t+1}}\right)$$

and letting $\mathcal{M}$ be the whole set of vectors $M = (m_1, \dots, m_t)$, so that

$$VM = \frac{m_1}{q^{\ell_1+1}} + \ldots + \frac{m_t}{q^{\ell_t+1}} = \frac{A_M}{H_M} \quad \text{with } (A_M, H_M) = 1.$$

Then, we get that

$$\tau(n) = \sum_{M\in\mathcal{M}} T_M \; e\left(\frac{A_M}{H_M} R(n)\right). \tag{6.1}$$

In [5], the second author proved that

$$\kappa(n) = \sum_{m_1,\dots,m_t} d^{(1)}_{m_1} \cdots d^{(t)}_{m_t} \; e(Q_{m_1}, \dots Q_{m_t}), \tag{6.2}$$

where

$$\sum_{m_1,\dots,m_t} \left| d^{(1)}_{m_1} \cdots d^{(t)}_{m_t} \right| < \infty.$$

Thus, combining (6.1) and (6.2), we obtain that

$$\kappa(n)\tau(n) = \sum_{M\in\mathcal{M}} \sum_{m_1,\dots,m_t} T_M d^{(1)}_{m_1} \cdots d^{(t)}_{m_t} \, e\left(\frac{A_M}{H_M} R(n) + Q_{m_1,\dots,m_t}(n)\right).$$

Let us sum the above over the positive integers $n \le x$. If $(m_1, \dots, m_t) \ne (0, \dots, 0)$, then the polynomial

$$\frac{A_M}{H_M} R(y) + Q_{m_1,\dots,m_t}(y)$$

has an irrational coefficient other than the constant term. Thus by Weyl's Theorem, we have that

$$\sum_{n\le x} e\left(\frac{A_M}{H_M} R(n) + Q_{m_1,\dots,m_t}(n)\right) = o(x),$$

while by using the theorem of I.M. Vinogradov, we get

$$\sum_{p\le x} e\left(\frac{A_M}{H_M}R(p)+Q_{m_1,\dots,m_t}(p)\right)=o(\pi(x)).$$

It remains to estimate, in the sum $\sum_{n\le x}\kappa(n)\tau(n)$ and in $\sum_{p\le x}\kappa(p)\tau(p)$, those terms for which $(m_1,\dots,m_t)=(0,\dots,0)$.

Since $d_0^{(\ell)}=\lambda(J_\ell)$, we conclude that

$$\begin{aligned}\sum_{n\le x}\kappa(n)\tau(n) &= \lambda(J_1)\cdots\lambda(J_t)\sum_{n\le x}\tau(n)+o(x),\\ \sum_{p\le x}\kappa(p)\tau(p) &= \lambda(J_1)\cdots\lambda(J_t)\sum_{p\le x}\tau(p)+o(\pi(x)).\end{aligned}$$

Then, by applying Lemma 5 of Bassily and Kátai [1], the proof of Theorem 2 follows immediately.

7 The proofs of the other theorems

The proof of Theorem 3 can be obtained by using the Frechet-Shohat Theorem stated in Section 4 of Kátai [5]. On the other hand, the proofs of Theorems 4 and 5 go essentially along the same lines as those of Theorems 1 and 2 proved in Bassily and Kátai [2] and will therefore be omitted.

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