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# ON THE INDEX OF COMPOSITION OF THE EULER FUNCTION AND OF THE SUM OF DIVISORS FUNCTION

JEAN-MARIE DE KONINCK and FLORIAN LUCA ✉

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**Abstract**

Given an integer $n \geq 2$, let $\lambda(n) := (\log n)/(\log \gamma(n))$, where $\gamma(n) = \prod_{p|n} p$, denote the index of composition of $n$, with $\lambda(1) = 1$. Letting $\phi$ and $\sigma$ stand for the Euler function and the sum of divisors function, we show that both $\lambda(\phi(n))$ and $\lambda(\sigma(n))$ have normal order 1 and mean value 1. Given an arbitrary integer $k \geq 2$, we then study the size of $\min\{\lambda(\phi(n)), \lambda(\phi(n+1)), \ldots, \lambda(\phi(n+k-1))\}$ and of $\min\{\lambda(\sigma(n)), \lambda(\sigma(n+1)), \ldots, \lambda(\sigma(n+k-1))\}$ as $n$ becomes large.

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## 1. Introduction

Given an integer $n \geq 2$, we define its *index of composition* by

$$\lambda(n) := (\log n)/(\log \gamma(n)),$$

where $\gamma(n)$ (often called the *kernel* of $n$) stands for the product of the distinct primes dividing $n$. For convenience, we let $\lambda(1) = \gamma(1) = 1$. In a sense, $\lambda(n)$ measures the level of compositeness of $n$. First introduced by Browkin [2] in 2000, the function $\lambda$ was further studied by De Koninck and Doyon [3] who examined its global and local behavior, namely by showing that its mean value is 1 and moreover by establishing that given any integer $k \geq 2$ and setting

$$Q_k(n) := \min\{\lambda(n), \lambda(n+1), \ldots, \lambda(n+k-1)\}, \tag{1}$$

and given any $\varepsilon > 0$, then

$$Q_k(n) > \frac{k}{k-1} - \varepsilon \tag{2}$$

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for infinitely many values of $n$, which is most likely optimal. Indeed, De Koninck and Doyon [3, p. 164] conjecture that $\limsup_{n\to\infty} Q_k(n) = k/(k-1)$ and show that the *abc* conjecture implies the validity of the above conjecture when $k = 3$. In this paper, we show that the above conjecture from [3] holds under the *abc* conjecture for all $k \geq 2$.

More recently, De Koninck and Kátai [4] as well as De Koninck *et al.* [5] have studied the distribution function of $(\lambda(n) - 1)\log n$ as $n$ runs through particular sets of integers, such as the shifted primes. The mean value of the function $\lambda(n)$ was also studied by Zhai [12].

In this paper, we also examine the global and local behavior of $\lambda(\phi(n))$ and $\lambda(\sigma(n))$, where $\phi$ and $\sigma$ stand for the Euler function and the sum of divisors function, respectively. More precisely, we first establish that each of $\lambda(\phi(n))$ and $\lambda(\sigma(n))$ have normal orders 1 and mean values 1. Then, given an integer $k \geq 2$, we discuss the behavior of the expressions

$$F_k(n) := \min\{\lambda(\phi(n)), \lambda(\phi(n+1)), \ldots, \lambda(\phi(n+k-1))\} \tag{3}$$

and

$$S_k(n) := \min\{\lambda(\sigma(n)), \lambda(\sigma(n+1)), \ldots, \lambda(\sigma(n+k-1))\} \tag{4}$$

and conjecture that, for any fixed $k$, $F_k(n)$ and $S_k(n)$ can become arbitrarily large, providing heuristic arguments in their favor.

In what follows, the letter $p$ always stands for a prime number. Moreover, given any integer $n \geq 2$, let $P(n)$ stand for the largest prime factor of $n$. We shall also write $\omega(n)$ for the number of distinct prime factors of $n$ and $\Omega(n)$ for the total number of prime factors of $n$ counting their multiplicity, with $\omega(1) = \Omega(1) = 0$. Finally, a positive integer $n$ is said to be *powerful* (or *square-full*) if $p^2 \mid n$ whenever the prime number $p$ divides $n$.

We write $\log_2 x$ for $\log\log x$ and we let $\log_k x = \log\log_{k-1} x$ for each integer $k \geq 3$. The input $x$ will always be assumed to be large enough so that the resulting iterated logarithms are greater than 1.

We use the Landau symbols $O$ and $o$ as well as the Vinogradov symbols $\ll$ and $\gg$ with their usual meanings.

## 2. Preliminary results

Henceforth, given any integer $n \geq 2$, we shall write

$$\phi(n) = A(n)B(n), \qquad \text{with } \gcd(A(n), B(n)) = 1, \tag{5}$$

where $A(n)$ is the square-full part of $\phi(n)$ and $B(n)$ its square-free part. To establish our results, we shall need the following lemmas.

LEMMA 1. *As* $x \to \infty$,

$$\#\{n \leq x \mid \Omega(n) > 10\log_2 x\} = O\left(\frac{x}{\log^2 x}\right).$$

PROOF. From [11, Lemma 13], uniformly for every positive integer $K$,

$$\sum_{n\le x:\Omega(n)\ge K} 1 \ll \frac{K}{2^K} x \log x.$$

Applying this with $K = \lfloor 10 \log_2 x \rfloor$ leads to the desired estimate. □

LEMMA 2. *The inequality $A(n) \le (\log x)^4$ holds for all positive integers $n \le x$ with $O(x/(\log x)^2)$ exceptions.*

PROOF. It is well known that the number of square-full numbers $n \le x$ is $O(\sqrt{x})$ (see, for example, [9, Theorem 14.4]). Given any $y \in [1, \sqrt{x}]$ and any square-full number $d \ge y$, it is clear that the number of positive integers $n \le x$ that are multiples of $d$ is at most $x/d$, and therefore by Abel's summation formula, we easily get that the number of $n \le x$ having a square-full divisor $d \ge y$ is $O(x/\sqrt{y})$. Taking $y = (\log x)^4$, we get the desired result. □

LEMMA 3. *For large $x$, the number of positive integers $n \le x$ such that*

$$\max\{\Omega(\phi(n)), \Omega(\sigma(n))\} > 110(\log_2 x)^2$$

*is $O(x/(\log x)^2)$.*

PROOF. By Lemma 2, we may assume that $A(n) < (\log x)^4$. Thus,

$$\phi(A(n)) \le A(n) \le \sigma(A(n)) < (\log x)^5,$$

and therefore

$$\max\{\Omega(\phi(A(n))), \Omega(\sigma(A(n)))\} < (5/\log 2) \log\log x < 10 \log_2 x.$$

By Lemma 1, we may further assume that $\Omega(B(n)) < 10 \log_2 x$. Thus, if

$$\max\{\Omega(\phi(n)), \Omega(\sigma(n))\} > 110(\log_2 x)^2,$$

it then follows that there exists a prime divisor $p$ of $n$ such that $\Omega(p \pm 1) > 10 \log_2 x$. Let $n = pm$. Then $p < x/m$, so that $p \pm 1 \le x/m + 1 \le 2x/m$. The number of such numbers $p$ is, by the argument from the proof of Lemma 1, at most a multiple of

$$\frac{K}{2^K}\frac{x \log x}{m},$$

where $K = \lfloor 10 \log_2 x \rfloor$. Summing up over all values of $m \le x$, the number of such numbers $n \le x$ is at most

$$\frac{K \log x}{2^K} \sum_{m\le x} \frac{1}{m} \ll \frac{x(\log x)^2 \log_2 x}{2^{\lfloor 10 \log_2 x \rfloor}} \ll \frac{x}{(\log x)^2},$$

because $10 \log 2 > 4$. □

LEMMA 4. *The estimate*

$$\#\{n \le x : p^2 \mid \sigma(n) \text{ for some } p > (\log x)^5\} = O\left(\frac{x}{(\log x)^2}\right)$$

*holds as $x \to \infty$. A similar estimate holds when $\sigma(n)$ is replaced by $\phi(n)$.*

PROOF. By Lemma 2, we may assume that $A(n) < (\log x)^4$. Hence,

$$\phi(A(n)) \le A(n) \le \sigma(A(n)) < (\log x)^5$$

for large $x$. If $p^2 \mid \sigma(n)$ or $p^2 \mid \phi(n)$ for some $p > (\log x)^5$, it follows that $p^2 \mid \sigma(B(n))$ or $p^2 \mid \phi(B(n))$, respectively. Now [1, Lemma 2] shows that

$$\#\{n \le x \mid \phi(B(n)) \equiv 0 \bmod p^2\} \ll \frac{x(\log_2 x)^2}{p^2},$$

and a straightforward adaptation of it shows that the same is true when $\phi$ is replaced by $\sigma$. Thus, the number of positive integers $n \le x$ such that either $p^2 \mid \sigma(n)$ or $p^2 \mid \phi(n)$ for some $p > (\log x)^5$ is, by the above inequality, at most a multiple of

$$x(\log_2 x)^2 \sum_{p > (\log x)^5} \frac{1}{p^2} < x(\log_2 x)^2 \int_{(\log x)^5}^{x^{1/2}} \frac{dt}{t^2} \ll \frac{x(\log_2 x)^2}{(\log x)^5} \ll \frac{x}{(\log x)^2}. \qquad \square$$

## 3. The normal order of $\lambda(\phi(n))$

Here, we prove the following result.

THEOREM 5. *For every $\varepsilon > 0$, the inequality $1 \le \lambda(\phi(n)) \le 1 + \varepsilon$ holds for all $n$ except for a set of asymptotic density zero. The same inequality holds when $\phi$ is replaced by $\sigma$.*

PROOF. We shall prove this result only for $\sigma$ since the proof for $\phi$ is entirely similar. Since $n \le \sigma(n) \ll n \log_2 n$ holds for all $n$, we have that

$$\log(\sigma(n)) = \log n + O(\log_3 n). \tag{6}$$

By Lemmas 2–4, for most $n$ we have that if $Q(n)$ is the largest prime $p$ such that $p^2 \mid \sigma(n)$ (equivalently, $Q(n) = P(\sigma(n)/\gamma(\sigma(n)))$), then $Q(n) < (\log n)^5$. Furthermore, $\Omega(\sigma(n)) < 110(\log_2 n)^2$. This shows that

$$\log(\gamma(\sigma(n))) \ge \log(\sigma(n)) - \Omega(\sigma(n)) \log(Q(n)) = \log n + O((\log_2 n)^3). \tag{7}$$

From estimates (6) and (7), we immediately get that for most $n$,

$$\lambda(\sigma(n)) = 1 + O\left(\frac{(\log_2 n)^3}{\log n}\right) = 1 + o(1), \quad \text{as } n \to \infty,$$

which is what we wanted to prove. $\square$

## 4. The mean value of $\lambda(\phi(n))$

In this section we prove the following result.

THEOREM 6. *The estimate*

$$\frac{1}{x}\sum_{n\le x}\lambda(\phi(n)) = 1 + o(1)$$

*holds as $x \to \infty$. The same holds when $\phi$ is replaced by $\sigma$.*

PROOF. Again, we shall give the proof only for $\sigma$ since for $\phi$ it is entirely similar. The arguments from Section 3 show that the estimates

$$\log(\sigma(n)) = \log x + O(\log_3 x) \quad \text{and} \quad \log(\gamma(\sigma(n))) = \log x + O((\log_2 x)^3)$$

both hold for all positive integers $n \le x$ with at most $O(x/(\log x)^2)$ exceptions. On the exceptional set, it is clear that $\lambda(\sigma(n)) \le \log x$. Hence,

$$\sum_{n\le x}\lambda(\sigma(n)) = \sum_{\substack{n\le x: Q(n)<(\log x)^5\\ \Omega(\sigma(n))<110(\log_2 x)^2}} \left(1 + O\left(\frac{(\log_2 x)^3}{\log x}\right)\right) + O\left(\frac{x}{(\log\ x)^2}\log x\right)$$

$$= x + O\left(\frac{x(\log_2 x)^3}{\log x}\right),$$

which is the desired estimate. □

## 5. The local behavior of $\lambda(\phi(n))$

We prove the analogue of [3, Theorem 3] for the case of the quantity $F_k(n)$ given by (3).

THEOREM 7. *Given any integer $k \ge 2$, for every $\varepsilon > 0$, there exist infinitely many $n$ such that*

$$F_k(n) > \frac{k}{k-1} - \varepsilon.$$

PROOF. We follow the method of [3, Proof of Theorem 3]. Let $y > k$ be sufficiently large so that the interval $[y, y + y^{2/3}]$ contains at least $k$ prime numbers. Let these be $y < p_1 < \cdots < p_k < y + y^{2/3}$. Observe that

$$\frac{p_k}{p_1} = 1 + O\left(\frac{1}{y^{1/3}}\right) = 1 + o(1) \quad (y \to \infty). \tag{8}$$

Let $a > 3$ be a large positive integer and let $n$ be such that $n \equiv -i \bmod p_i^a$ for all $i = 1, 2, \ldots, k$. This system is solvable by the Chinese remainder theorem and it therefore has a solution $n \in [M, 2M)$, where $M = \prod_{i=1}^k p_i^a$. Since

$$2M + O(1) \ge n + i > \phi(n+i) \gg \frac{n+i}{\log_2(n+i)} \ge \frac{M}{\log_2(2M+k)},$$

we get that

$$\log(\phi(n+i)) = n+i+O(\log_3 M) = \log M + O(\log_3 M), \quad i=1,2,\ldots,k, \tag{9}$$

whenever the $p_i$ are fixed and $a$ tends to infinity. However, note that since $n+i = p_i^a m_i$ for some positive integer $m_i$,

$$\phi(n+i) = p_i^{a-1}(p_i-1)n_i,$$

for some positive integer $n_i$ (here, $n_i = \phi(m_i)$ if $p_i \nmid m_i$ and $n_i = \phi(m_i)p_i/(p_i-1)$ if $p_i \mid m_i$, so that in any case $n_i \le m_i$ always holds). Therefore, in light of (9), for each $i = 1, 2, \ldots, k$,

$$\begin{aligned}
\log(\gamma(\phi(n+i))) &\le \log(p_i\gamma(p_i-1)\gamma(n_i)) \le \log(p_i^2 m_i) \\
&= \log\left(\frac{p_i^a m_i}{p_i^{a-2}}\right) = \log\left(\frac{n+i}{p_i^{a-2}}\right) \\
&= \log(\phi(n+i)) + O(\log_3 M) - (a-2)\log p_i \\
&= \log M + O(\log_3 M) - (a-2)\log p_i.
\end{aligned} \tag{10}$$

On the other hand, using (8), it is clear that

$$\log M = a\sum_{j=1}^{k}\log p_j = ka(1+o(1))\log p_i, \quad i=1,2,\ldots,k. \tag{11}$$

Combining (10) and (11), we obtain that

$$\begin{aligned}
\log(\gamma(\phi(n+i))) &\le \log M - \frac{\log M}{k}(1+o(1)) + O(\log_3 M) \\
&= \left(1-\frac{1}{k}+o(1)\right)\log M,
\end{aligned}$$

which together with estimate (9) shows that, for each $i = 1, 2, \ldots, k$,

$$\lambda(\phi(n+i)) = \frac{\log(\phi(n+i))}{\log(\gamma(\phi(n+i)))} \ge \frac{1}{1-(1/k)+o(1)} = \frac{k}{k-1}+o(1),$$

which implies the desired inequality. □

## 6. The local behavior of $\lambda(\sigma(n))$

Here, the method of proof of Theorem 7 does not work because if $p$ is a fixed prime and $a$ is a positive integer, then $\gamma(\sigma(p^a))$ is not small (in fact, it probably tends to infinity with $a$, and the *abc* conjecture predicts that it is as large as $p^{a(1-\varepsilon)}$ for every $\varepsilon > 0$ provided that $a$ is sufficiently large with respect to $\varepsilon$). However, the same result holds nevertheless.

THEOREM 8. *Given any integer $k \geq 2$, for every $\varepsilon > 0$, the inequality*

$$S_k(n) \geq \frac{k}{k-1} - \varepsilon$$

*holds for infinitely many positive integers $n$.*

We shall need the following well-known lemma, essentially due to Erdős [7].

LEMMA 9. *There exists a constant $\delta \in (0, 1)$ such that the estimate*

$$\#\{p \in [y, 2y] \mid P(p+1) < y^{\delta}\} \gg \pi(y)$$

*holds for large $y$.*

Specific values of $\delta$ are known from the work of several mathematicians but they are of no use to us.

PROOF. Let $\delta \in (0, 1)$ be as in Lemma 9, $y$ be large and $\varepsilon \in (0, 1-\delta)$. Let $U = \lfloor y^{\delta+\varepsilon} \rfloor$ and $V = kU$. Choose $p_1 < \cdots < p_V$ primes in $(y, 2y)$ such that

$$P(p_i + 1) < y^{\delta} \quad \text{for all } i = 1, 2, \ldots, V.$$

This is possible for large $y$ by Lemma 9 and the fact that $V = O(y^{\delta+\varepsilon}) = o(\pi(y))$ as $y \to \infty$. For $j = 1, 2, \ldots, k$ put

$$m_j = \prod_{i=U(j-1)+1}^{Uj} p_i.$$

Note that

$$\log m_j = \sum_{i=U(j-1)+1}^{Uj} \log p_i = U \log y + O(U) = (1 + o(1)) y^{\delta+\varepsilon} \log y$$

for all $j = 1, 2, \ldots, k$ as $y \to \infty$. Since $\sigma(m_j) = \prod_{p \mid m_j}(p+1)$, it follows, from the way we have chosen the prime factors of $m_j$, that

$$\gamma(\sigma(m_j)) \leq \prod_{p \leq y^{\delta}} p = \exp((1 + o(1)) y^{\delta}),$$

where the last estimate follows from the prime number theorem. Therefore

$$\log \gamma(\sigma(m_j)) \leq (1 + o(1)) y^{\delta} = o(\log(m_j))$$

for all $j = 1, 2, \ldots, k$ as $y \to \infty$. Now let $n$ be a positive integer such that $n + j \equiv 0 \bmod m_j$ for all $j = 1, 2, \ldots, k$. The above system is solvable by the Chinese remainder theorem and all its solutions are of the form $n = M\ell + N$, where $M = \prod_{j=1}^{k} m_j$ and $N \in [0, 1, \ldots, M-1]$ is the smallest nonnegative solution of the above system of congruences. We claim that there exists $\ell \in [y, 2y]$ such that the corresponding $n$ satisfies the fact that $(n+j)/m_j$ and $m_j$ are coprime for $j = 1, 2, \ldots, k$. Indeed, note that

$$(n+j) = M\ell + (N+j) = m_j((M/m_j)\ell + (N+j)/m_j),$$

so that

$$(n+j)/m_j = (M/m_j)\ell + (N+j)/m_j.$$

Clearly, $M/m_j$ and $m_j$ are coprime since $M$ is square-free. Thus, if $(n+j)/m_j$ and $m_j$ are, say, both divisible by the prime $p$, then this puts $\ell$ into a certain uniquely determined congruence class modulo $p$. The number of such $\ell$ in the interval $[y, 2y]$ is less than or equal to $y/p + 1$. Thus, the number of $\ell \in [y, 2y]$ for which the corresponding $n$ has the property that $(n+j)/m_j$ and $m_j$ are not coprime for some $j = 1, 2, \ldots, k$ is at most

$$y \sum_{p|M} \frac{1}{p} + \omega(M) \le \frac{ky^{1+\delta+\varepsilon}}{y} + ky^{\delta+\varepsilon} < 2ky^{\delta+\varepsilon}.$$

Since $\delta + \varepsilon < 1$ and since the interval $[y, 2y]$ contains at least $y - 1$ integers, we get that there are at least $y - 1 - 2ky^{\delta+\varepsilon} > 0$ integers $\ell \in [y, 2y]$ such that the corresponding $n$ does indeed have the property that $(n+j)/m_j$ and $m_j$ are coprime for all $j = 1, 2, \ldots, k$. Such an $n$ has the following properties:

$$\begin{aligned}\log(\sigma(n+j)) &= (1+o(1))\log n = (1+o(1))(\log M + \log y)\\ &= (k+o(1))y^{\delta+\varepsilon}\log y;\end{aligned}$$

further, since $(n+j)/m_j$ and $m_j$ are coprime,

$$\sigma(n+j) = \sigma(m_j)\sigma((n+j)/m_j),$$

so that

$$\begin{aligned}\log(\gamma(\sigma(n+j))) &\le \log(\gamma(\sigma(m_j))) + \log(\gamma(\sigma((n+j)/m_j)))\\ &= o(\log(m_j)) + (1+o(1))\log((n+j)/m_j)\\ &= (1+o(1))(\log n - \log m_j)\\ &= (1+o(1))(\log M + \log y - \log m_j)\\ &= (k-1+o(1))y^{\delta+\varepsilon}\log y,\end{aligned}$$

which yields

$$\lambda(\sigma(n+j)) \ge \frac{k}{k-1} + o(1)$$

for all $j = 1, 2, \ldots, k$ as $y \to \infty$, therefore establishing the desired conclusion. □

## 7. Heuristics

As we have already mentioned, [3, Theorem 3] shows that inequality (2) holds for infinitely many $n$, and it was conjectured that apart from the $\varepsilon$ this inequality is the best possible. Here, we prove that this is indeed so under the *abc* conjecture.

THEOREM 10. *For each integer $k \geq 2$, let $Q_k(n)$ be as in (1). The estimate*

$$\limsup_{n\to\infty} Q_k(n) = \frac{k}{k-1}$$

*holds under the abc conjecture.*

PROOF. Instead of recalling the *abc* conjecture, we recall the following consequence of it (see [6, 8], or [10]).

LEMMA 11 (The $ABC$ conjecture). *Let $f$ be a homogeneous polynomial with integer coefficients having no repeated irreducible factors. Then for every $\varepsilon > 0$ and coprime positive integers $m$ and $n$,*

$$\gamma(f(m,n)) \gg \max\{m,n\}^{d-2-\varepsilon},$$

*where $d$ is the degree of $f$ and the constant implied by the Vinogradov symbol above depends on both $f$ and $\varepsilon$.*

The classical *abc* conjecture is usually the above statement for the polynomial $f(X,Y) = XY(X+Y)$. To deduce Theorem 10 from Lemma 11, we may assume that $k \geq 3$ and look at the homogeneous polynomial

$$f(X,Y) = XY(Y-X)(2Y-X)(3Y-2X)\ldots((k-1)Y-(k-2)X),$$

which obviously has degree $k+1$ and no repeated factors. Note that

$$f(n,n+1) = n(n+1)(n+2)(n+3)\ldots(n+k-1),$$

so that by Lemma 11 we have that the inequality

$$\gamma(n(n+1)\ldots(n+k-1)) \gg n^{k-1-\varepsilon/2} \tag{12}$$

holds for every fixed $\varepsilon > 0$ where the implied constant depends on $\varepsilon$ and $k$. Now consider an integer $n$ such that

$$Q_k(n) \geq \frac{k}{k-1} + \varepsilon.$$

Then

$$\gamma(n+i) \leq (n+i)^{((k-1)/(k+(k-1)\varepsilon))} \ll n^{((k-1)/(k+(k-1)\varepsilon))}, \quad i = 0, 1, \ldots, k.$$

Multiplying all these relations for $i = 0, 1, \ldots, k-1$, we get that

$$\prod_{i=1}^{k} \gamma(n+i-1) \ll n^{((k(k-1))/(k+(k-1)\varepsilon))}.$$

But for $\varepsilon < 1/(k-1)$,

$$\frac{k(k-1)}{k+(k-1)\varepsilon} < k-1-\varepsilon,$$

because this last inequality is equivalent to $(k-1)^2 \geq k + (k-1)\varepsilon$, which is implied by $(k-1)^2 \geq k+1$ (because $\varepsilon \leq 1/(k-1)$), and this last inequality is equivalent to $k \geq 3$. Hence,

$$\gamma(n(n+1)\ldots(n+k-1)) \leq \prod_{i=1}^{k} \gamma(n+i-1) \ll n^{k-1-\varepsilon},$$

which compared with inequality (12) gives us an upper bound on $n$. This completes the proof of the theorem. □

We conjecture that, unlike $Q_k(n)$, both the amounts $F_k(n)$ and $S_k(n)$ should be unbounded and that in fact each of the inequalities $F_k(n) \gg \log n$ and $S_k(n) \gg \log n$ should hold for infinitely many positive integers $n$, where the implied constants depend on $k$. In what follows, we will treat only the case of $F_k(n)$. To see why, let us first look at the case $k = 2$.

If there existed infinitely many primes $p$ of the form $2^a \cdot 3^b + 1$, then it would follow that $F_2(n)$ is unbounded. Indeed, let $p = 2^a \cdot 3^b + 1$ be such a large prime and set $n = p - 1$. Then

$$\phi(n) = \phi(2^a \cdot 3^b) = 2^a \cdot 3^{b-1} \quad \text{and} \quad \phi(n+1) = 2^a \cdot 3^b,$$

so that $\lambda(\phi(n)) = ((a \log 2 + (b-1)\log 3)/(\log 2 + \log 3)) \gg \log n$ and similarly $\lambda(\phi(n+1)) \gg \log n$. Hence, $F_2(n) \gg \log n$, proving our claim. A computer check showed that the number of primes $p \leq x$ of the above form is equal to 66 for $x = 10^{10}$ and to 789 for $x = 10^{100}$.

Using essentially the same argument as above, let us show how one would go about constructing integers $n$ for which $F_k(n) \gg \log n$. Assume that

$$2 = p_1 < p_2 < \cdots < p_k$$

are the first $k$ prime numbers. Assume that $a_1, \ldots, a_k$ are such that $a_i > \log k / \log p_i$ and such that if we set

$$n = p_1^{a_1} p_2^{a_2} \ldots p_k^{a_k},$$

then $(n+i)/i$ is a prime number for all $i = 1, 2, \ldots, k$. Note that, from the conditions we imposed on the exponents $a_i$, the number $(n+i)/i$ is always an integer coprime to $i$. If this is the case, then

$$\phi(n+i) = \phi(i)\left(\frac{n+i}{i} - 1\right) = \frac{\phi(i)n}{i},$$

so that

$$\gamma(\phi(n+i)) = \log(p_1 \ldots p_k) = O(1) \quad \text{for all } i = 1, 2, \ldots, k.$$

Thus

$$\lambda(\phi(n+i)) \gg \log(\phi(n+i)) \gg \log n$$

for all such choices of $n$.

To back up our construction a little more, we give heuristic support to the existence of infinitely many positive integers $n$ of the above form. Let $X$ be a large positive integer. There are at least a multiple of $X^k$ $k$-tuples of integers $(a_1, \ldots, a_k)$ such that $(a_1, \ldots, a_k) \in (X, 2X)^k$. For each one of them, we assume, heuristically, that the probability of each one of the numbers $(n+i)/i$ being prime is roughly

$$1/\log((n+i)/i) \gg 1/X.$$

Of course, this cannot possibly be true for all such $k$-tuples $(a_1, \ldots, a_k)$ because the number $n/i$ might end up having all exponents divisible by the same odd prime in which case the expression $n/i + 1$ factors in an obvious way. To fix this, we may first fix $a_1, \ldots, a_{k-1}$ in an arbitrary manner, and then fix $a_k$ to be any prime in $(X, 2X)$ which does not divide any of $a_i$ for $i = 1, 2, \ldots, k-1$ (note that if $X$ is large, $a_k$ can be any prime in $(X, 2X)$ except for at most $k-1$ of them). Assuming further that the events that $(n+i)/i$ are prime are independent for $i = 1, 2, \ldots, k$, we conclude that if $X$ is large, for a suitable set of choices of $(a_1, \ldots, a_k) \in (X, 2X)^k$ of total cardinality at least a multiple of

$$X^{k-1}(\pi(2X) - \pi(X) - k + 1) \gg X^k/\log X,$$

the probability that all numbers $(n+i)/i$ are simultaneously prime is at least a multiple of $1/X^k$. Multiplying those two amounts, we get that the expected number of such primes is at least a multiple of $1/\log X$. Now letting $X = 2^\ell$ go to infinity through powers of 2 starting with a sufficiently large $2^{\ell_0}$, we get that the number of such numbers $n$ should be at least a multiple of $\sum_{\ell \ge \ell_0} 1/\ell$, hence, an infinite number of them.

Computationally, letting $k = 4$ and choosing

$$n = 2^8 \cdot 3^{30} \cdot 5^{20} = 5\,026\,638\,967\,154\,516\,601\,562\,500\,000\,000,$$

TABLE 1. Some values of $F_k(n)$.

| $k$ | $n$ | Number of digits of $n$ | $\lfloor F_k(n) \rfloor$ |
|---|---|---|---|
| 2 | $2^{44} \cdot 3^{40}$ | 33 | 40 |
| 2 | $2^{491} \cdot 3^{579}$ | 425 | 544 |
| 3 | $2^{77} \cdot 3^{213}$ | 125 | 159 |
| 4 | $2^{43} \cdot 3 \cdot 5^{7}$ | 19 | 17 |
| 4 | $2^{8} \cdot 3^{30} \cdot 5^{20}$ | 31 | 20 |
| 4 | $2^{12} \cdot 3^{29} \cdot 5^{281}$ | 214 | 144 |
| 5 | $2^{46} \cdot 3^{41} \cdot 5^{19}$ | 47 | 31 |
| 6 | $2^{42} \cdot 3^{6} \cdot 5^{5} \cdot 7^{4} \cdot 13^{24}$ | 58 | 16 |

one can check that $n+1$, $(n+2)/2$ and $(n+3)/3$ are all prime numbers. This allows us to obtain that

$$\phi(n) = 2^{10} \cdot 3^{29} \cdot 5^{19}, \quad \text{so that } \lambda(\phi(n)) = \frac{10 \log 2 + 29 \log 3 + 19 \log 5}{\log 2 + \log 3 + \log 5} \approx 20.3959,$$

$$\phi(n+1) = 2^{8} \cdot 3^{30} \cdot 5^{20}, \quad \text{so that } \lambda(\phi(n+1)) = \frac{8 \log 2 + 30 \log 3 + 20 \log 5}{\log 2 + \log 3 + \log 5} \approx 20.7845,$$

$$\phi(n+2) = \phi(2^{8} \cdot 3^{30} \cdot 5^{20} + 2) = \phi(2(2^{7} \cdot 3^{30} \cdot 5^{20} + 1)) = 2^{7} \cdot 3^{30} \cdot 5^{20}, \quad \text{so that } \lambda(\phi(n+2)) = \frac{7 \log 2 + 30 \log 3 + 20 \log 5}{\log 2 + \log 3 + \log 5} \approx 20.5807,$$

$$\phi(n+3) = \phi(2^{8} \cdot 3^{30} \cdot 5^{20} + 3) = \phi(3(2^{8} \cdot 3^{29} \cdot 5^{20} + 1)) = 2 \cdot 2^{8} \cdot 3^{29} \cdot 5^{20}, \quad \text{so that } \lambda(\phi(n+3)) = \frac{9 \log 2 + 29 \log 3 + 20 \log 5}{\log 2 + \log 3 + \log 5} \approx 20.6653,$$

thus establishing that

$$F_4(n) \approx 20.3959 = \min(20.3959, 20.7845, 20.5807, 20.6653).$$

More examples can be seen in Table 1.

As mentioned above, similar heuristics apply for $S_k(n)$. In fact, if instead one does not start with only the first $k$ primes $2 = p_1 < \cdots < p_k$, but with the first $2k$ primes and sets $n = p_1^{a_1} p_2^{a_2} \ldots p_{2k}^{a_{2k}}$ for some sufficiently large positive integers $a_i$ with $i = 1, 2, \ldots, 2k$, then one can further assume that $(n+i)/n$ and $(n-i)/n$ are both primes for all $i = 1, 2, \ldots, k$, and then with such $n$ one finds that the even stronger inequality $\min\{F_k(n), S_k(n)\} \gg \log n$ holds. We let the reader fill in the details of such a deduction as well as working out a heuristic that would predict that there should indeed be infinitely many such positive integers $n$.

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JEAN-MARIE DE KONINCK, Département de mathématiques et de statistique, Université Laval, Québec G1V OA6, Canada
e-mail: jmdk@mat.ulaval.ca

FLORIAN LUCA, Mathematical Institute, UNAM, Ap. Postal 61-3 (Xangari), CP 58 089, Morelia, Michoacán, Mexico
e-mail: fluca@matmor.unam.mx