Exponential sums involving the largest prime factor function

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May 10, 2010 Edition

1 Introduction

Let $P(n)$ stand for the largest prime factor of the integer $n \geq 2$ and set $P(1) = 1$. Let $\wp$ be the set of all prime numbers $p_1 < p_2 < \cdots$. A well known result of I.M. Vinogradov [7] asserts that, given any irrational number α, the sequence αp_n, $n = 1, 2, \ldots$, is uniformly distributed in $[0, 1]$. In 2005, Banks, Harman and Shparlinski [1] proved that for every irrational number α, the sequence $\alpha P(n)$, $n = 1, 2, \ldots$, is uniformly distributed mod 1. They did so by using the well known Weyl criteria (see the book of Kuipers and Niederreiter [5]) and thus by establishing that

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n \leq x} e(\alpha P(n)) = 0. \tag{1.1}$$

Let $\mathcal{M}$ stand for the set of all complex valued multiplicative functions and let $\widetilde{\mathcal{M}}$ be the subset of those functions $f \in \mathcal{M}$ such that $|f(n)| \leq 1$ for $n = 1, 2, \ldots$ Daboussi (see Daboussi and Delange [2]) proved that given $f \in \widetilde{\mathcal{M}}$ and any irrational number α, then

$$\lim_{x\to\infty} \sup_{f \in \widetilde{\mathcal{M}}} \frac{1}{x} \sum_{n \leq x} f(n) e(n\alpha) = 0,$$

where $e(z) := \exp\{2\pi i z\}$.

In this paper, we first generalize (1.1) by showing that for any irrational number α and any function $f \in \mathcal{M}_1$, we have $\sum_{n \leq x} f(n) e(\alpha P(n)) = o(x)$. We further show that this later estimate also holds if one replaces $e(\alpha P(n))$ by $T(P(n))$, where T is any function defined on primes satisfying $|T(p)| = 1$ for all primes p and such that $\sum_{p \leq x} T(p) = o(\pi(x))$, where $\pi(x)$ stands for the number of primes $\leq x$.

We then move our interest to shifted primes by establishing that (1.1) holds if one replaces $P(n)$ by $P(n-1)$, provided $f \in \mathcal{M}_1$ satisfies an additional condition.

Finally, we examine the counting function $E(x, q, a) := \#\{p \leq x : P(p-1) \equiv a \pmod{q}\}$. In [1], Banks, Harman and Shparlinski proved that

$$E(x, q, a) \ll \frac{\mathrm{li}(x)}{\phi(q)} \qquad (\log q \leq (\log x)^{1/3}),$$

where the constant implicit in $\ll$ is absolute, with $\mathrm{li}(x) := \displaystyle\int_2^x \frac{dt}{\log t}$ and ϕ stands for the Euler function, and mentioned that the matching lower bound $E(x, q, a) \gg \dfrac{\mathrm{li}(x)}{\phi(q)}$

should most likely hold as well, but could not prove it. Here we prove their guess to be true.

In what follows, $c, c_1, c_2, \ldots$ always denote absolute real constants.

2 Main results

Let $\mathcal{M}_1$ be the subset of those functions $f \in \mathcal{M}$ such that $|f(n)| = 1$ for $n = 1, 2, \ldots$

Theorem 1. *Given an irrational number α and a function $f \in \mathcal{M}_1$, then*

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} f(n) e(\alpha P(n)) = 0,$$

where $e(z) := \exp\{2\pi i z\}$.

Theorem 2. *Let $f \in \mathcal{M}_1$. Let $T : \wp \to \mathbb{C}$ be such that $|T(p)| = 1$ for each $p \in \wp$ and such that $\sum_{p\le x} T(p) = o(\pi(x))$, where $\pi(x)$ stands for the number of primes not exceeding x. Then*

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} f(n) T(P(n)) = 0.$$

Note that one can show that Theorems 1 and 2 remain valid when replacing $P(n)$ by $P_k(n)$, the k-th largest prime factor of n.

Theorem 3. *Given an arbitrary fixed number $A > 0$, there exists an absolute constant $c > 0$ such that, for all $x \ge 2$,*

$$E(x, q, a) \ge c \frac{li(x)}{\phi(q)} \qquad \left((a, q) = 1,\ q \le (\log x)^A\right).$$

Theorem 4. *Let $f \in \mathcal{M}_1$ and assume that $\displaystyle\sum_p \frac{1 - \Re\left(f(p) p^{-it}\right)}{p}$ converges for some $t \in \mathbb{R}$. Then, given any irrational number α,*

$$\lim_{x\to\infty} \sum_{n\le x} f(n)\, e(\alpha P(n-1)) = 0.$$

3 Preliminary results

The following two lemmas are essentially due to Halász [4]. We state them as follows.

Lemma 1. *Let $f \in \mathcal{M}$ with $|f(n)| \le 1$ for all $n \in \mathbb{N}$. Assume that the series $\displaystyle\sum_p \frac{1-\Re(f(p)p^{-ia_0})}{p}$ is convergent for some real number a_0. Then, there exists a constant $C_0 \in \mathbb{C}$ and a slowly oscillating function $L_0(u)$, with $|L_0(u)| = 1$, such that*

$$\sum_{n\le x} f(n) = C_0\, L_0(\log x)\, x^{1+ia_0} + o(x).$$

Remark. Observe that the constant C_0 is nonzero if there exists at least one integer $r \ge 0$ for which $f(2^r) \ne -1$.

Lemma 2. *Let $f \in \mathcal{M}$ with $|f(n)| \le 1$ for all $n \in \mathbb{N}$. Then,*

$$\sum_{n\le x} f(n) = o(x)$$

if

$$\sum_p \frac{1-\Re\left(f(p)p^{-ib}\right)}{p}$$

diverges for every real number b or if $f(2^r) = -1$ for $r = 1, 2, \dots$.

The next lemma, which may be of independent interest, plays a crucial role in what follows.

Lemma 3. *Let $(a(n))_{n\ge1}$ be a sequence of complex numbers of modulus 1 and set $A(x) := \sum_{n\le x} a(n)$. Also let $\tau \in \mathbb{R}$ and set $A_\tau(x) := \sum_{n\le x} a(n)n^{i\tau}$. If $A(x) = o(x)$, then $A_\tau(x) = o(x)$.*

Remark. As a consequence of Lemma 3, it follows that if $A_{\tau_1}(x) = o(x)$ for some real number τ_1, then $A_\tau(x) = o(x)$ for every real number τ.

Proof of Lemma 3. Since $A(x) = o(x)$, there exist decreasing functions $\varepsilon(x)$ and $\delta(x)$, both tending to 0 as $x \to \infty$, such that

$$|A(x+y) - A(x)| \le \delta(x)y, \tag{3.1}$$

uniformly for $\varepsilon(x)x \le y \le x$,

Now observe that

$$\begin{aligned} A_\tau(x+y) - A_\tau(x) &= x^{i\tau} \sum_{x<n\le x+y} a(n)e^{i\tau\log(n/x)} \\ &= x^{i\tau}(A(x+y) - A(x)) + O\left(|\tau| \sum_{x<n\le x+y} \log\frac{n}{x}\right). \end{aligned}$$

Therefore,

$$|A_\tau(x+y) - A_\tau(x)| \le |A(x+y) - A(x)| + c_1|\tau|\frac{y^2}{x}. \tag{3.2}$$

We shall now prove that

$$\limsup_{X\to\infty} \frac{|A_\tau(X)|}{X} = 0. \tag{3.3}$$

To do so, we first let $M > 0$ be an arbitrarily large integer and choose X large enough so that we have both $\delta(\frac{X}{M}) < \frac{1}{M^2}$ and $\varepsilon(\frac{X}{M}) < \frac{1}{M^2}$. Finally let $x = X/M$. Since

$$A_\tau(Mx) = A_\tau(x) + \sum_{j=2}^{M} (A_\tau(jx) - A_\tau((j-1)x)),$$

it follows, in light of (3.1) and (3.2), that

$$\begin{aligned} |A_\tau(Mx)| &\le |A_\tau(x)| + \sum_{j=2}^{M} |A_\tau(jx) - A_\tau((j-1)x)| \\ &\le x + \sum_{j=1}^{M-1} x\delta(jx) + c_1|\tau| x \sum_{j=1}^{M-1} \frac{1}{j} \\ &\le x + xM\delta(x) + c_2 x|\tau| \log M, \end{aligned}$$

from which it follows that

$$\frac{|A_\tau(Mx)|}{Mx} \le \frac{1}{M} + \delta(x) + c_2|\tau|\frac{\log M}{M},$$

which in turn implies that

$$\limsup_{X\to\infty} \frac{|A_\tau(X)|}{X} \le c_3|\tau|\frac{\log M}{M}.$$

Since M can be taken arbitrarily large, (3.3) follows, thus completing the proof of Lemma 3.

4 The proofs of Theorems 1 and 2

Let $f \in \mathcal{M}_1$, α an irrational number and $S(x) := \sum_{n\le x} f(n)$. Assume for now that f is completely multiplicative. We shall consider separately the two cases

$$\text{(i)} \ \lim_{x\to\infty} \frac{S(x)}{x} = 0, \qquad \text{(ii)} \ \frac{S(x)}{x} \not\to 0 \quad \text{as } x\to\infty.$$

It is well known (see Tenenbaum [6]) that

$$\psi(x,y) := \#\{n \le x : P(n) \le y\} = (1+o(1))x\rho(u) \qquad (x \to \infty), \tag{4.1}$$

where $\rho(u)$ stands for the Dickman function and $u := (\log x)/(\log y)$ is fixed.

Therefore, it is clear that, for a fixed positive $\delta < \frac{1}{2}$,

$$\begin{aligned} &\lim_{x\to\infty} \frac{1}{x}\left(\#\{n \le x : P(n) \le x^\delta\} + \#\{n \le x : P(n) > x^{1-\delta}\}\right) \\ &\quad = \lim_{x\to\infty} \frac{1}{x}\left(\psi(x, x^\delta) + x - \psi(x, x^{1-\delta})\right) \\ &\quad = \rho(1/\delta) + 1 - \rho(1/(1-\delta)) \ll \delta. \end{aligned} \tag{4.2}$$

So, let $0 < \delta < \frac{1}{2}$ be fixed. For some prime q, $x^\delta < q < x^{1-\delta}$, define

$$S_q(x) := \sum_{\substack{n \le x \\ P(n) < q}} f(n) \qquad \text{and} \qquad D_q = \prod_{q \le p \le x} p.$$

Observe that for any $n \le x$, one has $P(n) < q$ if and only if $\gcd(n, D_q) = 1$. Using the fact that f is completely multiplicative, it follows that

$$S_q(x) = \sum_{d \mid D_q} \mu(d) f(d) S(x/d). \tag{4.3}$$

Now consider the sum

$$\Sigma_1 := \Sigma_1(x) = \sum_{x^\delta < q < x^{1-\delta}} f(q) e(\alpha q) S_q(x/q).$$

It follows from (4.2) that

$$\left| \sum_{n \le x} f(n) e(\alpha P(n)) - \Sigma_1 \right| \le c_4 \delta x.$$

It follows from this last estimate that Theorem 1 will be proved (in this case) if we can show that $\Sigma_1 = \Sigma_1(x)$ tends to 0 as $x \to \infty$.

Now since $S(x) = o(x)$, there exists a function $\varepsilon_1(x)$ which tends to 0 as $x \to \infty$ and such that $|S(x)| \le \varepsilon_1(x) \cdot x$.

From (4.3) and the definition of Σ_1, we have

$$\begin{aligned} |\Sigma_1| &\le x \sum_{x^\delta < q < x^{1-\delta}} \frac{1}{q} \sum_{\substack{d \mid D_q \\ dq < x^{1-\delta^2}}} \frac{\varepsilon_1(x^{\delta^2})}{d} + x \sum_{\substack{d \mid D_q \\ x^{1-\delta^2} \le qd < x}} \frac{1}{qd} \\ &= x\Sigma_A + \Sigma_B, \end{aligned} \tag{4.4}$$

say. Clearly,

$$
\begin{aligned}
\Sigma_A &\leq \varepsilon_1(x^{\delta^2}) \sum_{x^\delta<q<x^{1-\delta}} \frac{1}{q} \prod_{q\leq p<x} \left(1+\frac{1}{p}\right) \\
&\leq c_5\varepsilon_1(x^{\delta^2}) \sum_{x^\delta<q<x^{1-\delta}} \frac{\log x}{q\log q} \\
&\leq c_6\varepsilon_1(x^{\delta^2}) \frac{1}{\delta}. \qquad (4.5)
\end{aligned}
$$

In order to estimate Σ_B, we proceed as follows. For a fixed prime q, each divisor d in the sum lies in $[z, x^{\delta^2}z]$, where $z = x^{1-\delta^2}/q$. Splitting this interval into dyadic subintervals of the form $[2^j z, 2^{j+1}z]$, we observe that

$$\sum_{\substack{d|D_q \\ d\in]2^j z, 2^{j+1}z[}} \frac{1}{d} \leq c_7 \prod_{p<q} \left(1-\frac{1}{p}\right) \leq \frac{c_8}{\log q}.$$

Since the maximum value of j in the above expression is $c_9\delta^2 \log x$, it follows that

$$\Sigma_B \leq c_{10}\delta^2 \sum_{x^\delta<q<x^{1-\delta}} \frac{\log x}{q\log q} \leq c_{11}\delta^2 \frac{\log x}{\delta \log x} = c_{11}\delta. \qquad (4.6)$$

Using (4.5) and (4.6) in (4.4), we obtain that

$$\left|\frac{\Sigma_1}{x}\right| \leq c_{11}\delta + c_6 \frac{\varepsilon_1(x^{\delta^2})}{\delta},$$

which implies that

$$\limsup_{x\to\infty} \frac{|\Sigma_1(x)|}{x} \ll \delta.$$

Since δ can be chosen arbitrarily small, it follows that $|\Sigma_1(x)|/x \to 0$ as $x \to \infty$, which completes the proof of Theorem 1 in case (i), when f is assumed to be completely multiplicative, a fact that we only used to deduce (4.3).

To drop this last condition, we proceed as follows. We define $f_1 = f_{1,x} \in \mathcal{M}$ as follows: $f_1(p^\alpha) = f(p^\alpha)$ if $p \notin [x^\delta, x^{1-\delta}]$ and $f_1(p^\alpha) = f(p)^\alpha$ otherwise. Set

$$S^{(1)}(x) := \sum_{n\leq x} f_1(n),$$

and, for $x^\delta < q < x^{1-\delta}$, let

$$S_q^{(1)}(x) := \sum_{d|D_q} \mu(d)f(d)S^{(1)}(x/d).$$

In light of these definitions, it is easy to see that

$$\left|S(x) - S^{(1)}(x)\right| \le x \sum_{x^\delta < q < x^{1-\delta}} \frac{1}{q^2} \ll x^{1-\delta}$$

and

$$\left|\sum_{n\le x} (f(n) - f_1(n)) e(\alpha P(n))\right| \ll \delta x + x^{1-\delta},$$

so that the Theorem is proved in case (i) without the restriction that f is completely multiplicative.

It remains to consider case (ii). In this case, it follows from Lemma 2 that there exists a real number τ for which $\sum_p \frac{1 - \Re(f(p)p^{-i\tau})}{p}$ converges. From Lemma 3, we have that, as $x \to \infty$,

$$\frac{1}{x}\sum_{n\le x} f(n)e(\alpha P(n)) \to 0 \qquad \text{and} \qquad \frac{1}{x}\sum_{n\le x} f(n)n^{-i\tau} e(\alpha P(n)) \to 0.$$

In light of these observations, it is sufficient to consider the case $\tau = 0$, that is

$$\sum_p \frac{1 - \Re(f(p))}{p} \qquad \text{is convergent.} \tag{4.7}$$

Let $f(p^r) = e(F(p^r))$ with $-\frac{1}{2} \le F(p^r) \le \frac{1}{2}$. It is clear that (4.7) holds if and only if

$$\sum_p \frac{F^2(p)}{p} < \infty. \tag{4.8}$$

Let Y be a fixed large number and set

$$A_{X,Y} := \sum_{Y<p<X} \frac{F(p)}{p}.$$

Further define the multiplicative functions $f_Y(n)$ and $g_Y(n)$ by

$$f_Y(p^r) := \begin{cases} f(p^r) & \text{if} \quad p \le Y, \\ 1 & \text{if} \quad p > Y \end{cases}$$

and

$$g_Y(p^r) := \begin{cases} f(p^r) & \text{if} \quad p > Y, \\ 1 & \text{if} \quad p \le Y. \end{cases}$$

It is clear that $f(n) = f_Y(n) \cdot g_Y(n)$.

Further let

$$G_Y(n) := \sum_{\substack{p^r \| n \\ p > Y}} F(p^r).$$

It follows from the Turán-Kubilius Inequality that

$$\sum_{n \le x} |G_Y(n) - A_{X,Y}|^2 \le c_{12} x \sum_{\substack{p \ge Y \\ r \ge 1}} \frac{F^2(p^r)}{p^r} = c_{12} x B_Y^2, \tag{4.9}$$

say. From (4.8), it follows that $B_Y \to 0$ as $Y \to \infty$. On the other hand, since $g_Y(n) = e(G_Y(n))$, it is clear, in light of (4.9), that

$$\sum_{n \le x} |g_Y(n) - e(A_{X,Y})|^2 \le c_{13} x B_Y^2.$$

Therefore,

$$\left| \sum_{n \le x} f(n) e(\alpha P(n)) - e(-A_{X,Y}) \sum_{n \le x} f_Y(n) e(\alpha P(n)) \right| \le c_{14} x B_Y. \tag{4.10}$$

We shall now establish that

$$\frac{1}{x} \sum_{n \le x} f_Y(n) e(\alpha P(n)) \to 0 \qquad (x \to \infty). \tag{4.11}$$

We further define the multiplicative function $\widetilde{f_Y}(n)$ by

$$\widetilde{f_Y}(p^r) := \begin{cases} 1 & \text{if} \quad p > Y^{1/r}, \\ f_Y(p^r) & \quad \text{otherwise.} \end{cases}$$

First observe that

$$\left| \sum_{n \le x} f_Y(n) e(\alpha P(n)) - \sum_{n \le x} \widetilde{f_Y}(n) e(\alpha P(n)) \right| \le \sum_{\substack{p^r \ge Y \\ p \le Y}} \frac{x}{p^r} \le \varepsilon_1(Y) x, \tag{4.12}$$

where $\varepsilon_1(Y) \to 0$ as $Y \to \infty$.

Let the function $h_Y(n)$ be the function defined implicitly by

$$\widetilde{f_Y}(n) = \sum_{d | n} h_Y(d).$$

It is easy to see that

$$h_Y(p) = \begin{cases} \widetilde{f_Y}(p) - 1 & \text{if} \quad p \le Y, \\ 0 & \text{if} \quad p > Y, \end{cases}$$

and that similarly $h_Y(p^r) = 0$ if $p > Y$.

On the other hand, since $h_Y(p^r) = \widetilde{f_Y}(p^r) - \widetilde{f_Y}(p^{r-1})$, it follows that $h_Y(p^r) = 0$ if $p^{r-1} > Y$.

From the definition of h_Y, it is clear that

$$\sum_{n \le x} \widetilde{f_Y}(n) e(\alpha P(n)) = \sum_{d \le x} h_Y(d) \sum_{dm \le x} e(\alpha P(dm)). \tag{4.13}$$

If $h_Y(d) \neq 0$, then $p^r \| d$ implies that $p < Y$ and $p^{r-1} \le Y$, so that $p^r \le Y^2$. Consequently, $d \le Y^{2\pi(Y)} \le Y^{2Y}$. Furthermore, $h_Y(d) \le 2^{\pi(Y)}$.

For a fixed positive integer d, we have

$$\sum_{m \le x/d} e(\alpha P(dm)) = \sum_{m \le x/d} e(dP(m)) + O\left(\sum_{\substack{m \le x/d \\ P(m) \le P(d)}} 1 \right). \tag{4.14}$$

Using the main result of Banks, Harman and Shparlinski [1], namely that for any fixed irrational number α,

$$\lim_{x \to \infty} \frac{1}{x} \sum_{n \le x} e(\alpha P(n)) = 0,$$

we have, using (4.14) in (4.13), that

$$\lim_{x \to \infty} \frac{1}{x} \sum_{n \le x} \widetilde{f_Y}(n) e(\alpha P(n)) = 0, \tag{4.15}$$

Hence, it follows from estimate (4.15), taking into account (4.12), that (4.11) is proved. Finally, gathering (4.10) and (4.11), Theorem 1 is proved.

Theorem 2 can be established along the lines of the proof of Theorem 1 and its proof will therefore be omitted.

5 The proof of Theorem 3

Let $0 < \eta_1 < \eta_2 < \frac{1}{2}$. It is clear that

$$\begin{aligned} E(x, Q, a) &\ge \sum_{\substack{x^{\eta_1} < Q < x^{\eta_2} \\ Q \equiv a \pmod q}} \pi(x; Q, 1) - \sum_{\substack{Q < Q' \\ x^{\eta_1} < Q < x^{\eta_2} \\ Q \equiv a \pmod q}} \pi(x; QQ', 1) \\ &= \Sigma_1 - \Sigma_2. \end{aligned} \tag{5.1}$$

say, where as usual $\pi(x; b, a) := \#\{p \le x : p \equiv a \pmod b\}$. It follows from the Bombieri-Vinogradov Theorem that

$$\Sigma_1 = \mathrm{li}(x) \sum_{\substack{x^{\eta_1} < Q < x^{\eta_2} \\ Q \equiv a \pmod q}} \frac{1}{Q-1} + O\left(\frac{x}{(\log x)^A} \right), \tag{5.2}$$

assuming that $x^{\eta_2} \le \dfrac{\sqrt{x}}{(\log x)^{2A+5}}$, a condition which is equivalent to

$$\frac{1}{2} - \eta_2 \ge (2A+5)\frac{\log\log x}{\log x}. \tag{5.3}$$

Summing over Q allows us to write (5.2) as

$$\Sigma_1 = \left(\log \frac{\eta_2}{\eta_1}\right) \frac{\mathrm{li}(x)}{\phi(q)} + O\left(\frac{x}{(q\log x)^D}\right) \tag{5.4}$$

uniformly for $q \le (\log x)^c$, where D is any preassigned value.

In order to estimate Σ_2, we use standard sieve techniques. Actually Σ_2 represents the number of solutions of $p-1 = bQQ' \le x$, where b, Q, Q' vary as follows:

$$Q \equiv a \pmod{q}, \quad Q \in [x^{\eta_1}, x^{\eta_2}], \quad Q < Q', \quad b = 1, 2, 3, \ldots$$

We first fix b and Q, and we assume that there is at least one pair of numbers p, Q' which is a solution of $p-1 = bQQ' \le x$, in which case we have $b < x^{1-2\eta_1}$ and $bQ < x^{1-\eta_1}$. Let η_1 be close to $1/2$. Then we have

$$E_{b,Q} := \#\{p, Q' \text{ such that } p-1 = bQQ' \le x,\ Q \equiv a \pmod{q}\} \le c_{15}\frac{x}{\log^2 x\ \phi(bQ)}. \tag{5.5}$$

Using the well known estimate $\sum_{b\le y} 1/\phi(b) \le c_{16}\log y$, it follows from (5.5) that

$$\begin{aligned} \Sigma_2 = \sum_{b,Q} E_{b,Q} &\le c_{15}\frac{x}{\log^2 x} c_{16} \sum_{x^{\eta_1} < Q < x^{\eta_2}} \frac{\log(x/Q^2)}{Q-1} \\ &\le c_{17}\frac{x}{\log x}\frac{1}{\phi(q)}(1-2\eta_1)\log\frac{\eta_2}{\eta_1}. \end{aligned} \tag{5.6}$$

Choosing η_2 so that it satisfies (5.3) and η_1 so that $c_{17}(1-2\eta_1) < \frac{1}{2}$, and then gathering (5.4) and (5.6) in (5.1), we obtain that

$$E(x,q,a) \ge \frac{1}{2}\left(\log\frac{\eta_2}{\eta_1}\right)\frac{\mathrm{li}(x)}{\phi(q)},$$

thus completing the proof of Theorem 3.

6 The proof of Theorem 4

Again using the analogue of Lemma 3, namely in the form that

$$\lim_{x\to\infty} \frac{1}{x}\sum_{n\le x} f(n) n^{i\tau} e(\alpha P(n-1)) = 0 \quad \Longleftrightarrow \quad \lim_{x\to\infty}\frac{1}{x}\sum_{n\le x} f(n) e(\alpha P(n-1)) = 0,$$

we may assume that $\tau = 0$, that is that

$$\sum_p \frac{1 - \Re(f(p))}{p} < \infty.$$

Arguing as in the proof of case (ii) of Theorem 1, we reduce the problem to the proof that the expression

$$\sum_{n \le x} \widetilde{f_Y}(n) e(\alpha P(n-1)) = \sum_{d \le x} h_Y(d) \sum_{m \le x/d} e(\alpha P(dm-1)) \tag{6.1}$$

is $o(x)$ as $x \to \infty$.

First let us define

$$\psi(x, y; a, q) := \#\{n \le x : P(n) \le y,\ n \equiv a \pmod q\}.$$

Since, in the first sum on the right hand side of (6.1), d runs over a finite set of integers which does not change as $x \to \infty$, it is enough to prove that

$$\lim_{X \to \infty} \frac{1}{X} \sum_{m \le X} e(\alpha P(dm-1)) = 0. \tag{6.2}$$

We have $P(dm-1) = q$ if $dm - 1 = q\nu$, $P(\nu) \le q$, that is $q\nu + 1 \equiv 0 \pmod d$, $\nu \equiv \ell_q \pmod d$, $P(\nu) \le q$, $\nu \le x/q$. This quantity is precisely $\psi(\frac{xd}{q}, q; \ell_q, d)$.

It follows that

$$\sum_{m \le X} e(\alpha P(dm-1)) = \sum_{q < xd} e(\alpha q) \psi(\frac{xd}{q}, q; \ell_q, d).$$

Let $\varepsilon > 0$ be an arbitrary real number. It follows from (4.2) that

$$\sum_{m \le x} e(\alpha P(dm-1)) = \sum_{x^\varepsilon < q < x^{1-\varepsilon}} e(\alpha q) \psi(\frac{xd}{q}, q; \ell_q, d) + R_x, \tag{6.3}$$

where $|R_x| \le \varepsilon x$. It has been established by Granville [3] that, if $\gcd(a, d) = 1$ and $d^{1+\varepsilon} \le y \le x$, then

$$\psi(x, y; a, d) \sim \frac{1}{d} \psi(x, y) \qquad (x \to \infty). \tag{6.4}$$

Observing that

$$\psi(\frac{xd}{q}, q) = (1 + o(1)) \rho \left(\frac{\log xd}{\log q} - 1 \right) \frac{xd}{q} \qquad (x \to \infty),$$

we have, in light of (6.4), that the right hand side of (6.3) is, as $x \to \infty$, equal to

$$xd \sum_{x^\varepsilon < q < x^{1-\varepsilon}} \rho\left(\frac{\log xd}{\log q} - 1\right)\frac{e(\alpha q)}{q} + o(1)xd \sum_{x^\varepsilon < q < x^{1-\varepsilon}} \rho\left(\frac{\log xd}{\log q} - 1\right)\frac{1}{q} + R_x$$
$$= S_1(x) + S_2(x) + R_x.$$

In order to prove (6.2), it remains to show that

$$S_1(x) = o(x) \qquad \text{and} \qquad S_2(x) = o(x). \tag{6.5}$$

First we set

$$J_x := \left[\frac{1}{1-\varepsilon} - 1 + \frac{\log d}{\log x}, \frac{1}{\varepsilon} - 1 + \frac{\log d}{\log x}\right].$$

If $q \in [x^\varepsilon, x^{1-\varepsilon}]$, then $\dfrac{\log xd}{\log q} - 1 \in J_x$. On the other hand, note that $J_x \subseteq \left[\frac{1}{1-\varepsilon}, \frac{1}{\varepsilon}\right]$, and that in this interval, ρ is bounded, and therefore,

$$S_2(x) \ll o(1)xd \sum_{x^\varepsilon < q < x^{1-\varepsilon}} \frac{1}{q} \ll o(1)x\log(1/\varepsilon) = o(x) \qquad (x \to \infty), \tag{6.6}$$

which proves the second estimate in (6.5).

To estimate $S_1(x)$, we proceed as follows. First set

$$B(y) := \sum_{x^\varepsilon \le q < y} \frac{e(\alpha q)}{q}.$$

By using the theorem of I.M. Vinogradov according to which

$$\max_{2x^\varepsilon \le y \le x} \frac{1}{\pi(y)}\left|\sum_{x^\varepsilon \le q < y} e(\alpha q)\right| = \delta(x) \to 0 \qquad \text{as } x \to \infty,$$

we obtain immediately that

$$\max_{2x^\varepsilon \le y \le x} |B(y)| = \delta_1(x) \to 0 \qquad \text{as } x \to \infty.$$

On the other hand, since

$$\sum_{x^\varepsilon \le q < y} \frac{1}{q} \le \log\left(\frac{\log y}{\varepsilon \log x}\right) \text{ holds for } x^\varepsilon < y \le 2x^\varepsilon,$$

it follows that

$$\max_{x^\varepsilon \le y \le x} |B(y)| = \delta_2(x) \to 0 \qquad \text{as } x \to \infty.$$

From the definitions of $S_1(x)$ and $B(y)$, we have

$$\begin{aligned} S_1(x) &= xd \int_{x^\varepsilon}^{x^{1-\varepsilon}} \rho\left(\frac{\log xd}{\log u} - 1\right) dB(u) \\ &= xd\ \rho\left(\frac{\log xd}{\log u} - 1\right) B(u)\Big|_{x^\varepsilon}^{x^{1-\varepsilon}} \\ &\qquad + xd \int_{x^\varepsilon}^{x^{1-\varepsilon}} B(u)\rho'\left(\frac{\log xd}{\log u} - 1\right) \frac{\log xd}{u(\log u)^2}\, du. \end{aligned} \tag{6.7}$$

Since both $\rho(u)$ and $\rho'(u)$ are bounded in J_x, it follows from (6.7) and the above bounds on $B(u)$ that

$$\left|\frac{1}{x} S_1(x)\right| \le d\ o(1) + d\ o(1) \int_{x^\varepsilon}^{x^{1-\varepsilon}} \frac{\log xd}{u(\log u)^2}\, du \qquad (x \to \infty). \tag{6.8}$$

On the other hand,

$$\int_{x^\varepsilon}^{x^{1-\varepsilon}} \frac{1}{u(\log u)^2}\, du = \int_{\varepsilon \log x}^{(1-\varepsilon)\log x} \frac{dv}{v^2} = \frac{1}{v}\Big|_{\varepsilon \log x}^{(1-\varepsilon)\log x} = \left(\frac{1}{\varepsilon} - \frac{1}{1-\varepsilon}\right) \frac{1}{\log x}. \tag{6.9}$$

Gathering (6.6), (6.8) and (6.9) completes the proof of (6.5), as required. Since $\varepsilon > 0$ is arbitrary, it follows from (6.3) that

$$\frac{1}{x} \sum_{n \le x} e(\alpha P(dm-1)) \to 0 \quad (x \to \infty)$$

for every d, thus proving (6.2) and thereby (6.1), which completes the proof of Theorem 4.

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