On the local distribution of the number of prime factors of $(n, \varphi_k(n))$

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Abstract

Let φ stand for Euler's totient function and let $\omega(n)$ stand for the number of distinct prime factors of n, with $\omega(1) = 0$. Given an arbitrary non negative integer r, Ram Murty and Kumar Murty have obtained an asymptotic estimate for $\#\{n \le x : \omega((n, \varphi(n))) = r\}$. For each positive integer k, letting $\varphi_k(n)$ be the k-fold iterate of φ, we obtain an asymptotic estimate for $\#\{n \le x : \omega((n, \varphi_k(n))) = r\}$.

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§1. Introduction

Given an integer $n \ge 2$, we write $\omega(n)$ for the number of distinct prime factors of n, with $\omega(1) = 0$, $p(n)$ for the smallest prime factor of n and $P(n)$ for the largest prime factor of n, with $p(1) = P(1) = 1$. Let φ stand for Euler's totient function and, for each positive integer k, let $\varphi_k(n)$ be the k-fold iterate of φ, i.e. $\varphi_0(n) = n$, $\varphi_{j+1}(n) = \varphi(\varphi_j(n))$ for $j = 0, 1, 2, \ldots$. Throughout this paper p and q, with or without subscripts, denote prime numbers. We shall use the abbreviation $x_1 = \log x$, $x_2 = \log x_1$, and so on. For each real number $y \ge 2$ and positive integer n, define the functions $A(n, y)$ and $B(n, y)$ by

$$n = \prod_{\substack{p^\alpha \| n \\ p \le y}} p^\alpha \cdot \prod_{\substack{p^\alpha \| n \\ p > y}} p^\alpha = A(n, y) \cdot B(n, y).$$

Moreover, for all real numbers $2 \le y \le x$, let

$$\mathcal{F}_{x,y} := \{n \le x : A(n, y) = 1\} = \{n \le x : p(n) > y\} \quad \text{and} \quad \Phi(x, y) = \#\mathcal{F}_{x,y}.$$

Erdős [4] proved that

$$(1) \quad \#\{n \le x : (n, \varphi(n)) = 1\} = (1 + o(1))\Phi(x, x_2) = (1 + o(1))\frac{e^{-\gamma} x}{x_3} \qquad (x \to \infty),$$

where γ stands for Euler's constant.

Further results of this type were obtained by Kátai and Subbarao [6], Bassily, Kátai and Wijsmuller [1], and Indlekofer and Kátai [5]. Given an arbitrary non negative integer r, R. Murty and K. Murty [9] proved that

$$(2) \qquad \#\{n \le x : \omega((n, \varphi(n))) = r\} = (1 + o(1))\frac{e^{-\gamma} x \, x_4^r}{r! \, x_3},$$

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the particular case $r=0$ yielding Erdős' result (1). V. Kumar Murty [8] proved that

$$\#\{n \le x : (n,\varphi(n)) = k\} = (1+o(1))\frac{e^{-\gamma}x}{kx_3}. \tag{3}$$

Our main goal here is to generalize (2) by studying the local distribution of $\omega((n,\varphi_k(n)))$.

§2. Main results

For $2 \le y \le x$, let

$$T(x,y) := \#\{n \le x : n \in \mathcal{F}_{x,y} \text{ and } (n,\varphi_k(n)) > 1\}.$$

Theorem 1. *Let k be a positive integer. Then, as $x \to \infty$,*

$$T(x,x_2^k) \ll \frac{1}{x_3}\Phi(x,x_2^k).$$

Let k be a fixed positive integer and $\varepsilon > 0$ an arbitrary fixed number, and write $H = H_x$ for the least common multiple of all the prime powers $< x_2^{k(1-\varepsilon)}$. In other words,

$$H = H_x = \prod_{p < x_2^{k(1-\varepsilon)}} p^{\alpha_p},$$

where each α_p is the unique positive integer satisfying

$$p^{\alpha_p} \le x_2^{k(1-\varepsilon)} < p^{\alpha_p+1}.$$

Theorem 2. *Let k, ε and H be as above. Then, as $x \to \infty$,*

$$\frac{1}{\Phi(x,x_2^k)}\#\{n \le x : n \in \mathcal{F}_{x,x_2^k} \text{ and } \varphi_k(n) \not\equiv 0 \pmod{H}\} \ll \frac{1}{x_1^{0.8}}.$$

Theorem 3. *Let $k \ge 1$ and $r \ge 0$ be fixed integers, and let*

$$R_r(x) := \#\{n \le x : \omega((n,\varphi_k(n)) = r\}.$$

Then

$$R_r(x) = (1+o(1))\frac{e^{-\gamma}x}{kx_3}\cdot\frac{x_4^r}{r!}$$

uniformly for $r \le Bx_4$ for any fixed positive constant B.

3. Preliminary lemmas

Lemma 1. *Let $\ell \in \{-1,1\}$ and $1 \le k \le x$. Then there exists a positive constant c_1 such that*

$$\sum_{\substack{p \le x \\ p \equiv \ell \pmod k}} \frac{1}{p} \le \frac{c_1 x_2}{\varphi(k)}.$$

Proof. See Bassily, Kátai and Wijsmuller [1].

Let Q be a fixed prime; let $\kappa_0, \kappa_1, \ldots$ be the sequence of completely additive functions defined on the primes by

$$\kappa_0(p) := \begin{cases} 1 & \text{if } p = Q, \\ 0 & \text{if } p \neq Q, \end{cases} \qquad \kappa_{j+1}(p) := \sum_{q \mid p-1} \kappa_j(q).$$

Moreover, let

$$S_j(y) := \sum_{p \le y} \kappa_j(p).$$

Lemma 2. *Let m and j be arbitrary positive integers, $y > e^{2^{j-3}Q^2}$. Then there exists a positive constant $c_2 = c_2(m,j)$ such that*

$$S_j(y) \le \frac{c_2 y (\log\log y)^{j-1}}{(\log y) Q^{(m-1)/m}}.$$

Proof. For a proof, see Indlekofer and Kátai [5].

Lemma 3. *Let $2 \le y \le x$. Then, setting $\rho(y) := \prod_{p<y}(1-1/p)$,*

$$\Phi(x,y) = x\rho(y) + O\left(x \exp\left\{ -\frac{1}{2} \frac{x_1}{\log y} \right\} \right).$$

Proof. This result is a consequence of Theorems 5.1 and 6.2 of the book of Tenenbaum [10].

Lemma 4. *Let f be a non negative strongly additive function and set*

$$A(x) := \sum_{p \le x} \frac{f(p)}{p-1} \quad \text{and} \quad B^2(x) := \sum_{p \le x} \frac{f^2(p)}{p}.$$

Then, for each real number $\alpha \in [0,2[$, there exists a constant $D = D(\alpha)$ such that the inequality

$$\sum_{p-1 \le x} |f(p-1) - A(x)|^\alpha \le D \cdot \frac{x}{x_1} \cdot B^\alpha(x)$$

holds uniformly for $x \ge 2$.

Proof. This is a variant of Lemma 4.18 in the book of Elliott [3].

Let Q be a given prime number. We say that $(Q \to) p_1 \to \ldots \to p_k$ is a chain of primes if $Q|p_1 - 1$, $p_j|p_{j+1} - 1$ $(j = 1, 2, \ldots, k-1)$.

It is clear that $Q \not| \varphi_k(m)$ implies that $p_k \not| m$ whenever p_k is the last element of an arbitrary chain of primes $Q \to p_1 \to \ldots \to p_k$.

Let $\varepsilon > 0$ be a fixed number, Q be a prime or a prime power, $Q \le x_2^{k(1-\varepsilon)}$. Let $\tau_1, \tau_2, \ldots, \tau_k$ be a sequence of strongly additive functions defined on the primes as follows:

$$\tau_1(p) = \begin{cases} 1 & \text{if } Q|p-1 \text{ and } p > Q^3, \\ 0 & \text{otherwise;} \end{cases}$$

$$\tau_{j+1}(p) = \sum_{\substack{q < p^{1/3} \\ q|p-1}} \tau_j(q) \qquad (j = 2, 3, \ldots, k-1).$$

Moreover, for each real number $u \le x$, define the strongly additive function f_u by

$$f_u(n) = \sum_{\substack{q < u^{1/3} \\ q|n}} \tau_{k-1}(q).$$

It follows from Lemma 4 that

$$\sum_{u \le p \le 2u} |f_u(p-1) - A_u|^\alpha \le D \frac{u}{\log u} \cdot B_u^\alpha, \tag{4}$$

where

$$A_u = \sum_{q < u^{1/3}} \frac{\tau_{k-1}(q)}{q-1} \qquad \text{and} \qquad B_u^2 = \sum_{q < u^{1/3}} \frac{\tau_{k-1}^2(p)}{p}.$$

By using the Bombieri-Vinogradov inequality (see Bombieri [2] or Vinogradov [11]) and the Siegel-Walfisz Theorem, as well as standard sieve inequalities, one can deduce that there exist positive constants c_3 et c_4 such that, for $u \ge e^{2^k Q^2}$,

$$A_u \ge c_3 \frac{(\log\log u)^k}{Q} \qquad \text{and} \qquad B_u^2 \le c_4 \frac{(\log\log u)^{2k-1}}{Q^2} \tag{5}$$

uniformly for $Q \le (\log\log u)^{k(1-\delta)} \le x_2^{k(1-\varepsilon)}$, where δ and ε are arbitrarily small but fixed numbers. The inequalities (5) can be deduced by using the method used in the proof of Lemma 2. We shall therefore omit the details.

From (4) and (5), and setting as usual $\mathrm{li}(u) := \int_2^u \frac{dt}{\log t}$, we can deduce that

$$\frac{1}{\mathrm{li}(u)} \#\{p \in [u, 2u] : f_u(p-1) = 0\} = O\left(B_u^\alpha \cdot A_u^{-\alpha}\right) = O\left((\log\log u)^{-\alpha/2}\right),$$

which tends to 0 if one chooses α arbitrarily close to 2.

Since $\kappa_k(p) = 0$ for $p \in [u, 2u]$ implies that $f_u(p-1) = 0$, it follows that

$$\frac{1}{\mathrm{li}(u)} \#\{p \in [u, 2u] : \kappa_k(p) = 0\} \to 0 \qquad (u \to \infty). \tag{6}$$

Lemma 5. *Let k be a positive integer, $\varepsilon > 0$ an arbitrary small number, Q a prime power, $Q \le x_2^{k-\varepsilon}$. Then*

$$U_k(Q) := \#\{n \in \mathcal{F}_{x,x_2^k} : Q \not| \varphi_k(n)\} \ll \frac{x}{x_1^{0.9}}.$$

Proof. Assume that $Q \not| \varphi_k(n)$. Then $p_k \not| n$ for every prime p_k which is the last element of the chain of primes $(Q \to) p_1 \to p_2 \to \ldots \to p_k$. Moreover $p_k \not| n$ whenever $\tau_k(p_k) > 0$. It follows that

$$U_k(Q) \ll x \prod_{p < x_2^k} \left(1 - \frac{1}{p}\right) \rho_k(Q, x), \tag{7}$$

where

$$\rho_k(Q, x) = \prod_{\substack{U_0 < p < x \\ \tau_k(p) > 0}} \left(1 - \frac{1}{p}\right) \qquad \text{with} \qquad U_0 := e^{2^k Q^2}. \tag{8}$$

Now, from (6) and (8), it follows that

$$-\log \rho_k(Q, x) = \sum_{\substack{U_0 < p < x \\ \tau_k(p) > 0}} \frac{1}{p} > 0.95 \log\log x, \tag{9}$$

since for $u \in [U_0, x]$, (6) holds uniformly. Combining (7) and (9), the proof of Lemma 5 is complete.

§4. The proofs of Theorems 1 and 2

Assume that $n \in \mathcal{F}_{x,x_2^k}$ and that $(n, \varphi_k(n)) > 1$. Then there exists some prime $q|n$ with $q > x_2^k$ such that $q|\varphi_k(n)$. The fact that $q|\varphi_k(n)$ implies that either $p_1|\varphi_{k-1}(n)$ for some prime $p_1 \equiv 1 \pmod q$ or $q^2|\varphi_{k-1}(n)$. Continuing this argument, we obtain one of the following two situations:

(a) there exists a chain of primes $(q = p_0) \to p_1 \to \ldots \to p_k$, $qp_k|n$,

(b) there exists a positive integer $t < k$ such that

$$(q = p_0) \to p_1 \to \ldots \to p_t,\ p_t|\varphi_{k-t}(n),\ p_t^2|\varphi_{k-t-1}(n),\ p_j|p_{j+1}-1\ (j = 0, 1, \ldots, t-1),\ q|n.$$

Let us first estimate the number of those $n \in \mathcal{F}_{x,x_2^k}$ for which (a) holds. We need to find an appropriate upper bound for the expression

$$\Sigma := \sum_{\substack{q > x_2^k \\ q \to p_1 \to \ldots \to p_k (\le x)}} \Phi\left(\frac{x}{qp_k}, x_2^k\right) = \Sigma_1 + \Sigma_2 + \Sigma_3, \tag{10}$$

where in Σ_1 we sum over those $q > x_1^{1/5}$, in Σ_2 over those $q \in [x_2^k, x_1^{1/5}]$ and $p_k > x^{1/3}$, and in Σ_3 over all other possible pairs of primes q and p_k.

Using the trivial inequality $\Phi(x,y) \le x$, and by repeated application of Lemma 1, it follows that

(11) $$\Sigma_1 \le x \sum_{q > x_1^{1/5}} \frac{1}{q} \sum_{q \to p_1 \to \ldots \to p_k} \frac{1}{p_k} \ll x x_2^k \sum_{q > x_1^{1/5}} \frac{1}{q^2} \ll \frac{x}{x_1^{0.19}},$$

say.

Furthermore, again using $\Phi(x,y) \le x$, we have

$$\Sigma_2 \le x \sum_{\substack{x_2^k < q < x_1^{1/5} \\ q \to p_1 \to \ldots \to p_k \\ x^{1/3} < p_k < x}} \frac{1}{q p_k}.$$

For a fixed prime $q \in [x_2^k, x_1^{1/5}]$ with $q := Q$, it follows from Lemma 2 that

$$\sum_{x^{1/3} < p_k < x} \frac{1}{p_k} \le \sum_{t=0}^{T_0} \frac{1}{2^t x^{1/3}} S_k(2^t x^{1/3}) \ll \frac{x_2^{k-1}}{q^{(m-1)/m}},$$

where m is any arbitrary fixed number. Therefore, choosing $m > k$, we obtain that

(12) $$\Sigma_2 \ll x \cdot x_2^{k-1} \sum_{q > x_2^k} \frac{1}{q^{2-1/m}} \ll \frac{x}{x_2^{1-k/m}} \ll \frac{x}{x_3}.$$

It remains to estimate Σ_3. In this case, we have $q < x_1^{1/5}$ and $p_k < x^{1/3}$. Clearly, there exists an absolute constant c_8 such that

(13) $$\Phi\left(\frac{x}{q p_k}, x_2^k\right) \le \frac{c_8}{q p_k} \Phi(x, x_2^k).$$

On the other hand,

(14) $$\Phi(x, x_2^k) \ll \frac{x}{x_3}.$$

Since

$$\sum_{q, p_k} \frac{1}{q p_k} \ll x_2^k \sum_{q > x_2^k} \frac{1}{q^2} \ll \frac{1}{x_3},$$

it follows from (13) and (14) that

(15) $$\Sigma_3 \le \frac{x}{x_3^2}.$$

Substituting (11), (12) and (15) in (10), it follows that $\Sigma \ll \frac{x}{x_3^2}$, which clears the case when (a) holds.

Now, by the method used in Bassily, Kátai and Wijsmuller [1], one can prove that the contribution of those n for which (b) holds is less than $\frac{1}{x_3}\Phi(x, x_2^k)$, thus completing the proof of Theorem 1.

Theorem 2 is a direct consequence of Lemma 5. Hence we shall omit its proof.

§5. The proof of Theorem 3

Let $k \geq 1$ be a fixed integer. Let $1 \leq r \leq Bx_4$, where B is an arbitrary constant. Let $E(n)$ stand for $(n, \varphi_k(n))$. In what follows, A runs through integers $A \geq 2$ whose prime factors do not exceed x_2^k. Now, first observe that

$$\#\{n \leq x : A(n, x_2^k) > x_1\} \ll \sum_{\substack{A > x_1 \\ P(A) \leq x_2^k}} \Phi(\frac{x}{A}, x_2^k) \ll x \exp\left\{-\frac{1}{2k}\frac{x_2}{x_3}\right\}. \tag{16}$$

Hence, it is clear that in order to prove Theorem 3, we may drop those integers n for which $A(n, x_2^k) > x_1$.

By using Theorem 1, the number of integers $n = A\nu \leq x$, with A fixed and $\nu \in \mathcal{F}_{\frac{x}{A}, x_2^k}$, and such that $E(\nu) > 1$ is $O\left(\frac{x}{Ax_3^2}\right)$, since

$$\Phi(\frac{x}{A}, x_2^k) \approx \frac{x}{A} \prod_{p < x_2^k} \left(1 - \frac{1}{p}\right) \approx \frac{x}{Ax_3}.$$

Summing over all A's, it follows that the total number of these particular integers is less than

$$\sum_A \frac{x}{Ax_3^2} \ll \frac{x}{x_3}.$$

We may therefore also drop all this category of integers.

Hence, let $A \leq x_1$. By Theorem 2, we obtain that $\varphi_k(\nu) \equiv 0 \pmod{H_{x/A}}$ for all but $O\left(\frac{x}{Ax_3x_1^{0.8}}\right)$ integers. Hence, by summing over all possible $A \leq x_1$, we obtain a quantity of integers which is less than $O\left(\frac{x}{x_1^{0.8}}\right)$. Now, observe that $H_{x/A}$ is a multiple of

$$M := \prod_{p < \frac{1}{2}x_2^{k(1-\varepsilon)}} p, \tag{17}$$

and write $A = A_1 \cdot A_2$, where $P(A_1) \leq \frac{1}{2}x_2^{k(1-\varepsilon)}$ and $p(A_2) > \frac{1}{2}x_2^{k(1-\varepsilon)}$. We shall now consider all the remaining integers n. For these integers n, we have $A_1 | E(n)$ and there exists a divisor $D | A_2$ such that $E(n) = A_1 D$.

Let

$$T = \sum_{p < \frac{1}{2}x_2^{k(1-\varepsilon)}} \frac{1}{p} \qquad \text{and} \qquad S = \sum_{\frac{1}{2}x_2^{k(1-\varepsilon)} < p < x_2^k} \frac{1}{p}.$$

It is clear that $S < c_9\varepsilon$ for some positive constant c_9 and that there exists a constant $c_{10} = c_{10}(k)$ such that

$$T = x_4 + c_{10} + O\left(\frac{1}{x_3}\right). \tag{18}$$

Furthermore,

$$\sum_{\substack{P(A_1)<\frac{1}{2}x_2^{k(1-\varepsilon)}\\ \omega(A_1)=h}} \frac{1}{A_1} = (1+o(1))\frac{T^h}{h!} \tag{19}$$

uniformly for $0 \le h \le Bx_4$.

Thus,

$$\begin{aligned} R_r(x) &= \#\{n \le x : \omega(E(n)) = r\} \\ &= \sum_{\substack{A=A_1A_2<x_1\\ E(\nu)=1}} \sum_{D|A_2} \#\{n = A\nu : E(n) = A_1D,\ \omega(A_1D) = r\} + O\left(\frac{x}{x_3}\right). \end{aligned} \tag{20}$$

In order to estimate the right hand side of (20), we first consider the case $D = 1$, that is when $\omega(E(n)) = \omega(A_1) = r$. In this case the contribution of the corresponding integers is

$$\sum_{\substack{A_1A_2<x_1\\ \omega(A_1)=r}} \#\{n = A\nu \le x : E(n) = A_1\} = \Sigma_1 + \Sigma_2,$$

where

$$\begin{aligned} \Sigma_1 &= \#\{n = A_1\nu \le x : A_1 < x_1,\ \omega(A_1) = r,\ E(n) = A_1\}, \\ \Sigma_2 &= \#\{n = A_1A_2\nu \le x : A_2 > 1,\ A_1A_2 < x,\ \omega(A_1) = r,\ E(n) = A_1\}. \end{aligned}$$

Thus, recalling the definition of M given in (17), we have

$$\begin{aligned} \Sigma_1 &= \sum_{\substack{A_1<x_1\\ \omega(A_1)=r}} \#\{\nu \le x/A_1 : E(\nu) = 1,\ \varphi_k(\nu) \equiv 0 \pmod M\} \\ &= (1+o(1)) \sum_{\substack{A_1<x_1\\ \omega(A_1)=r}} \frac{x}{A_1}\rho(x_2^k) = (1+o(1))x\rho(x_2^k)\frac{T^r}{r!}. \end{aligned} \tag{21}$$

On the other hand, in light of Lemma 3,

$$\begin{aligned} \Sigma_2 &\ll \sum_{\substack{A_1A_2<x_1,\ A_2>1\\ \omega(A_1)=r}} \#\{n = A_1A_2\nu \le x : E(n) = A_1\} \\ &\ll x\rho(x_2^k) \sum_{\omega(A_1)=r} \frac{1}{A_1} \sum_{A_2>1} \frac{1}{A_2} \\ &\ll x\rho(x_2^k)\frac{T^r}{r!} \sum_{A_2>1} \frac{1}{A_2}. \end{aligned} \tag{22}$$

Observing that

$$\sum_{A_2>1} \frac{1}{A_2} = \prod_{\frac{1}{2}x_2^{k(1-\varepsilon)}<p<x_2^k} \left(1 + \frac{1}{p-1}\right) - 1 \ll \varepsilon,$$

it follows from (22) that

$$\Sigma_2 \ll \varepsilon\Sigma_1. \tag{23}$$

Gathering (21) with (18) and taking into account (23), it follows that the contribution to $R_r(x)$ of those integers with corresponding $D = 1$ will be the main one, namely $(1+o(1))\frac{e^{-\gamma}x}{kx_3}\cdot\frac{x_4^r}{r!}$.

It remains to show that the contribution of those integers with corresponding $D > 1$ is negligible in the estimation of $R_r(x)$. Hence, we now consider the cases in (20) where $D > 1$. For these, we have

$$\omega(A_1) = r_1(< r), \quad \omega(D) = r - r_1.$$

For fixed A_1 and D, $A_2 \equiv 0 \pmod D$, the contribution to $R_r(x)$ is less than

$$\frac{x}{A_1}\rho(x_2^k)\sum_{A_2\equiv 0 \pmod D}\frac{1}{A_2} \ll \frac{x}{A_1 D}\rho(x_2^k),$$

so that by summing over all those $D > 1$, $\omega(A_1) = r_1$, $\omega(D) = r - r_1$, the total contribution to $R_r(x)$ is less than

$$x\rho(x_2^k)\sum_{\omega(A_1)=r_1}\frac{1}{A_1}\sum_{\omega(D)=r-r_1}\frac{1}{D} \ll x\rho(x_2^k)\frac{T^{r_1}}{r_1!}\frac{(c\varepsilon)^{r-r_1}}{(r-r_1)!}.$$

Since

$$\frac{T^{r_1}}{r_1!}\frac{(c\varepsilon)^{r-r_1}}{(r-r_1)!} \le \frac{T^r}{r!}\left(\frac{rc\varepsilon}{T}\right)^{r-r_1}\frac{1}{(r-r_1)!},$$

and since

$$\sum_{r_1<r}\left(\frac{rc\varepsilon}{T}\right)^{r-r_1}\frac{1}{(r-r_1)!} \le \exp\left\{\frac{rc\varepsilon}{T}\right\} - 1 < 2Bc\varepsilon,$$

provided ε is small, it does indeed follow that the contribution of this category of integers does not contribute to the asymptotic estimate of $R_r(x)$.

This completes the proof of Theorem 3, except for the case $r = 0$ which is a particular case of (3).

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