On a theorem of Daboussi related to the set of Gaussian integers II

JEAN-MARIE DE KONINCK and IMRE KÁTAI

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Abstract

Let W stand for the union of finitely many convex bounded domains in $\mathbb{C}$. Given $x > 0$, we denote by xW the set $\{xz : z \in W\}$. Let $G = \mathbb{Z}[i]$ be the set of Gaussian integers and set $G^* := G \setminus \{0\}$. Given a complex number $z = u + iv$, where $u, v \in \mathbb{R}$, let $\{z\} = \{u\} + i\{v\}$, where $\{x\}$ stands for the fractional part of x. Let $E := \{w : 0 \le \Re(w) < 1,\ 0 \le \Im(w) < 1\}$. We say that the sequence of complex numbers $z_1, z_2, \ldots$ is uniformly distributed mod E if $\lim_{N\to\infty} \frac{1}{N}\#\{n \le N : \Re(\{z_n\}) < u,\ \Im(\{z_n\}) < v\} = uv$ for every pair of real numbers $u, v \in]0, 1]$. Let $\mathcal{T}$ be the set of those functions $t : G^* \to \mathbb{C}$ for which $t(\alpha) + F(\alpha)$ is uniformly distributed mod E in limit on xW (as $x \to \infty$) for every additive arithmetical function F, and such that the discrepancy does not depend on F. We prove that if $P(z) \in \mathbb{C}[z]$ is a polynomial of positive degree, whose leading coefficient is a and such that the numbers 1, $\Re(a)$ and $\Im(a)$ are rationally independent, then $P \in \mathcal{T}$.

1 Introduction

Let W stand for the union of finitely many convex bounded domains in $\mathbb{C}$. Given $x > 0$, we denote by xW the set $\{xz : z \in W\}$, and observe that with the Lebesgue measure $|\cdot|$, we have $|xW| = x^2|W|$. Let $G = \mathbb{Z}[i]$ be the set of Gaussian integers and set $G^* := G \setminus \{0\}$. Finally, let $\mathcal{M}$ be the set of multiplicative functions defined on G^* and let $\mathcal{M}^*$ be the subset of $\mathcal{M}$ made of those $g \in \mathcal{M}$ satisfying $|g(\alpha)| \le 1$ for all $\alpha \in G^*$. Let χ be an arbitrary additive character, that is a function $\chi : G \to \{z : |z| = 1\}$ for which $\chi(0) = 1$ and $\chi(\alpha_1 + \alpha_2) = \chi(\alpha_1)\chi(\alpha_2)$ for all $\alpha_1, \alpha_2 \in G$. Using the standard notation $e(u) = e^{2\pi i u}$, we set $\chi(1) = e(A)$ and $\chi(i) = e(B)$, and then denote by $\mathcal{A}$ the set of those χ's for which at least one of A and B is irrational. We proved in [1] that, given $\chi \in \mathcal{A}$ and $g \in \mathcal{M}^*$,

$$\lim_{x\to\infty} \frac{1}{|xW|} \sum_{\beta \in xW} g(\beta)\chi(\beta) = 0,$$

where the convergence is uniform in g, thereby generalizing a previous result of Daboussi and Delange [2].

This paper is essentially a continuation of the results obtained in [1].

2 The main result

Given a complex number $z = u + iv$, where $u, v \in \mathbb{R}$, let $\{z\} = \{u\} + i\{v\}$, where $\{x\}$ stands for the fractional part of x. Let $E := \{w : 0 \leq \Re(w) < 1,\ 0 \leq \Im(w) < 1\}$. We say that the sequence of complex numbers $z_1, z_2, \ldots$ is uniformly distributed mod E if

$$\lim_{N\to\infty} \frac{1}{N} \#\{n \leq N : \Re(\{z_n\}) < u,\ \Im(\{z_n\}) < v\} = uv$$

for every pair of real numbers $u, v \in]0, 1]$.

A result of H. Weyl states that (see [3]) that the sequence z_n is uniformly distributed mod E if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} e(k\Re(z_n) + \ell\Im(z_n)) = 0$$

for each pair $(k, \ell) \in \mathbb{Z} \times \mathbb{Z} \setminus \{(0, 0)\}$.

For each real positive number x, let $N(x) := \#\{\alpha \in xW \cap G^*\}$ and further let $h : G^* \to \mathbb{C}$. For $u, v \in]0, 1]$, let

$$F_x(u, v) := \frac{1}{N(x)} \#\{z \in xW \cap G^* : \Re(\{h(z)\}) < u,\ \Im(\{h(z)\}) < v\}.$$

We say that h is uniformly distributed mod E in limit on xW for $x \to \infty$ if

$$\lim_{x\to\infty} F_x(u, v) = uv \qquad \text{holds for } 0 < u \leq 1,\ 0 < v \leq 1. \tag{2.1}$$

Let $\mathcal{T}$ be the set of those functions $t : G^* \to \mathbb{C}$ for which $t(\alpha) + F(\alpha)$ is uniformly distributed mod E in limit on xW (as $x \to \infty$) for every additive arithmetical function F, and such that the discrepancy does not depend on F.

Theorem 1. *Let $P(z) \in \mathbb{C}[z]$ a polynomial of positive degree k. Let a be the coefficient of z^k in $P(z)$. Assume that the numbers 1, $\Re(a)$ and $\Im(a)$ are rationally independent. Then $P \in \mathcal{T}$.*

3 Preliminary lemmas

Lemma 1. *Let $\wp = \{\rho_1, \rho_2, \ldots, \rho_r\}$ be a finite set of Gaussian primes, with $|\rho_1| \leq |\rho_2| \leq \ldots \leq |\rho_r|$ such that no two of them are associates. Let χ be an additive character. Set $T(x) := \sum_{\beta\in xW} g(\beta)\chi(P(\beta))$ and let*

$$T_1(x) := \sum_{\substack{\rho\gamma\in xW\\ \rho\in\wp}} g(\rho\gamma)\chi(P(\rho\gamma)), \qquad T_2(x) := \sum_{\substack{\rho\gamma\in xW\\ \rho\in\wp}} g(\rho)g(\gamma)\chi(P(\rho\gamma)).$$

Then,

$$|T_1(x) - T_2(x)| \leq \frac{cx^2}{|\rho_1|^2} A_\wp,$$

where $A_{\wp} = \sum_{j=1}^{r} \frac{1}{|\rho_j|^2}$.

Lemma 2. (Weyl) *Let* $f(x) = \alpha_k x^k + \ldots + \alpha_1 x + \alpha_0$ *be a polynomial with real coefficients* $\alpha_0, \alpha_1, \ldots, \alpha_k$ *and such that*

$$\left|\alpha_k - \frac{h}{q}\right| \leq \frac{1}{q^2}, \qquad (h, q) = 1.$$

Then,

$$\sum_{x=1}^{P} e(f(x)) \ll P^{1+\varepsilon} q^{\varepsilon} \left(\frac{1}{P} + \frac{1}{q} + \frac{q}{P^k}\right)^{2^{1-k}}.$$

Proof. This result is due to H. Weyl and is stated (and proved) as Lemma 3.6 in the book of Hua [3]. □

Lemma 3. (Erdős-Turan-Koksma) *Let* (x_n)*, where* $n = 1, 2, \ldots, N$*, be a sequence of points in* $\mathbb{R}^s$ *and let* G *be an arbitrary positive integer. Then, the discrepancy* $D_N(x_n)$ *is less than*

$$2s^2 3^{s+1} \left(\frac{1}{G} + \sum_{0 < \|h\| \leq G} \frac{1}{R(h)} \left|\frac{1}{N} \sum_{n=1}^{N} e(< h, x_n >)\right|\right),$$

where $< \cdot, \cdot >$ *denotes the usual inner product on* $\mathbb{R}^s$*,* $\|h\| = \max_{i=1,\ldots,s} |h_i|$ *for integral lattice points* $h = (h_1, \ldots, h_s)$*, and* $R(h) = \prod_{j=1}^{s} \max(|h_j|, 1)$.

Proof. For a proof of this result, see the book of Kuipers and Niederreiter [4]. □

4 The proof of the main result

The case $k = 1$ follows essentially from our Theorem 1 proved in [1]. Hence, we may assume that $k \geq 2$.

The first part of the proof follows exactly the same reasoning as that of the proof of Theorem 1 in [1].

Indeed, applying Lemma 1 with $a(\gamma) = g(\gamma)$ and $b(\gamma) = \sum_{\substack{\rho \in \wp \\ \rho \in \frac{xW}{\gamma}}} g(\rho)\chi(P(\rho\gamma))$, we get that

$$T_2(x) = \sum_{\substack{\gamma \in G^* \\ \gamma \in \cup_{\rho \in \wp} \frac{xW}{\rho}}} a(\gamma) b(\gamma),$$

with

$$|T_1(x) - T_2(x)| \leq \frac{cx^2}{|\rho_1|^2} A_{\wp}. \tag{4.1}$$

By the Cauchy-Schwarz Inequality, we obtain that

$$|T_2(x)| \le \left(\sum_{\gamma} |a(\gamma)|^2\right)^{1/2} \cdot \left(\sum_{\gamma} |b(\gamma)|^2\right)^{1/2} = \Sigma_1^{1/2} \cdot \Sigma_2^{1/2}, \tag{4.2}$$

say.

On the one hand, it is clear that

$$\Sigma_1 \ll x^2. \tag{4.3}$$

On the other hand, Σ_2 can be written as

$$\Sigma_2 = \sum_{\gamma} \sum_{\substack{\rho\in\wp \\ \rho\gamma\in xW}} 1 + \sum_{\nu\neq j} \sum_{\gamma\in S_{\nu,j}} g(\rho_\nu)\overline{g(\rho_j)}\chi(P(\rho_\nu\gamma))\overline{\chi(P(\rho_j\gamma))}, \tag{4.4}$$

where $S_{\nu,j} = \dfrac{xW}{\rho_\nu} \cap \dfrac{xW}{\rho_j}$. Now, since χ is an additive character, it follows that

$$\chi(P(\rho_\nu\gamma))\overline{\chi(P(\rho_j\gamma))} = \chi(P(\rho_\nu\gamma) - P(\rho_j\gamma)). \tag{4.5}$$

Assume that

$$\chi(z) := e(k\Re(z) + \ell\Im(z)), \qquad (k,\ell) \in \mathbb{Z}\times\mathbb{Z}\setminus\{(0,0)\}.$$

Then, set

$$B_{\nu,j} := \sum_{\gamma\in S_{\nu,j}} \chi(P(\rho_\nu\gamma) - P(\rho_j\gamma)). \tag{4.6}$$

In light of the estimates (4.1) through (4.6), it is clear that it is sufficient to prove that

$$\lim_{x\to\infty} \frac{1}{N(x)} |B_{\nu,j}| = 0. \tag{4.7}$$

To do so, we argue as in the proof of Theorem 1 of [1].

Let $\rho_\nu, \rho_j \in \wp$ be fixed, $\rho_\nu \neq \rho_j$. Further let $a = A + Bi$, $\rho_\nu^k - \rho_j^k = P + Qi$, $U = K(AP - QB) + L(AQ + BP)$. We must have that $U \neq 0$. Indeed, since

$$U = (KP + LQ)A + (LP - KQ)B,$$

and $KP + LQ = 0$, $LP - KQ = 0$ would imply that $\dfrac{P}{Q} = \dfrac{K}{L}$, $\dfrac{P}{Q} = -\dfrac{L}{K}$, that is either $K = L$ or $K = -L$.

If $K = L$, then $K \neq 0$, and $KP + LQ = 0$, $LP - KQ = 0$, which would apply that $P + Q = 0$, $P - Q = 0$, implying that $P = Q = 0$.

If $K=-L$, then $K\neq 0$, and so $P-Q=0$, $P+Q=0$ would follow, which is also impossible.

Since $A, B, 1$ are rationally independent, it follows that U is irrational and therefore that $k!U$ is an irrational number.

Let $0\leq\lambda<1$ be the unique (irrational) number such that $e(k!U)=e(\lambda)$ and let $q_1<q_2<\dots$ be a sequence of positive integers such that

$$q_\nu\|\lambda q_\nu\|<1 \quad \text{holds for } \nu=1,2,3,\dots$$

holds.

Let

$$Y(x)=\max_{q_\nu<\log x} q_\nu$$

and

$$B_{\nu,j}^{(x)}:=\frac{1}{Y(x)}\sum_{\gamma\in S_{\nu,j}}\sum_{\ell=0}^{Y(x)-1}\chi\left(P(\rho_\nu(\gamma+\ell))-P(\rho_j(\gamma+\ell))\right).$$

First, letting $N(x)=\#\{\gamma\in S_{\nu,j}\}$, we observe that $|B_{\nu,j}-B_{\nu,j}^{(x)}|=o(N(x))$ as $x\to\infty$. Thus in order to prove (4.7), we only need to prove that

$$\lim_{x\to\infty}\frac{1}{N(x)}\left|B_{\nu,j}^{(x)}\right|=0. \tag{4.8}$$

Now let $N^{(0)}$ be the number of those γ for which $\gamma+\ell\in S_{\nu,j}$, $(\ell=0,1,\dots,Y(x)-1)$. If $\gamma\in S_{\nu,j}$ and $\gamma+\ell\notin S_{\nu,j}$ for at least one $\ell\in\{0,1,\dots,Y(x)-1\}$, then either $\gamma\rho_\nu$ or $\gamma\rho_j$ is close to the boundary of xW. Since W is a finite union of convex domains, the length of the boundary of xW is $O(x)$, which implies that

$$0\leq N(x)-N^{(0)}(x)\ll xY(x)=o(N(x)) \qquad (x\to\infty).$$

We shall now prove that

$$\max_{\gamma\in S_{\nu,j}}\frac{1}{Y(x)}\left|\sum_{\ell=0}^{Y(x)-1}\chi(Q(\ell))\right|\to 0 \qquad (x\to\infty), \tag{4.9}$$

where

$$Q(\ell)=Q_j(\ell)=P(\rho_\nu(\gamma+\ell))-P(\rho_j(\gamma+\ell)).$$

To prove (4.9), we shall use Lemma 2. But in order to do so, we first observe that $Q(\ell)=a(\rho_\nu^\ell-\rho_j^\ell)\ell^k+\dots$

Let $R(\ell)=K\Re Q(\ell)+L\Im Q(\ell)$. Then $R(\ell)$ is a polynomial of degree k, of which the coefficient of the main term is $K\Re a(\rho_\nu^k-\rho_j^k)+L\Im a(\rho_\nu^k-\rho_j^k)$.

Thus,

$$T:=\sum_{\ell=0}^{Y(x)-1}\chi(Q(\ell))=\sum_{\ell=0}^{Y(x)-1}e(R(\ell)).$$

Applying Lemma 3 with $P = Y(x)$, $f = R$, $\alpha_k = \lambda$, we may conclude that

$$|T| \ll Y(x)^{1+2\varepsilon} \left(\frac{1}{Y(x)}\right)^{2^{1-k}},$$

where $\varepsilon > 0$ is an arbitrary small constant. Thus, $T/Y(x) \to 0$ as $x \to \infty$ uniformly in γ, which completes the proof of (4.9) and therefore of (4.8). The estimates being uniform in t, Theorem 1 follows from Lemma 3.

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Jean-Marie De Koninck
Dép. de mathématiques et de statistique
Université Laval
Québec
Québec G1V 0A6
Canada
jmdk@mat.ulaval.ca

Imre Kátai
Computer Algebra Department
Eötvös Loránd University
1117 Budapest
Pázmány Péter Sétány I/C
Hungary
katai@compalg.inf.elte.hu

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