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ON THE ASYMPTOTIC VALUE OF THE IRRATIONAL FACTOR

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Dedicated to Professor Paolo Ribenboim on the occasion of his 80th birthday.

RÉSUMÉ. Les fonctions

$$I(n) := \prod_{p^\alpha \| n} p^{1/\alpha} \quad \text{et} \quad R(n) := \prod_{p^\alpha \| n} p^{\alpha-1}$$

ont été introduites par Atanassov. Nous montrons qu'il existe une constante positive c telle que $\sum_{n\le x} I(n) = cx^2 + O\left(x^{3/2}\Delta(x)\right)$, où $\Delta(x) = o(x)$, améliorant ainsi un résultat récent de Alkan, Ledoan et Zaharescu.

ABSTRACT. The functions

$$I(n) := \prod_{p^\alpha \| n} p^{1/\alpha} \quad \text{and} \quad R(n) := \prod_{p^\alpha \| n} p^{\alpha-1}$$

were introduced by Atanassov. We show that there exists a positive constant c such that $\sum_{n\le x} I(n) = cx^2 + O\left(x^{3/2}\Delta(x)\right)$, where $\Delta(x) = o(x)$, thereby improving a recent result of Alkan, Ledoan and Zaharescu.

1. Introduction

In 1996 and 2002, Atanassov [2], [3] studied the following arithmetic functions

$$I(n) := \prod_{p^\alpha \| n} p^{1/\alpha} \quad \text{and} \quad R(n) := \prod_{p^\alpha \| n} p^{\alpha-1}$$

where $n = p_1^{\alpha_1} \cdots p_r^{\alpha_r}$ stands for the usual factorization of n. These functions satisfy simple properties such as

$$I(n)R(n)^2 \ge n\,.$$

Some properties are less trivial such as the following, proved by Panaitopol [4]:

(1.1)
$$\sum_{n=1}^{\infty} \frac{1}{I(n)R(n)\varphi(n)} < e^2,$$

where φ stands for the Euler function. In the same paper, Panaitopol also proved that the arithmetic function

$$G(n) = \prod_{\nu=1}^{n} I(\nu)^{1/n}$$

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satisfies the inequalities

$$e^{-7}n < G(n) < n,$$

and further asked if there exists an absolute constant $c_1 > 0$ such that

$$G(n) = c_1 n + O(\sqrt{n})\,. \tag{1.2}$$

Recently, Alkan, Ledoan and Zaharescu [1] proved (1.2) and moreover established that there exists a positive constant c_2 such that, as $x \to \infty$,

$$\sum_{n \le x} I(n) = c_2 x^2 + O\left(x^{3/2}(\log x)^{9/4}\right). \tag{1.3}$$

In this paper, we improve estimate (1.3) by proving the following result.

Theorem 1.1. *There exists a positive constant c_3 such that, as $x \to \infty$,*

$$\sum_{n \le x} I(n) = c_2 x^2 + O\left(x^{3/2}\Delta(x)\right), \tag{1.4}$$

where

$$\Delta(x) = \exp\left\{-c_3(\log x)^{3/5}(\log\log x)^{-1/5}\right\}.$$

2. Proof of the Theorem

Clearly it is enough to prove that

$$\sum_{n \le x} \frac{I(n)}{n} = c_2 x + O\left(\sqrt{x}\,\Delta(x)\right). \tag{2.1}$$

Define $f(n) := \dfrac{I(n)}{n}$, in which case one easily checks that

$$\sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \frac{\zeta(s)}{\zeta(2s)} U(s)\,, \tag{2.2}$$

where

$$U(s) = \prod_p \left(1 + \frac{g(p^2)}{p^{2s}} + \frac{g(p^3)}{p^{3s}} + \cdots\right),$$

and

$$g(p) = 0, \quad g(p^2) = \frac{1}{p^{1/2}}, \quad g(p^3) = \frac{1}{p^{2/3}} - \frac{1}{p^{1/2}}, \quad g(p^4) = \frac{1}{p^{1/2}} - \frac{1}{p^{2/3}} + \frac{1}{p^{3/4}},$$

and so on, so that in general, one easily sees that

$$\left|g(p^\alpha)\right| \le \frac{1}{\sqrt{p}} \quad (\alpha \ge 2)\,. \tag{2.3}$$

First observe that

$$\sum_{k > x} \frac{|g(k)|}{k} \ll \frac{1}{x^{3/4}}\,, \tag{2.4}$$

that

$$\sum_{k>x} \frac{|g(k)|}{\sqrt{k}} = O(1)\,, \tag{2.5}$$

and also that

$$\sum_{k>x^\alpha} \frac{|g(k)|}{k} \ll \frac{1}{x^{3\alpha/4}} \quad (0<\alpha<1). \tag{2.6}$$

To prove (2.4), we use (2.3), thus allowing us to write

$$\begin{aligned} \sum_{k>x} \frac{g(k)}{k} &\ll \sum_{n>\sqrt{x}} \frac{1}{n^{2+1/2}} \ll \int_{\sqrt{x}}^{\infty} u^{-2-1/2}\,du \\ &\ll n^{-1-1/2}\Big|_{\sqrt{x}}^{\infty} \ll \frac{1}{x^{1/2+1/4}}\,, \end{aligned}$$

which proves (2.4). One can then prove (2.5) and (2.6) in a similar manner.

Now set $S(x) := \sum_{n\le x} f(n)$. In light of (2.2), of the fact that

$$\frac{\zeta(s)}{\zeta(2s)} = \sum_{n=1}^{\infty} \frac{\mu^2(n)}{n^s}$$

and of the well-known estimate

$$\sum_{n\le x} \mu^2(n) = \frac{6}{\pi^2}x + O\left(\sqrt{x}\exp\left\{-c_3(\log x)^{3/5}(\log\log x)^{-1/5}\right\}\right)$$

(see Walfisz [5]), we get that, for any fixed $0<\alpha<1$,

$$\begin{aligned} S(x) &= \sum_{mk\le x} \mu^2(m)g(k) = \sum_{k\le x} g(k) \sum_{m\le x/k} \mu^2(m) \\ &= \sum_{k\le x^\alpha} g(k) \sum_{m\le x/k} \mu^2(m) + O\left(\sum_{x^\alpha<k\le x} |g(k)|\cdot\frac{x}{k}\right) \\ &= \frac{6}{\pi^2}\,x \sum_{k\le x^\alpha} \frac{g(k)}{k} + O\left(\sqrt{x}\sum_{k\le x^\alpha} \frac{|g(k)|}{k^{1/2}} \exp\left\{-c_3\left(\log\frac{x}{k}\right)^{3/5}\left(\log\log\frac{x}{k}\right)^{-1/5}\right\}\right) \\ &\quad + O\left(\sum_{x^\alpha<k\le x} |g(k)|\cdot\frac{x}{k}\right). \end{aligned} \tag{2.7}$$

Calling upon (2.4), (2.5) and (2.6), we see that (2.7) yields (2.1) and therefore (1.4), thus completing the proof of the Theorem.

3. Final remarks

Observe that the upper bound in (1.1) can be improved. In fact, one can easily show that the indicated series C satisfies the following interesting inequalities:

$$\prod_p \left(1 + \frac{p^2}{(p-1)(p^3-1)}\right) < C < \prod_p \left(1 + \frac{1}{(p-1)^2}\right), \tag{3.1}$$

implying in particular that $2.0482 < C < 2.8264$, which improves (1.1).

To prove the first inequality in (3.1), first observe that

$$\begin{aligned} I(n)R(n)\varphi(n) &= \prod_{p^\alpha \| n} p^{1/\alpha} \cdot p^{\alpha-1} \cdot p^{\alpha-1} \cdot (p-1) \\ &= \prod_{p^\alpha \| n} p^{1/\alpha+\alpha-2} \cdot p \cdot p^{\alpha-1} \cdot (p-1) \\ &\geq \prod_{p^\alpha \| n} p \cdot p^{\alpha-1}(p-1) \\ &= \gamma(n)\varphi(n), \end{aligned}$$

where $\gamma(n) := \prod_{p|n} p$ with $\gamma(1) = 1$, from which it follows that

$$\sum_{n=1}^{\infty} \frac{1}{I(n)R(n)\varphi(n)} \leq \sum_{n=1}^{\infty} \frac{1}{\gamma(n)\varphi(n)} = \prod_p \left(1 + \frac{1}{(p-1)^2}\right).$$

The second inequality in (3.1) follows using the trivial inequality $I(n) \leq n$ and the fact that $R(n) = n/\gamma(n)$, so that

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{I(n)R(n)\varphi(n)} &\geq \sum_{n=1}^{\infty} \frac{\gamma(n)}{n^2\varphi(n)} \\ &= \prod_p \left(1 + \frac{p}{p^2(p-1)} + \frac{p}{p^4 \cdot p(p-1)} + \cdots\right) \\ &= \prod_p \left(1 + \frac{p^2}{(p-1)(p^3-1)}\right), \end{aligned}$$

which completes the proof of (3.1).

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