Exponential sums and arithmetic functions at polynomial values

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Édition du 16 juillet 2011

Abstract

Let $\mathcal{M}_1$ stand for the set of all complex valued multiplicative functions satisfying $|f(n)| \le 1$ for all $n \in \mathbb{N}$. Let $Q(n) = \alpha_k n^k + \alpha_{k-1} n^{k-1} + \cdots + \alpha_1 n$ be a polynomial with real coefficients and such that at least one among $\alpha_k, \dots, \alpha_1$ is an irrational number. Given polynomials $F_1(x), \dots, F_s(x) \in \mathbb{Z}[x]$ and strongly multiplicative functions $g_1, \dots, g_s$ satisfying certain conditions, consider the sum $S_f(x) := \sum_{n \le x} f(n)\ell(n)e(Q(n))$, where $\ell(n) := g_1(F_1(n)) \cdots g_s(F_s(n))$. We prove that $\sup_{f \in \mathcal{M}_1} \frac{|S_f(x)|}{x} \to 0$ as $x \to \infty$ and obtain an analogue result when sums run over primes.

Subject classification numbers: 11L07, 11N37
Key words: exponential sums, multiplicative functions

1 Introduction

Let $\mathcal{M}_1$ stand for the set of all complex valued multiplicative functions such that $|f(n)| \le 1$ for all $n \in \mathbb{N}$. Daboussi (see Daboussi and Delange [1]) proved that given any irrational number α,

$$\lim_{x\to\infty} \sup_{f\in\mathcal{M}_1} \frac{1}{x} \sum_{n \le x} f(n)e(n\alpha) = 0,$$

where $e(r) := \exp\{2\pi i r\}$. A paper by the second author [3] contains a survey of some generalizations of this result.

A well known result of I.M. Vinogradov [6] asserts that if $Q(n) = \alpha_k n^k + \alpha_{k-1} n^{k-1} + \cdots + \alpha_1 n$ is a polynomial with real coefficients and such that at least one among $\alpha_k, \dots, \alpha_1$ is an irrational number, then

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p \le x} e(Q(p)) = 0. \tag{1.1}$$

Under the same conditions, the second author [4] proved that

$$\sup_{f\in\mathcal{M}_1} \left| \frac{1}{x} \sum_{n\le x} f(n)e(Q(n)) \right| \to 0 \qquad \text{as} \qquad x \to \infty, \tag{1.2}$$

[1] Work supported in part by a grant from NSERC.

thereby generalizing a famous result of H. Daboussi (see [1]).

Recently, we [2] also considered such types of sums but on shifted primes. In particular, letting f be a multiplicative function such that $|f(n)| = 1$ for all $n \in \mathbb{N}$ and such that, for some real number τ,

$$\sum_p \frac{1 - \Re(f(p)p^{-i\tau})}{p} < \infty,$$

and T be a function defined on prime numbers satisfying $|T(p)| = 1$ for each prime p and

$$\lim_{x\to\infty} \frac{1}{\pi(x; d, -1)} \sum_{\substack{p\le x \\ p\equiv -1 \pmod{d}}} T(p) = 0$$

for every fixed integer $d > 0$, we established that

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p\le x} f(p+1)T(p) = 0.$$

Here, we further generalize some of these results. First, let $f \in \mathcal{M}_1$ and let $Q(x)$ be as above. Given polynomials $F_1(x), \ldots, F_s(x) \in \mathbb{Z}[x]$ (which take only positive values at positive arguments) and strongly multiplicative functions $g_1, \ldots, g_s$, consider the arithmetic function

$$\ell(n) := g_1(F_1(n)) \cdots g_s(F_s(n)). \tag{1.3}$$

We shall study the sum

$$S_f(x) := \sum_{n\le x} f(n)\ell(n)e(Q(n)) \tag{1.4}$$

as well as an analog sum running on prime numbers.

2 Some notations and the general set up

For each integer $n \ge 2$, let $\omega(n)$ stand for the number of distinct prime factors of n and let $P(n)$ be the largest prime factor of n. As usual, μ will stand for the Möbius function.

A strongly additive (resp. multiplicative) function f is an additive (resp. multiplicative) function for which $f(p^a) = f(p)$ for each positive integer a and each prime p. For instance, the function ω is a strongly additive function, while the function $\varphi(n)/n$, where φ stands for the Euler function, is a strongly multiplicative function.

We use $\pi(x)$ to denote the number of primes $p \le x$, while $\pi(x; k, \ell)$ will stand for the number of primes $p \le x$ such that $p \equiv \ell \pmod{k}$. As usual, we define the logarithmic integral by $\mathrm{li}(x) := \displaystyle\int_2^x \frac{dt}{\log t}$. In this paper, c denotes an absolute constant but not necessarily the same at each occurrence, while the letters p and q, with or without subscript, always stand for primes.

Let $F_1(x), \ldots, F_s(x)$ be polynomials with integer coefficients which take only positive values at positive arguments. For $j = 1, \ldots, s$, let $\rho_j(d)$ stand for the number of solutions of $F_j(n) \equiv 0 \pmod{d}$. Moreover, let $\rho(d_1, \ldots, d_s)$ be the number of solutions of the congruence system $F_j(n) \equiv 0 \pmod{d_j}$, $j = 1, \ldots, s$.

Let $g_1, \ldots, g_s$ be complex valued multiplicative functions each satisfying the following four conditions:

(i) $|g_j(n)| = 1$ for all $n \in \mathbb{N}$;

(ii) g_j is strongly multiplicative;

(iii) $\lim_{p \to \infty} g_j(p) = 1$;

(iv) $\displaystyle\sum_p \frac{\Re(1 - g_j(p))\rho_j(p)}{p} < \infty$.

Given $K > 1$, then for each $j = 1, \ldots, s$, we shall write $g_j(n) = g_j(n|K)h_j(n|K)$, where

$$g_j(n|K) = \prod_{\substack{p|n \\ p \le K}} g_j(p) \qquad \text{and} \qquad h_j(n|K) = \prod_{\substack{p|n \\ p > K}} g_j(p).$$

Let also $t_j(n|K)$ be the Möbius transform of $g_j(n|K)$, that is the function defined implicitly by

$$g_j(n|K) = \sum_{d|n} t_j(n|K).$$

It is clear that $t_j(n|K)$ is a multiplicative function defined on prime powers by

$$t_j(p^\alpha|K) = \begin{cases} g_j(p) - 1 & \text{if } \alpha = 1 \text{ and } p \le K, \\ 0 & \text{otherwise.} \end{cases}$$

Consequently,

$$g_j(n|K) = \sum_{\substack{d|n \\ P(d) \le K}} t_j(d|K).$$

Observe that in this last summation, it is enough to let the sum run over the squarefree divisors d.

We now define the strongly multiplicative functions $f_1, \ldots, f_s$ implicitly by

$$g_j(p) = e^{if_j(p)} \text{ choosing } -\pi < f_j(p) \le \pi.$$

Note that it follows from the above condition (iii) that

$$\lim_{p \to \infty} f_j(p) = 0 \qquad (j = 1, \ldots, s).$$

Observe also that condition (iv) can be rewritten in the form

$$\sum_p \frac{f_j^2(p)\rho_j(p)}{p} < \infty.$$

Moreover, for $j = 1, \ldots, s$, let

$$A_j(x|K) := \sum_{K<q\leq x} \frac{f_j(q)\rho_j(q)}{q} \qquad (j = 1, \ldots, s)$$

and

$$A_j^*(x|K) := \sum_{K<q\leq x^{1/5}} \frac{f_j(q)\rho_j(q)}{q-1} \qquad (j = 1, \ldots, s).$$

Finally, let

$$\widetilde{f}_j(n|K) = \sum_{\substack{p|n \\ p>K}} f_j(p) \qquad (j = 1, \ldots, s).$$

3 The main results

Theorem 1. *Let $F_1, \ldots, F_s$ and $g_1, \ldots, g_s$ be defined as in Section 2, with ℓ being defined by (1.3), while $S_f(x)$ is defined in (1.4). Then*

$$\sup_{f\in\mathcal{M}_1} \frac{|S_f(x)|}{x} \to 0 \qquad \textit{as } x \to \infty.$$

Theorem 2. *With the notation of Theorem 1, we have*

$$\left| \frac{1}{li(x)} \sum_{q\leq x} \ell(q) e(Q(q)) \right| \to 0 \qquad \textit{as } x \to \infty.$$

4 Preliminary lemmas

Lemma 1. *As $x \to \infty$, for $j = 1, \ldots, s$,*

$$\sum_{n\leq x} \left(\widetilde{f}_j(F_j(n)|K) - A_j(x|K) \right)^2 \leq cx \sum_{q>K} \frac{f_j^2(q)\rho_j(q)}{q} + o(x) \tag{4.1}$$

and

$$\sum_{q\leq x} \left(\widetilde{f}_j(F_j(q)|K) - A_j^*(x|K) \right)^2 \leq c\,\mathrm{li}(x) \sum_{q>K} \frac{f_j^2(q)\rho_j(q)}{q} + o(\mathrm{li}(x)). \tag{4.2}$$

Proof. The proof of (4.1) and (4.2) can be established by a classical method of Turán extended by Kubilius, with the additional use of the Bombieri-Vinogradov inequality in the case of (4.2) using the fact that $f_j(p) \to 0$ as $p \to \infty$. One can obtain these basic concepts in the recent book of Tenenbaum [5]. $\square$

As an immediate consequence of Lemma 1, we obtain the following result.

Lemma 2. *There is a function $\delta(K)$ which tends to 0 as $K \to \infty$ such that, for $j = 1, \ldots, s$,*

$$\frac{1}{x} \sum_{n \le x} \left| h_j(F_j(n)|K) e^{-iA_j(x|K)} - 1 \right| \le \delta(K)$$

and

$$\frac{1}{\mathrm{li}(x)} \sum_{q \le x} \left| h_j(F_j(q)|K) e^{-iA_j^*(x|K)} - 1 \right| \le \delta(K).$$

From this lemma, the following follows immediately.

Lemma 3. *With $\delta(K)$ as in Lemma 2, then, for $j = 1, \ldots, s$,*

$$\frac{1}{x} \sum_{n \le x} \left| g_j(F_j(n)) - g_j(F_j(n)|K) e^{iA_j(x|K)} \right| \le \delta(K) \tag{4.3}$$

and

$$\frac{1}{\mathrm{li}(x)} \sum_{q \le x} \left| g_j(F_j(q)) - g_j(F_j(q)|K) e^{iA_j(x|K)} \right| \le \delta(K).$$

5 Proof of Theorem 1

Let K be a large number. It follows from (4.3) that

$$\frac{1}{x} \left| S_f(x) - e^{i(A_1(x|K) + \cdots + A_s(x|K))} S_f^{(K)}(x) \right| \le c\delta(K), \tag{5.1}$$

where

$$S_f^{(K)}(x) = \sum_{n \le x} f(n) \ell_K(n) e(Q(n))$$

with

$$\ell_K(n) = g_1(F_1(n)|K) \cdots g_s(F_s(n)|K).$$

Observe that in (5.1), the constant c does not depend on K.

We claim that it is enough to prove that

$$\sup_{f \in \mathcal{M}_1} \frac{|S_f^{(K)}(x)|}{x} \to 0 \qquad \text{as } x \to \infty. \tag{5.2}$$

Indeed, using (5.2) in (5.1), it follows that

$$\sup_{f\in\mathcal{M}_1} \frac{|S_f^{(K)}(x)|}{x} \le c\delta(K) \qquad \text{as } x\to\infty,$$

thereby implying that

$$\lim_{x\to\infty} \sup_{f\in\mathcal{M}_1} \frac{|S_f^{(K)}(x)|}{x} \le c\delta(K).$$

Since this last inequality holds for every K and since $\delta(K)\to 0$ as $K\to\infty$, the theorem will follow.

So, let us prove (5.2).

First, we define

$$\mathcal{B}(d_1,\dots,d_s) = \{n\in\mathbb{N} : F_j(n)\equiv 0 \pmod{d_j} \text{ for } j=1,\dots,s\},$$

so that

$$S_f^{(K)}(x) = \sum_{\substack{d_1,\dots,d_s \\ \mu^2(d_j)=1 \text{ for } j=1,\dots,s \\ P(d_j)\le K \text{ for } j=1,\dots,s}} t_1(d_1|K)\cdots t_s(d_s|K) \sum_{\substack{n\le x \\ n\in\mathcal{B}(d_1,\dots,d_s)}} f(n)e(Q(n)).$$

Observe that in the above sum, $t_1(d_1|K)\cdots t_s(d_s|K)\neq 0$ for only finitely many choices of $d_1,\dots,d_s$, the number of choices depending only on K. So, let $D=d_1\cdots d_s$. If $n_0\in\mathcal{B}(d_1,\dots,d_s)$, then $n\in\mathcal{B}(d_1,\dots,d_s)$ whenever $n\equiv n_0 \pmod{d_1\cdots d_s}$. This means that $\mathcal{B}(d_1,\dots,d_s)$ is a collection of arithmetical progressions mod D. We can therefore write that there exists a positive integer J such that

$$\mathcal{B}(d_1,\dots,d_s) = \{n : n\equiv \ell_j \pmod{D},\ j=1,\dots,J\}. \tag{5.3}$$

We shall therefore focus our attention on one of the above arithmetic progressions, in which case we obtain

$$\begin{aligned}\sum_{\substack{n\le x \\ n\equiv \ell_j \pmod{D}}} f(n)e(Q(n)) &= \frac{1}{D}\sum_{a=0}^{D-1}\sum_{n\le x} f(n)e\left(Q(n)+a\left(\frac{n-\ell_j}{D}\right)\right) \\ &= \frac{1}{D}\sum_{a=0}^{D-1} e\left(-\frac{a\ell_j}{D}\right)\sum_{n\le x} f(n)e\left(Q(n)+\frac{an}{D}\right).\end{aligned}$$

Let us apply estimate (1.2) with $Q(n)$ replaced by $Q_a(n)=Q(n)+\dfrac{an}{D}$. Since $Q_a(n)$ has an irrational coefficient, it follows that

$$\sup_{f\in\mathcal{M}_1}\frac{1}{x}\left|\sum_{n\le x} f(n)e(Q_a(n))\right| \to 0 \qquad \text{as } x\to\infty.$$

Since we only have to consider a finite number of sums (recall that we have only J congruences), this completes the proof of Theorem 1.

6 The proof of Theorem 2

Proceeding essentially as we did in the proof of Theorem 1, we may write

$$\frac{1}{\mathrm{li}(x)}\sum_{q\le x}\left|\ell(q)-e^{i(A_1^*(x|K)+\cdots+A_s^*(x|K))}\ell_K(q)\right|\le c\delta(K),$$

so that it is enough to prove that, for every fixed K,

$$\sum_{q\le x}\ell_K(q)e(Q(q))=o(\mathrm{li}(x))\qquad\text{as } x\to\infty. \tag{6.1}$$

The left hand side of (6.1) can be written as

$$\sum_{\substack{d_1,\dots,d_s\\ \mu^2(d_j)=1\ \text{for } j=1,\dots,s\\ P(d_j)\le K\ \text{for } j=1,\dots,s}} t_1(d_1|K)\cdots t_s(d_s|K)\sum_{\substack{p\le x\\ p\in\mathcal{B}(d_1,\dots,d_s)}} e(Q(p)),$$

where $\mathcal{B}(d_1,\dots,d_s)$ is as in (5.3). Therefore

$$\begin{aligned}\sum_{\substack{p\le x\\ p\in\mathcal{B}(d_1,\dots,d_s)}} e(Q(p)) &= \sum_{j=0}^{J}\sum_{\substack{p\le x\\ p\equiv\ell_j\ (\mathrm{mod}\ D)}} e(Q(p))\\ &= \sum_{\substack{j=0\\ (\ell_j,D)=1}}^{J}\sum_{\substack{p\le x\\ p\equiv\ell_j\ (\mathrm{mod}\ D)}} e(Q(p))+O(\omega(D)).\end{aligned}$$

Now, since

$$\sum_{\substack{p\le x\\ p\equiv\ell_j\ (\mathrm{mod}\ D)}} e(Q(p))=\frac{1}{D}\sum_{a=0}^{D-1}e\left(-\frac{a\ell_j}{D}\right)\sum_{p\le x}e\left(Q(p)+\frac{ap}{D}\right),$$

it follows, in light of the Vinogradov estimate (1.1), that the last sum on the right hand side of (6.1) is $o(\mathrm{li}(x))$ as $x\to\infty$, thus completing the proof of Theorem 2.

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JMDK, le 16 juillet 2011; fichier: exponential-sums-poly-July2011.tex