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ESTHETIC NUMBERS

JEAN-MARIE DE KONINCK AND NICOLAS DOYON

Dedicated to Professor Paulo Ribenboim on the occasion of his 80th birthday.

RÉSUMÉ. Soit $q \geq 2$ un entier. Nous disons qu'un entier positif $n = d_1 d_2 \cdots d_r$, où les d_i sont les chiffres de n en base q, est q-esthétique si $|d_i - d_{i-1}| = 1$ pour tout entier i tel que $2 \leq i \leq r$. Soit $N_q(r)$ le nombre de nombres q-esthétiques à r chiffres. Nous obtenons une expression explicite pour $N_q(r)$ et nous étudions son comportement asymptotique lorsque $r \to \infty$.

ABSTRACT. Given an integer $q \geq 2$, we say that a positive integer $n = d_1 d_2 \cdots d_r$, where the d_i's are the digits of n in base q, is q-esthetic if $|d_i - d_{i-1}| = 1$ for each integer i with $2 \leq i \leq r$. Letting $N_q(r)$ stand for the number of r digit q-esthetic numbers, we obtain an explicit expression for $N_q(r)$ and we also study its asymptotic behavior as $r \to \infty$.

1. Introduction and preliminary observations

Given an integer $q \geq 2$, we say that a positive integer $n = d_1 d_2 \cdots d_r$, where the d_i's are the digits of n in base q, is *q-esthetic* (or simply *esthetic*) if $|d_i - d_{i-1}| = 1$ for each integer i with $2 \leq i \leq r$. For convenience, we say that the numbers $1, 2, \ldots, q-1$ are q-esthetic.

Our first goal will be to study the function $N_q(r)$ which represents the number of r digit q-esthetic numbers. First observe that, using a computer, one can generate the following table.

r	1	2	3	4	5	6	7	8	9
$N_2(r)$	1	1	1	1	1	1	1	1	1
$N_3(r)$	2	3	4	6	8	12	16	24	32
$N_4(r)$	3	5	8	13	21	34	55	89	144
$N_5(r)$	4	7	12	21	36	63	108	189	324
$N_6(r)$	5	9	16	29	52	94	169	305	549
$N_7(r)$	6	11	20	37	68	126	232	430	792
$N_8(r)$	7	13	24	45	84	158	296	557	1045
$N_9(r)$	8	15	28	53	100	190	360	685	1300
$N_{10}(r)$	9	17	32	61	116	222	424	813	1556

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In the case $q = 2$, it is clear that

(1) $$N_2(r) = 1, \qquad \text{for } r = 1, 2, \ldots$$

In the case $q = 3$, it is quite easy to show that

(2) $$N_3(r) = \begin{cases} 2^{(r+1)/2} = \sqrt{2} \cdot (\sqrt{2})^r & \text{if } r \text{ is odd,} \\ 3 \cdot 2^{(r/2)-1} = (3/2) \cdot (\sqrt{2})^r & \text{if } r \text{ is even.} \end{cases}$$

When $q = 4$, we will see that

(3) $$N_4(r) = F_{r+3},$$

where F_i denotes the i-th term of the Fibonacci sequence 1, 1, 2, 3, 5, 8, 13, 21, ...

For $q \geq 5$, the situation becomes more complicated. Hence, our first task will be to establish an exact formula for $N_q(r)$. From this formula, it will be clear that for each integer $q \geq 3$, there exists a real number $\alpha = \alpha_q$ such that $1 < \alpha < 2$ and $N_q(r) \approx \alpha^r$, and moreover that the sequence $\{\alpha_q\}$ increases with q and tends to 2 as $q \to \infty$.

2. The linear algebra set up

Let $q \geq 2$ be a fixed integer and let $N_q(r, i)$ denote the number of r digit q-esthetic numbers whose last digit is i, for $0 \leq i \leq q-1$. Consider the vector $u = (0, 1, 1, \ldots, 1)$ of length q and let $M = M_q = (m_{ij})$ be the $q \times q$ matrix defined by

$$m_{i,j} = \begin{cases} 1 & \text{if } |i - j| = 1, \\ 0 & \text{if } |i - j| \neq 1, \end{cases}$$

so that for example

$$M_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad M_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad M_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

In view of (1), and since $N_2(r, 0) = \frac{1+(-1)^{r-1}}{2}$, while $N_2(r, 1) = \frac{1-(-1)^r}{2}$, we shall assume from now on that $q \geq 3$. The following three relations are immediate consequences of the definition of $N_q(r, i)$:

$$\begin{cases} N_q(r, 0) = N_q(r - 1, 1), \\ N_q(r, q - 1) = N_q(r - 1, q - 2), \\ N_q(r, i) = N_q(r - 1, i - 1) + N_q(r - 1, i + 1), \quad \text{for } 1 \leq i \leq q - 2. \end{cases}$$

It follows that, for $i = 0, 1, 2, \ldots, q - 1$,

$$N_q(r, i) = \left(M^{r-1} u\right)_{i+1}, \quad \text{where } r = 1, 2, 3, \ldots$$

that is, the $(i + 1)$-th component of the vector $M^{r-1}u$, so that

$$N_q(r) = \sum_{i=0}^{q-1} N_q(r, i) = \sum_{i=0}^{q-1} \left(M^{r-1} u\right)_{i+1}, \qquad \text{where } r = 1, 2, 3, \ldots$$

3. A preliminary lemma

Lemma 1. *Given a positive integer $k \le q$, let $\theta_k = k\pi/(q+1)$. Then*

(a) $\displaystyle \frac{1}{\sin\theta_k}\sum_{j=2}^{q}\sin(j\theta_k) = \chi(q,k),$

(b) $\displaystyle \sum_{j=1}^{q}\sin^2(j\theta_k) = \frac{q+1}{2},$

where

$$\chi(q,k) := \begin{cases} -1 & \text{if } k \text{ is even or if } k = \frac{q+1}{2} \text{ and } q \equiv 3 \pmod 4, \\ 0 & \text{if } k = \frac{q+1}{2} \text{ and } q \equiv 1 \pmod 4, \\ \dfrac{\cos\theta_k}{1-\cos\theta_k} & \text{otherwise.} \end{cases}$$

Proof. To simplify the notation, let $\theta = \theta_k$. In order to prove (a), we study separately the cases $k = \frac{q+1}{2}$ and $k \ne \frac{q+1}{2}$. If $k = \frac{q+1}{2}$, then $\theta = \frac{\pi}{2}$ and in this case,

$$\frac{1}{\sin\theta}\sum_{j=2}^{q}\sin(j\theta) = \sum_{j=2}^{q}\sin(j\theta) = \begin{cases} -1 & \text{if } q \equiv 3 \pmod 4, \\ 0 & \text{if } q \equiv 1 \pmod 4. \end{cases} \tag{4}$$

If $k \ne \frac{q+1}{2}$, we first evaluate $A(\theta) := \sum_{j=0}^{q}\sin(j\theta)$. It is clear that

$$A(\theta) = \sum_{j=0}^{q}\frac{e^{ij\theta}-e^{-ij\theta}}{2i} = \frac{1}{2i}\left(\frac{1-e^{ik\pi}}{1-e^{ik\pi/(q+1)}} - \frac{1-e^{-ik\pi}}{1-e^{-ik\pi/(q+1)}}\right).$$

On the one hand, if k is even, this last expression is 0, since $e^{ik\pi} = e^{-ik\pi} = 1$, in which case

$$\frac{1}{\sin\theta}\sum_{j=2}^{q}\sin(j\theta) = \frac{1}{\sin\theta}\left(\sum_{j=0}^{q}\sin(j\theta) - \sin\theta\right) = \frac{1}{\sin\theta}(0-\sin\theta) = -1. \tag{5}$$

On the other hand, if k is odd, $e^{ik\pi} = e^{-ik\pi} = -1$, in which case,

$$A(\theta) = \frac{1}{i}\left(\frac{1}{1-e^{ik\pi/(q+1)}} - \frac{1}{1-e^{-ik\pi/(q+1)}}\right) = \frac{\sin\theta}{1-\cos\theta},$$

so that

$$\begin{aligned} \frac{1}{\sin\theta}\sum_{j=2}^{q}\sin(j\theta) &= \frac{1}{\sin\theta}\left(\sum_{j=0}^{q}\sin(j\theta) - \sin\theta\right) \\ &= \frac{1}{\sin\theta}\left(\frac{\sin\theta}{1-\cos\theta} - \sin\theta\right) \\ &= \frac{\cos\theta}{1-\cos\theta}. \end{aligned} \tag{6}$$

Combining (4) and (6) as well as the special case (5), we obtain (a).

Finally, using the identity

$$1 + 2\cos\phi + 2\cos(2\phi) + \cdots + 2\cos(n\phi) = \frac{\sin\left(\left(n+\frac{1}{2}\right)\phi\right)}{\sin(\phi/2)}, \quad \text{for } 0 < \phi < 2\pi,$$

and the identity $\sin^2\phi = \frac{1}{2}(1 - \cos 2\phi)$, we obtain

$$\begin{aligned}
\sum_{j=1}^{q} \sin^2(j\theta) &= \sum_{j=1}^{q} \left(\frac{1}{2} - \frac{1}{2}\cos(2j\theta)\right) \\
&= \frac{q}{2} - \frac{1}{4}\left(\frac{\sin(2q+1)\theta}{\sin\theta} - 1\right) \\
&= \frac{q}{2} - \frac{1}{4}\left(\frac{\sin\left(2k\pi - \frac{k\pi}{q+1}\right)}{\sin(k\pi/(q+1))} - 1\right) \\
&= \frac{q}{2} - \frac{1}{4}\left(-\frac{\sin(k\pi/(q+1))}{\sin(k\pi/(q+1))} - 1\right) \\
&= \frac{q+1}{2},
\end{aligned}$$

which proves (b). □

4. A sequence of polynomials and their roots

Consider the sequence of polynomials $p_0(x), p_1(x), p_2(x), \ldots$ defined by

$$p_0(x) = 1, \quad p_1(x) = x \quad \text{and} \quad p_j(x) = xp_{j-1}(x) - p_{j-2}(x), \qquad \text{for } j \geq 2.$$

These polynomials are similar to Chebyshev polynomials which have been extensively studied (see for instance Rivlin [1]). One can establish that the p_j's may also be defined by

$$p_j(x) = \sum_{\nu=0}^{[j/2]} (-1)^\nu \binom{j-\nu}{\nu} x^{j-2\nu},$$

which quickly reveals the first few terms of the sequence: 1, x, $x^2 - 1$, $x^3 - 2x$, $x^4 - 3x^2 + 1$, $x^5 - 4x^3 + 3x$, $x^6 - 5x^4 + 6x^2 - 1$, $x^7 - 6x^5 + 10x^3 - 4x$.

We now move to find the roots of $p_n(x)$ for any fixed positive integer n.

Proposition 2. *Given a positive integer n, the roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ of $p_n(x)$ are given by*

$$\alpha_k = 2\cos\left(\frac{k\pi}{n+1}\right), \qquad \textit{for } k = 1, 2, \ldots, n.$$

Proof. Given a real number $\alpha \neq 0$ such that $|\alpha| < 2$, consider the 2×2 matrix

$$K = K(\alpha) := \begin{pmatrix} 0 & 1 \\ -1 & \alpha \end{pmatrix}.$$

It follows from the definition of $p_n(\alpha)$ that

$$K^n \begin{pmatrix} 1 \\ \alpha \end{pmatrix} = \begin{pmatrix} p_n(\alpha) \\ p_{n+1}(\alpha) \end{pmatrix}, \qquad \text{for } n = 0, 1, 2, \ldots \tag{7}$$

The characteristic equation of K is $\lambda^2 - \alpha\lambda + 1 = 0$, yielding the eigenvalues

$$\lambda = \frac{\alpha}{2} \pm i\frac{\sqrt{4-\alpha^2}}{2}\cdot$$

Since clearly $|\lambda| = 1$, these values can be written as $e^{i\theta}$ and $e^{-i\theta}$, with

$$\theta := \arctan\left(\frac{\sqrt{4-\alpha^2}}{\alpha}\right).$$

The corresponding eigenvectors of K are then $(1, e^{i\theta})$ and $(1, e^{-i\theta})$.

It follows that we can write the matrix K as $K = QDQ^{-1}$, where D is the diagonal matrix $D = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$ and $Q = \begin{pmatrix} 1 & 1 \\ e^{i\theta} & e^{-i\theta} \end{pmatrix}$. Therefore,

$$\begin{aligned} K^n \begin{pmatrix} 1 \\ \alpha \end{pmatrix} &= (QDQ^{-1})^n \begin{pmatrix} 1 \\ \alpha \end{pmatrix} \\ &= QD^nQ^{-1} \begin{pmatrix} 1 \\ \alpha \end{pmatrix} \\ &= -\frac{1}{2i\sin\theta} Q \begin{pmatrix} e^{i(n-1)\theta} - \alpha e^{in\theta} \\ \alpha e^{-in\theta} - e^{-i(n-1)\theta} \end{pmatrix}. \end{aligned}$$

Combining this relation with (7), we obtain

$$\begin{pmatrix} p_n(\alpha) \\ p_{n+1}(\alpha) \end{pmatrix} = -\frac{1}{2i\sin\theta} \begin{pmatrix} 1 & 1 \\ e^{i\theta} & e^{-i\theta} \end{pmatrix} \begin{pmatrix} e^{i(n-1)\theta} - \alpha e^{in\theta} \\ \alpha e^{-in\theta} - e^{-i(n-1)\theta} \end{pmatrix},$$

thus allowing us to obtain an explicit value for $p_n(\alpha)$, namely

$$p_n(\alpha) = -\frac{1}{2i\sin\theta}\left(e^{i(n-1)\theta} - e^{-i(n-1)\theta} + \alpha\left(e^{-in\theta} - e^{in\theta}\right)\right) \tag{8}$$

for $n = 0, 1, 2, \ldots$ Hence,

$$p_n(\alpha) = 0 \iff \sin(n-1)\theta = \alpha \sin n\theta. \tag{9}$$

Besides this, since $e^{-i\theta} + e^{i\theta} = \alpha$, it follows that $\alpha = 2\cos\theta$. Therefore, from (9), we obtain that

$$\begin{aligned} p_n(\alpha) = 0 &\iff \sin(n-1)\theta = 2\cos\theta\sin n\theta \\ &\iff \cos\theta\sin n\theta + \cos n\theta\sin\theta = 0 \\ &\iff \sin(n+1)\theta = 0 \\ &\iff \theta = \frac{k\pi}{n+1}, \qquad \text{for } k = 1, 2, \ldots, n, \end{aligned}$$

since $\theta \in]0, \pi/2[$. □

5. Identifying the eigenvalues of the matrix M_q

We first show that the q eigenvalues of $M = M_q$ coincide with the q roots of $p_q(x)$.

Proposition 3. *The real number α is an eigenvalue of the matrix $M = M_q$ if and only if $p_q(\alpha) = 0$.*

Proof. Denoting by $I = I_q$ the identity matrix and given a real number α, it is clear that

$$(M-\alpha I)\begin{pmatrix} p_0(\alpha) \\ p_1(\alpha) \\ \vdots \\ p_{q-1}(\alpha) \end{pmatrix} = \begin{pmatrix} \alpha - \alpha \\ p_0(\alpha) - \alpha p_1(\alpha) + p_2(\alpha) \\ p_1(\alpha) - \alpha p_2(\alpha) + p_3(\alpha) \\ \vdots \\ p_{q-3}(\alpha) - \alpha p_{q-2}(\alpha) + p_{q-1}(\alpha) \\ p_{q-2}(\alpha) - \alpha p_{q-1}(\alpha) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ -p_q(\alpha) \end{pmatrix},$$

so that

$$(M - \alpha I)\begin{pmatrix} p_0(\alpha) \\ p_1(\alpha) \\ \vdots \\ p_{q-1}(\alpha) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \qquad \Longleftrightarrow \qquad p_q(\alpha) = 0.$$

But as we saw in Proposition 2, the roots of the polynomial $p_q(x)$ are all simple and distinct. Hence, it follows that the q eigenvalues of the matrix M_q are all accounted for, thus completing the proof of Proposition 3. □

6. A formula for $N_q(r, i)$

Let α be an eigenvalue of $M = M_q$ and consider its corresponding eigenvector

$$v_\alpha := (1, p_1(\alpha), p_2(\alpha), \ldots, p_{q-1}(\alpha)),$$

already mentioned in the proof of Proposition 3. Since M is a real symmetric matrix, its eigenvalues are all real (as shown in Propositions 2 and 3) and its eigenvectors are orthogonal. A well known result in linear algebra (see for instance Propositions 15.6 and 15.9 of Smith [2]) guarantees that the q vectors $\dfrac{v_\alpha}{\|v_\alpha\|}$ form an orthonormal basis of $\mathbf{R}^q$. Hence, we may write

$$u = \sum_{\alpha,\, p_q(\alpha)=0} \left(\frac{v_\alpha \cdot u}{\|v_\alpha\|^2}\right) v_\alpha,$$

so that, for each integer $r \geq 1$,

$$\text{(10)} \qquad M^{r-1}u = \sum_{\alpha\, p_q(\alpha)=0} \frac{v_\alpha \cdot u}{\|v_\alpha\|^2}\, M^{r-1}v_\alpha = \sum_{\alpha,\, p_q(\alpha)=0} \frac{v_\alpha \cdot u}{\|v_\alpha\|^2}\, \alpha^{r-1}v_\alpha = \sum_{k=1}^{q} \frac{v_{\alpha_k} \cdot u}{\|v_{\alpha_k}\|^2} v_{\alpha_k}\alpha_k^{r-1},$$

where

$$\text{(11)} \qquad v_{\alpha_k} = (1, p_1(\alpha_k), p_2(\alpha_k), \ldots, p_{q-1}(\alpha_k)), \qquad \text{for } k = 1, 2, \ldots, q.$$

In light of (8), we have that for each root $\alpha = \alpha_k$ of $p_q(x)$, with the notation $\theta = k\pi/(q+1)$,

(12)

$$p_j(\alpha) = \frac{\sin(j-1)\theta - \alpha \sin j\theta}{-\sin\theta} = \frac{\sin(j-1)\theta - 2\cos\theta \sin j\theta}{-\sin\theta} = \frac{\sin(j+1)\theta}{\sin\theta},$$

so that by using (11) and relation (a) of Lemma 1, we obtain

$$\text{(13)} \qquad v_{\alpha_k} \cdot u = \sum_{j=1}^{q-1} p_j(\alpha) = \frac{1}{\sin\theta} \sum_{j=1}^{q-1} \sin(j+1)\theta = \frac{1}{\sin\theta} \sum_{j=2}^{q} \sin j\theta = \chi(q,k).$$

Similarly, using relation (b) of Lemma 1, we obtain

$$\text{(14)} \qquad \|v_{\alpha_k}\|^2 = \sum_{j=0}^{q-1} p_j^2(\alpha) = \frac{1}{\sin^2\theta} \sum_{j=1}^{q} \sin^2(j\theta) = \frac{q+1}{2\sin^2\theta}.$$

By substituting (13) and (14) in (10), and letting $\theta_k = \theta_k(q) = k\pi/(q+1)$, we obtain

$$\text{(15)} \qquad M^{r-1}u = \frac{2^r}{q+1} \sum_{k=1}^{q} \sin^2\theta_k \cos^{r-1}\theta_k\, \chi(q,k)\, v_{\alpha_k},$$

so that by combining (11), (12) and (15), it follows that

$$N_q(r,i) = \frac{2^r}{q+1} \sum_{k=1}^{q} \sin\theta_k \sin((i+1)\theta_k) \cos^{r-1}\theta_k\, \chi(q,k), \qquad \text{for } 0 \leq i \leq q-1,$$

Hence,

$$\text{(16)} \qquad N_q(r) = \sum_{i=0}^{q-1} N_q(r,i) = \frac{2^r}{q+1} \sum_{k=1}^{q} \sin\theta_k \cos^{r-1}\theta_k\, \chi(q,k) \sum_{j=1}^{q} \sin(j\theta_k).$$

Now, in light of formula (a) of Lemma 1, we easily obtain

$$\begin{aligned}\sum_{j=1}^{q}\sin(j\theta_k) &= \sin\theta_k(1+\chi(q,k))\\ &= \begin{cases} 0 & \text{if } k \text{ is even or if } k=\frac{q+1}{2} \text{ and } q\equiv 3 \pmod 4,\\ \sin\theta_k & \text{if } k=\frac{q+1}{2} \text{ and } q\equiv 1 \pmod 4,\\ \dfrac{\sin\theta_k}{1-\cos\theta_k} & \text{otherwise.}\end{cases}\end{aligned}$$

From this, it is easily seen that (16) can be written as

$$\begin{aligned}N_q(r) &= \frac{2^r}{q+1}\sum_{\substack{k=1\\ k \text{ odd},\ k\neq\frac{q+1}{2}}}^{q}\cos^r\theta_k\frac{\sin^2\theta_k}{(1-\cos\theta_k)^2}\\ &= \frac{2^r}{q+1}\sum_{\substack{k=1\\ k \text{ odd},\ k\neq\frac{q+1}{2}}}^{q}\cos^r\theta_k\left(\cot\theta_k+\csc\theta_k\right)^2.\end{aligned} \tag{17}$$

Using this formula and a computer, one can easily extend indefinitely the lines and columns of the table of Section 1.

Observe that, for small values of q, relation (17) can be considerably simplified. Thus, by setting $q = 3$ in (17), we indeed obtain (2). Setting $q = 4$ in (17), we obtain

$$\begin{aligned}N_4(r) &= \frac{2^{-r}}{5}\Big((5+2\sqrt{5})(1+\sqrt{5})^r+(5-2\sqrt{5})(1-\sqrt{5})^r\Big)\\ &= \frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{r+3}-\left(\frac{1-\sqrt{5}}{2}\right)^{r+3}\right\}\\ &= F_{r+3},\end{aligned}$$

as claimed in (3). Finally, setting $q = 5$ in (17), we easily obtain

$$N_5(r)=\begin{cases} 4\cdot 3^{(r-1)/2}=(4/\sqrt{3})\cdot(\sqrt{3})^r & \text{if } r \text{ is odd,}\\ (7/3)\cdot(\sqrt{3})^r & \text{if } r \text{ is even.}\end{cases}$$

7. The asymptotic behavior of $N_q(r)$

The asymptotic behaviors of $N_q(r)$ and $N_q(r,i)$ do have some nice properties. We begin with the former. First of all, observe that

$$\cos\theta_1>\cos\theta_2>\cdots>\cos\theta_{q-1}>\cos\theta_q \quad \text{and} \quad \cos\theta_q=-\cos\theta_1. \tag{18}$$

On the one hand, if q is even, then the dominating term of the sum appearing in (17) is the one with index $k = 1$. Setting $\alpha = 2\cos\theta_1 = 2\cos(\pi/(q + 1))$, that is the largest root of $p_q(x)$, it follows from (17) and (18) that, as $r \to \infty$,

$$N_q(r)=(1+o(1))\frac{\alpha^r}{q+1}\left(\cot\theta_1+\csc\theta_1\right)^2, \qquad \text{for } q \text{ even.}$$

On the other hand, if q is odd, the sum appearing in (17) has two dominating terms, namely those of index $k = 1$ and index $k = q$. In this case, it follows from (17) and (18) that, as $r \to \infty$,

$$\begin{aligned} N_q(r) &= \frac{\alpha^r}{q+1}\left(\frac{\sin\theta_1}{1-\cos\theta_1}\right)^2 + \frac{(-\alpha)^r}{q+1}\left(\frac{\sin\theta_1}{1+\cos\theta_1}\right)^2 \\ &= \begin{cases} (1+o(1))\, 2\dfrac{\alpha^r}{q+1}(\csc^2\theta_1 + \cot^2\theta_1) & \text{if } r \text{ is even,} \\ (1+o(1))\, 4\dfrac{\alpha^r}{q+1}\csc\theta_1\cot\theta_1 & \text{if } r \text{ is odd.} \end{cases} \end{aligned}$$

Hence, for all $q \geq 3$, $N_q(r) \approx \alpha^r$ for some $\alpha = \alpha_q \in]1, 2[$, and $\lim_{q\to\infty} \alpha_q = 2$, as claimed in Section 1.

We can also study the behaviour of $N_q\left(\left[cq^2\right]\right)$ when c is a fixed real number and $q \to +\infty$. Indeed, if $r = \left[cq^2\right]$, then, as $q \to \infty$,

$$\begin{aligned} \cos^r\theta_k &= \left(\cos\left(\frac{k\pi}{q+1}\right)\right)^{\left[cq^2\right]} \\ &= \left(1 + O\left(\frac{k^2}{q^2}\right)\right)\left(\cos\left(\frac{k\pi}{q+1}\right)\right)^{cq^2} \\ &= \left(1 + O\left(\frac{k^2}{q^2}\right)\right)\left(1 - \frac{k^2\pi^2}{2(q+1)^2} + O\left(\frac{k^4}{q^4}\right)\right)^{cq^2} \\ &= \left(1 + O\left(\frac{k^2}{q^2}\right)\right)\exp\left\{-\frac{ck^2\pi^2}{2}\right\}. \end{aligned} \tag{19}$$

Furthermore,

$$\frac{\sin^2\theta_k}{(1-\cos\theta_k)^2} = \frac{1+\cos\theta_k}{1-\cos\theta_k} = \frac{4}{\theta_k^2}\left(1 + O\left(\theta_k^2\right)\right) = \frac{4(q+1)^2}{k^2\pi^2}\left(1 + O\left(\frac{k^2}{q^2}\right)\right). \tag{20}$$

Combining (19) and (20) in (17), we obtain

$$\begin{aligned} N_q(r) &= (1+o(1))2^r q\sum_{k=1}^{q}\frac{4}{k^2\pi^2}\exp\left\{-\frac{ck^2\pi^2}{2}\right\} \\ &= (1+o(1))2^r q\sum_{k=1}^{\infty}\frac{4}{k^2\pi^2}\exp\left\{-\frac{ck^2\pi^2}{2}\right\}. \end{aligned} \tag{21}$$

Thus if we set

$$f(c) = \sum_{k=1}^{\infty}\frac{4}{k^2\pi^2}\exp\left\{-\frac{ck^2\pi^2}{2}\right\}$$

and $r = \left[cq^2\right]$, (21) can be written as

$$N_q(r) = (1+o(1))2^r q f(c) \qquad \text{as } q \to +\infty.$$

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J.-M. DE KONINCK, DÉP. DE MATHÉMATIQUES ET DE STATISTIQUE, U. LAVAL, QUÉBEC, G1V 0A6.
jmdk@mat.ulaval.ca

N. DOYON, DÉP. DE MATHÉMATIQUES ET DE STATISTIQUE, U. LAVAL, QUÉBEC, G1V 0A6.
nicodoyon77@hotmail.com