THE PRODUCT OF EXPONENTS IN THE FACTORIZATION OF CONSECUTIVE INTEGERS

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Abstract. For each integer $n \geq 2$, let $\beta(n)$ stand for the product of the exponents in the prime factorization of n. Given an arbitrary integer $k \geq 2$, let n_k be the smallest positive integer n such that $\beta(n+1) = \beta(n+2) = \cdots = \beta(n+k)$. We prove that there exist positive constants c_1 and c_2 such that, for all integers $k \geq 2$,

$$\exp\left\{c_1 \frac{\log^3 k}{\log\log k}\right\} < n_k < \exp\{c_2\, k\ \log^3 k\ \log\log k\}.$$

§1. *Introduction.* It has been proved by Heath-Brown [**1**] that there exist infinitely many positive integers n such that $\tau(n) = \tau(n+1)$, where $\tau(n)$ stands for the number of positive divisors of n. No one has yet proved whether

$$\tau(n) = \tau(n+1) = \tau(n+2)$$

for infinitely many positive integers n.

In this paper, we prove such a result for a similar but slightly smaller arithmetic function. Writing an integer $n \geq 2$ as the product of its prime factors in the usual form $n = q_1^{\alpha_1} q_2^{\alpha_2} \cdots q_r^{\alpha_r}$, where the q_i are the distinct prime factors of n and the α_i positive integers, let $\beta(n) = \alpha_1 \alpha_2 \cdots \alpha_r$. Note that $\beta(n)$ stands for the number of divisors of $n/\gamma(n)$, where $\gamma(n) := \prod_{p|n} p$. While the order of $\tau(n)$ is $\log n$, the function $\beta(n)$ has an asymptotic mean value. Indeed, it can easily be shown that

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\leq x} \beta(n) = \frac{\zeta(2)\zeta(3)}{\zeta(6)} \approx 1.9436,$$

where ζ stands for the Riemann zeta function, a result which essentially follows from the fact that

$$\sum_{n=1}^{\infty} \frac{\beta(n)}{n^s} = \frac{\zeta(s)\zeta(2s)\zeta(3s)}{\zeta(6s)} \quad (s > 1).$$

Given any arbitrary integer $k \geq 2$, let n_k be the smallest positive integer n such that

$$\beta(n+1) = \beta(n+2) = \cdots = \beta(n+k). \tag{1}$$

A computer search indicates that the first values of n_k are as follows.

k	2	3	4	5	6	7
n_k	2	4	843	74 848	671 345	8 870 024

We also believe that $n_8 = 1770\,019\,255\,373\,287\,038\,727\,484\,868\,192\,109\,228\,823$.

Here, not only do we prove that n_k exists for each integer $k \geq 2$, but we also find upper and lower bounds for the value of n_k. More precisely, we prove the following result.

THEOREM. *There exist positive constants c_1 and c_2 such that, for all $k \geq 2$,*

$$\exp\left\{c_1 \frac{\log^3 k}{\log\log k}\right\} < n_k < \exp\{c_2\, k\, \log^3 k\, \log\log k\}.$$

§2. *Preliminary lemmas.* Throughout this paper, we assume that $k \geq 2$ is a large integer. Given an integer $n \geq 2$, we let $P(n)$ and $p(n)$ stand for its largest and smallest prime factor, respectively, and write $\omega(n)$ for the number of distinct prime factors of n and $\Omega(n)$ for the total number of prime factors of n counting their multiplicity, with $\omega(1) = \Omega(1) = 0$. We write $[m_1, m_2, \ldots, m_k]$ for the least common multiple of the integers $m_1, m_2, \ldots, m_k$.

Given a prime p and a non-zero integer ℓ, we let $\nu_p(\ell)$ stand for the unique non-negative integer α such that $p^\alpha \| \ell$.

We use the Vinogradov symbols $\gg$ and $\ll$ as well as the Landau symbols O and o with their regular meanings and $c_1, c_2, \ldots$ for computable positive constants which are labelled increasingly throughout the paper.

LEMMA 1. *As $k \to \infty$,*

$$[\beta(1), \ldots, \beta(k)] \leq \exp\left\{(1 + o(1)) \frac{\log k\, \log\log k}{\log 2}\right\}.$$

Proof. Let $t(n)$ be the largest possible multiplicity of any prime factor of n. For each $j = 1, 2, \ldots, t(n)$, let $\omega_j(n)$ be the number of distinct prime factors p of n such that $p^j \| n$. Clearly,

$$\omega(n) = \sum_{j=1}^{t(n)} \omega_j(n), \qquad \Omega(n) = \sum_{j=1}^{t(n)} j\omega_j(n) \quad \text{and} \quad \beta(n) = \prod_{j=1}^{t(n)} j^{\omega_j(n)}.$$

Furthermore, if p is a prime and $\alpha_p(n) := \nu_p(\beta(n))$, then

$$\alpha_p(n) = \sum_{i \geq 1} \sum_{\substack{j \geq 1 \\ p^i \| j}} i\omega_j(n).$$

Since $p^i \geq ip$ for all $i \geq 1$, we obtain

$$\alpha_p(n) \leq \frac{1}{p} \sum_{i \geq 1} \sum_{\substack{j \geq 1 \\ p^i \| j}} p^i \omega_j(n) \leq \frac{1}{p} \sum_{j \geq 1} j\omega_j(n) = \frac{\Omega(n)}{p} \leq \frac{\log n}{p \log 2}.$$

The above argument shows that

$$\max_{1 \leq j \leq k} \nu_p(\beta(j)) \leq \frac{\log k}{p \log 2}.$$

Hence, using the prime number theorem,

$$[\beta(1), \ldots, \beta(k)] \,\Big|\, \prod_{p \le \log k/\log 2} p^{\lfloor \log k/(p \log 2)\rfloor},$$

and, therefore,

$$\begin{aligned}[\beta(1), \ldots, \beta(k)] &\le \prod_{p \le \log k/\log 2} p^{\log k/(p \log 2)} \\ &= \exp\left\{\frac{\log k}{\log 2} \sum_{p \le \log k/\log 2} \frac{\log p}{p}\right\} \\ &= \exp\left\{\frac{1}{\log 2}(1 + o(1))(\log k)(\log\log k)\right\}\end{aligned}$$

as $k \to \infty$, thus completing the proof of Lemma 1. □

LEMMA 2. *If $q_1, \ldots, q_t$ are distinct prime factors of $\beta(n)$, then*

$$n \ge p(n)^{q_1+\cdots+q_t}.$$

Proof. There exists a partition $I_1 \cup I_2 \cup \cdots \cup I_s$ of $\{1, \ldots, t\}$ and s distinct prime numbers $p_1, \ldots, p_s$ such that $p_i^{\prod_{j\in I_i} q_j}$ is a divisor of n. Hence,

$$n \ge \prod_{i=1}^{s} p_i^{\prod_{j\in I_i} q_j} \ge p(n)^{\sum_{i=1}^{s} \prod_{j\in I_i} q_j},$$

and the conclusion of the lemma now follows from the inequality

$$\prod_{i=1}^{\ell} a_i \ge \sum_{i=1}^{\ell} a_i$$

which is valid for all $\ell \ge 1$ and $a_i \ge 2$ for $i = 1, \ldots, \ell$, applied to each one of the products $\prod_{j\in I_i} q_j$. □

§3. *Proof of the upper bound.* Set

$$\mu_k := [\beta(1), \beta(2), \ldots, \beta(k)],$$

and let $a_1, \ldots, a_k$ be positive integers such that:

(i) for $i = 1, 2, \ldots, k$, all prime factors of a_i belong to $[k^2, 2k^2]$;
(ii) $(a_i, a_j) = 1$ for $i \ne j$;
(iii) $\beta(a_i) = \mu_k/\beta(i)$ for all $i = 1, 2, \ldots, k$.

We now justify that it is possible to choose such integers a_i if k is large. Indeed, let $i \le k$ and write

$$\frac{\mu_k}{\beta(i)} = q_{j_1} q_{j_2} \ldots q_{j_{\ell_i}},$$

where the q_{j_s} are primes which are not necessarily distinct. Note that

$$\ell_i \le \Omega(\mu_k) \le \frac{\log \mu_k}{\log 2} \ll \log k \ \log\log k, \tag{2}$$

by Lemma 1. Since the interval $[k^2, 2k^2]$ contains $\gg k^2/\log k$ primes, it follows that it is possible to choose $a_i = \prod_{s=1}^{\ell_i} p_{i,s}^{q_{j_s}}$, with the set $P_i := \{p_{i,1}, p_{i,2}, \dots, p_{i,\ell_i}\}$ consisting of ℓ_i distinct primes p all contained in the interval $[k^2, 2k^2]$ and such that $P_i \cap P_j = \emptyset$ for all $i \ne j$. (This is because, by (2), we only need $O(k \log k \ \log\log k)$ elements in $\bigcup_{i=1}^{k} P_i$, and we have $\gg k^2/\log k$ admissible elements in $[k^2, 2k^2]$ to choose from.)

Furthermore, note that

$$a_i \le ((2k^2)^{\log k/\log 2})^{\ell_i} \le \exp\{c_3(\log k)^3(\log\log k)\}, \tag{3}$$

for some appropriate positive constant c_3. Let

$$N_k = [1, 2, \dots, 2k]$$

and choose a positive integer n_0 such that $n_0 \equiv 0 \pmod{N_k^2}$ and $n_0 + i \equiv a_i \pmod{a_i^2}$ for $i = 1, 2, \dots, k$, whose existence is guaranteed by the Chinese remainder theorem, after observing that $(N_k, a_i) = 1$ for each $i \in \{1, 2, \dots, k\}$. Put

$$n = n_0 + mM \quad \text{where } M = N_k^2 \prod_{i=1}^{k} a_i^2,\ m \ge 0,$$

where n_0 is the smallest positive integer satisfying the above system of congruences.

Note that

$$n + i = i \cdot a_i \cdot (c_{i,0} + mM_i),$$

where $c_{i,0} = (n_0 + i)/(ia_i)$ and $M_i = M/(ia_i)$. Note also that $c_{i,0} + mM_i$ is coprime to ia_i for each $i = 1, 2, \dots, k$. In fact, this number is congruent to one modulo every prime factor of ia_i. Therefore, if m is such that $c_{i,0} + mM_i$ is square-free for all $i = 1, 2, \dots, k$, then $\beta(n+i) = \beta(ia_i) = \beta(i)\beta(a_i) = \mu_k$ for all $i = 1, 2, \dots, k$, which shows that, for such an m, the corresponding n satisfies $n \ge n_k$. To complete the proof of the upper bound, we show that there exists $m \le M$ such that the above k numbers $c_{i,0} + mM_i$ $(i = 1, 2, \dots, k)$ are all square-free.

Assume that, for some i, we have $c_{i,0} + mM_i \equiv 0 \pmod{p^2}$ for some prime $p < 2M$. Note that $p > 2k$. For every such prime p, the number of positive integers $m \le M$ satisfying the above congruence is at most $M/p^2 + 1$. Thus, the total number of such possibilities over all such primes $p < 2M$ and indices $i \in \{1, 2, \dots, k\}$ is at most

$$\sum_{i=1}^{k} \sum_{2k<p<2M} \left(\frac{M}{p^2} + 1\right) = kM \sum_{p>2k} \frac{1}{p^2} + k\pi(2M)$$
$$\le k\pi(2M) + O\left(\frac{M}{\log k}\right). \tag{4}$$

Observing that $N_k = \exp\{\psi(2k)\}$, where $\psi(x) := \sum_{p^\alpha \le x} \log p$, by applying the prime number theorem, we obtain

$$M \ge N_k^2 = \exp\{2\psi(2k)\} = \exp\{4(1+o(1))k\},$$

from which it follows that $\log M \ge 4(1+o(1))k$ as $k \to \infty$. Hence, using this inequality and the prime number theorem again, we deduce that

$$\pi(2M) = (1+o(1))\frac{2M}{\log M} \le \left(\frac{1}{2} + o(1)\right)\frac{M}{k} \quad \text{as } k \to \infty.$$

Thus, $\pi(2M) \le 2M/3k$ if k is sufficiently large, which shows that the right-hand side of inequality (4) is at most $3M/4$ if k is sufficiently large.

Therefore, if k is large, there exist at least $M/4$ positive integers $m \le M$ (in particular, at least one of them), such that $c_{i,0} + mM_i$ is free of squares of primes $p < 2M$ for all $i = 1, 2, \dots, k$. However, note that such integers are necessarily square-free as, if not, there must exist a prime $p \ge 2M$ such that $p^2 \mid c_{i,0} + mM_i$ for some $i \in \{1, 2, \dots, k\}$, leading to

$$(2M)^2 \le p^2 \le c_{i,0} + mM_i \le M + M^2,$$

which is impossible for any $M > 1$.

This shows that, in light of (3),

$$n_k \le n_0 + M^2 \le 2M^2 = 2N_k^4 \prod_{i=1}^{k} a_i^4 \le \exp\{4\, c_3\, k\, \log^3 k\, \log\log k + O(k)\},$$

which completes the proof of the upper bound.

§4. *Proof of the lower bound.* By the results from §3, we know that n_k exists. We now let

$$t_k := \left\lfloor \frac{\log(k/2)}{\log 2} \right\rfloor - 1 \quad \text{and} \quad r_k := \prod_{p \le t_k} p.$$

We start with the following lemma.

LEMMA 3. *For each integer $k \ge 2$, ρ_k is a multiple of r_k.*

Proof. First we prove that, for any positive integer α such that $2^\alpha \le k/2$,

$$\alpha - 1 \text{ divides } \rho_k. \tag{5}$$

Since $2^\alpha \le k/2$, there exists $i \in \{0, 1, 2, \dots, \lfloor k/2 \rfloor\}$ such that $2^\alpha \mid n + i$, so that $2^{\alpha-1} \| n + (i + 2^{\alpha-1})$. Now

$$j := i + 2^{\alpha-1} \le \left\lfloor \frac{k}{2} \right\rfloor + 2^{\alpha-1} \le \frac{k}{2} + \frac{k}{4} < k,$$

implying that $j \le k-1$, and since $2^{\alpha-1} \| n + j$ and $j \le k-1$, we deduce that $\alpha - 1 \mid \beta(n_k + j)$, thus establishing (5).

It follows that, if $p \le \lfloor \log(k/2)/\log 2 \rfloor - 1$, then $p \mid \rho_k$. Indeed, since $2^{p+1} \le k/2$, we see that, by (5), $(p+1-1) | \rho_k$, from which Lemma 3 follows. □

Observe that, by Lemma 3 and the prime number theorem, we have

$$\rho_k \geq r_k = \exp\{(1+o(1))t_k\} = \exp\{c_4(1+o(1))\log k\} \quad (k \to \infty), \tag{6}$$

where $c_4 = 1/\log 2$.

We are now ready to prove the lower bound. Let $n = n_k$. For each integer $i \in \{1, 2, \ldots, k\}$, write $n + i = a_i b_i$, where $P(a_i) \leq k$ and $p(b_i) \geq k+1$. We first use a classical argument of Erdős to show that one of the a_i is "small". For each prime $p \leq k$, select some $j = j(p) \in \{1, 2, \ldots, k\}$ such that $\nu_p(a_j) = \max\{\nu_p(a_\ell) : 1 \leq \ell \leq k\}$, and consider the sets

$$S := \{j(p) \mid p \leq k\} \quad \text{and} \quad T := \{1, 2, \ldots, k\} \backslash S.$$

Note that $\#T = k - \#S = k - \pi(k) \geq k/2$ for $k \geq 8$.

Step 1. If k is large, then

$$\prod_{i \in T} a_i \leq k^{3k}. \tag{7}$$

Proof. This follows by observing that

$$\begin{aligned} \prod_{i \in T} a_i &\leq \prod_{p \leq k} p^{\sum_{1 \leq \alpha \leq \log k/\log p} \lfloor \frac{k}{p^\alpha} \rfloor + 1} \\ &\leq k! \prod_{p \leq k} p^{\log k/\log p} \leq k! \mathrm{e}^{\pi(k) \log k} < k^{3k} \end{aligned}$$

if k is large. □

Let i_0 be some element of T such that $a_{i_0} = \min\{a_i \mid i \in T\}$. Note that

$$\prod_{i \in T} a_i \geq a_{i_0}^{\#T} \geq a_{i_0}^{k/2}. \tag{8}$$

From (7) and (8), we obtain

$$a_{i_0} \leq k^6. \tag{9}$$

Step 2. There exists a positive constant c_5 such that

$$\omega(\beta(a_{i_0})) < c_5 \sqrt{\frac{\log k}{\log \log k}} \tag{10}$$

for all $k \geq 2$.

Proof. Indeed, write $\beta(a_{i_0}) = q_1^{\lambda_1} \ldots q_s^{\lambda_s}$, where the q_i are distinct primes and the λ_j are positive integers. From Lemma 2,

$$a_{i_0} \geq 2^{\sum_{i=1}^s q_i}.$$

Using (9), we obtain, by taking logarithms, that

$$\frac{6 \log k}{\log 2} \geq \sum_{i=1}^s q_i \gg s^2 \log s,$$

where we have used the prime number theorem. This last inequality implies that

$$s \ll \sqrt{\frac{\log k}{\log\log k}},$$

which proves Step 2. □

We are now ready to complete the proof of the lower bound.

Note that $\beta(b_{i_0}) = \rho_k/\beta(a_{i_0})$. Therefore, using (10) and Lemma 3,

$$\begin{aligned}\ell := \omega(\beta(b_{i_0})) &= \omega(\rho_k) - \omega(\beta(a_{i_0}))\\ &\geq \pi(t_k) - c_5\sqrt{\frac{\log k}{\log\log k}} \geq c_6\frac{\log k}{\log\log k},\end{aligned} \tag{11}$$

where we can take $c_6 = 1/(2\log 2)$ provided that k is large enough.

Let $q_1, \ldots, q_\ell$ be all of the distinct prime factors of $\beta(b_{i_0})$. By Lemma 2,

$$\begin{aligned}n + k \geq b_{i_0} \geq (k+1)^{\sum_{i=1}^{\ell} q_i} &\geq \exp\{c_7\ell^2 \ \log\ell \ \log(k+1)\}\\ &\geq \exp\left\{c_8\frac{\log^3 k}{\log\log k}\right\},\end{aligned}$$

where one can take $c_7 = 1/3$ and $c_8 = c_7c_6^2$, provided that k is sufficiently large, thus completing the proof of the lower bound and of the theorem.

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