Journal of Integer Sequences, Vol. 11 (2008),
Article 08.1.5

Positive Integers n Such That $\sigma(\phi(n)) = \sigma(n)$

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Abstract

In this paper, we investigate those positive integers n for which the equality $\sigma(\phi(n)) = \sigma(n)$ holds, where σ is the sum of the divisors function and ϕ is the Euler function.

1 Introduction

For a positive integer n we write $\sigma(n)$ and $\phi(n)$ for the sum of divisors function and for the Euler function of n, respectively. In this note, we study those positive integers n such that

$$\sigma(\phi(n)) = \sigma(n)$$

holds. This is sequence A033631 in Sloane's Online Encylopedia of Integer Sequences. Let $\mathcal{A}$ be the set of all such positive integers n and for a positive real number x we put $\mathcal{A}(x) = \mathcal{A} \cap [1, x]$. Our result is the following.

Theorem 1. *The estimate*

$$\#\mathcal{A}(x) = O\left(\frac{x}{(\log x)^2}\right)$$

holds for all real numbers $x > 1$.

The above upper bound might actually be the correct order of magnitude of $\#\mathcal{A}(x)$. Indeed, note that if m is such that

$$\sigma(2\phi(m)) = 2\sigma(m) \tag{1}$$

and if $q > m$ is a Sophie Germain prime, that is a prime number q such that $p = 2q + 1$ is also prime, then $n = mp \in \mathcal{A}$. The numbers $m = 2318,\ 2806,\ 5734,\ 5937,\ 7198,\ 8097,\ \ldots$ all satisfy relation (1), and form Sloane's sequence A137733. More generally, if k and m are positive integers such that

$$\sigma(2^k\phi(m)) = 2^k\sigma(m) \tag{2}$$

and $q_1 < \ldots < q_k$ are primes with $p_i = 2q_i + 1$ also primes for $i = 1, \ldots, k$ and $q_1 > m$, then $n = p_1 \ldots p_k m \in \mathcal{A}$. Now recall that the *Prime K-tuplets Conjecture* of Dickson (see, for instance, [2, 4, 8]) asserts that, except in cases ruled out by obvious congruence conditions, K linear forms $a_i n + b_i$, $i = 1, \ldots, K$, take prime values simultaneously for about $cx/(\log x)^K$ integers $n \le x$, where c is a positive constant which depends only on the given linear forms. Under this conjecture (applied with $K = 2$ and the linear forms n and $2n+1$), we obtain that there should be $\gg x/(\log x)^2$ Sophie Germain primes $q \le x$, which suggests that $\#\mathcal{A}(x) \gg x/(\log x)^2$. We will come back to the Sophie Germain primes later.

Throughout, we use the Vinogradov symbols $\gg$ and $\ll$ and the Landau symbols O and o with their regular meanings. We use log for the natural logarithm and p, q and r with or without subscripts for prime numbers.

2 Preliminary Results

In this section, we point out a subset $\mathcal{B}(x)$ of all the positive integers $n \le x$ of cardinality $O(x/(\log x)^2)$. For the proof of Theorem 1 we will work only with the positive integers $n \in \mathcal{A}(x)\backslash\mathcal{B}(x)$. Further, x_0 is a sufficiently large positive real number, where the meaning of sufficiently large may change from a line to the next.

We put

$$y = \exp\left(\frac{\log x}{\log\log x}\right).$$

For a positive integer n we write $P(n)$ for the largest prime factor of n. It is well known that

$$\Psi(x, y) = \#\{n \le x \mid P(n) \le y\} = x\exp(-(1 + o(1))u\log u) \qquad (u \to \infty), \tag{3}$$

where $u = \log x/\log y$, provided that $u \le y^{1/2}$ (see [1], Corollary 1.3 of [6], or Chapter III.5 of [9]). In our case, $u = \log\log x$, so, in particular, the condition $u \le y^{1/2}$ is satisfied for $x > x_0$. We deduce that

$$u\log u = (\log\log x)(\log\log\log x).$$

Thus, if we set $\mathcal{B}_1(x) = \{n \le x \mid P(n) \le y\}$, then

$$\begin{aligned}\#\mathcal{B}_1(x) &= \Psi(x,y) = x\exp\left(-(1+o(1))(\log\log x)(\log\log\log x)\right) \\ &< \frac{x}{(\log x)^{10}} \qquad \text{for } x > x_0. \end{aligned} \tag{4}$$

We now let

$$z = (\log x)^{26}$$

and for a positive integer n we write $\rho(n)$ for its largest *squarefull divisor*. Recall that a positive integer m is squarefull if $p^2 \mid m$ whenever p is a prime factor of m. It is well known that if we write $\mathcal{S}(t) = \{m \le t \mid m \text{ is squarefull}\}$, then

$$\#\mathcal{S}(t) = \frac{\zeta(3/2)}{\zeta(3)} t^{1/2} + O(t^{1/3}),$$

where ζ is the Riemann Zeta-Function (see, for example, Theorem 14.4 in [7]). By partial summation, we easily get that

$$\sum_{\substack{m \ge t \\ m \text{ squarefull}}} \frac{1}{m} \ll \frac{1}{t^{1/2}}. \tag{5}$$

We now let $\mathcal{B}_2(x)$ be the set of positive integers $n \le x$ such that one of the following conditions holds:

(i) $\rho(n) \ge z$,

(ii) $p \mid n$ for some prime p such that $\rho(p \pm 1) \ge z$,

(iii) there exist primes r and p such that $p \mid n$, $p \equiv \pm 1 \pmod r$ and $\rho(r \pm 1) \ge z$.

We will find an upper bound for $\#\mathcal{B}_2(x)$. Let $\mathcal{B}_{2,1}(x)$ be the set of those $n \in \mathcal{B}_2(x)$ for which (i) holds. We note that for every $n \in \mathcal{B}_{2,1}(x)$ there exists a squarefull positive integer $d \ge z$ such that $d \mid n$. For a fixed d, the number of such $n \le x$ does not exceed x/d. Hence,

$$\#\mathcal{B}_{2,1}(x) \le \sum_{\substack{d \ge z \\ d \text{ squarefull}}} \frac{x}{d} \ll \frac{x}{(\log x)^{13}}, \tag{6}$$

where we have used estimate (5) with $t = z$. Now let $\mathcal{B}_{2,2}(x)$ be the set of those $n \in \mathcal{B}_2(x)$ for which (ii) holds. We note that each $n \in \mathcal{B}_{2,2}(x)$ has a prime divisor p such that $p \equiv \pm 1 \pmod d$, where d is as above. Given d and p, the number of such $n \le x$ does not exceed x/p. Summing up over all choices of p and d we get that

$$\begin{aligned}\#\mathcal{B}_{2,2}(x) &\le \sum_{\substack{d \ge z \\ d \text{ squarefull}}} \sum_{\substack{p \equiv \pm 1 \pmod d \\ p \le x}} \frac{x}{p} \ll x \sum_{\substack{d \ge z \\ d \text{ squarefull}}} \frac{\log\log x}{\phi(d)} \\ &\ll x(\log\log x)^2 \sum_{\substack{d \ge z \\ d \text{ squarefull}}} \frac{1}{d} \ll \frac{x(\log\log x)^2}{(\log x)^{13}}, \end{aligned} \tag{7}$$

where in the above estimates we used aside from estimate (5), the fact that the estimate

$$\sum_{\substack{p\equiv a \pmod b\\ p\le x}} \frac{1}{p} \le \frac{1}{p_1(a,b)} + O\left(\frac{\log\log x}{\phi(b)}\right) \tag{8}$$

holds uniformly in a, b and x when $b \le x$, where a and b are coprime and $p_1(a,b)$ is the smallest prime number $p \equiv a \pmod b$ (note that $p_1(1,b) \ge b+1$ and $p_1(-1,b) = p_1(b-1,b) \ge b-1 \ge \phi(b)$ for all $b \ge 2$), together with the well known minimal order $\phi(n)/n \gg 1/\log\log x$, valid for n in the interval $[1,x]$.

Let $\mathcal{B}_{2,3}(x)$ be the set of those $n \in \mathcal{B}_2(x)$ for which (iii) holds. Then there exists r such that $r \equiv \pm 1 \pmod d$ for some d as above, as well as $p \mid n$ such that $r \mid p-1$ or $r \mid p+1$. Given d, r and p, the number of such $n \le x$ does not exceed x/p, and now summing up over all choices of d, r and p, we get that

$$\begin{aligned}
\#\mathcal{B}_{2,3}(x) &\le \sum_{\substack{d\ge z\\ d \text{ squarefull}}} \sum_{\substack{r\equiv \pm 1 \pmod d\\ r\le x}} \sum_{\substack{p\equiv \pm 1 \pmod r\\ p\le x}} \frac{x}{p} \\
&\ll x \sum_{\substack{d\ge z\\ d \text{ squarefull}}} \sum_{\substack{r\equiv \pm 1 \pmod d\\ r\le x}} \frac{\log\log x}{\phi(r)} \\
&\ll x(\log\log x)^2 \sum_{\substack{d\ge z\\ d \text{ squarefull}}} \sum_{\substack{r\equiv \pm 1 \pmod d\\ r\le x}} \frac{1}{r} \\
&\ll x(\log\log x)^3 \sum_{\substack{d\ge z\\ d \text{ squarefull}}} \frac{1}{\phi(d)} \\
&\ll x(\log\log x)^4 \sum_{\substack{d\ge z\\ d \text{ squarefull}}} \frac{1}{d} \ll \frac{x(\log\log x)^4}{(\log x)^{13}},
\end{aligned} \tag{9}$$

where in the above estimates we used again estimate (5), estimate (8) twice as well as the minimal order of the Euler function on the interval $[1,x]$.

Hence, using estimates (6)–(9), we get

$$\#\mathcal{B}_2(x) \le \#\mathcal{B}_{2,1}(x) + \#\mathcal{B}_{2,2}(x) + \#\mathcal{B}_{2,3}(x) < \frac{x}{(\log x)^{10}} \qquad \text{for } x > x_0. \tag{10}$$

We now put

$$w = 10\log\log x$$

and set

$$S(w,x) = \sum_{\substack{\omega(m)\ge w\\ m\le x}} \frac{1}{m},$$

where $\omega(m)$ denotes the number of distinct prime factors of the positive integer m. Note that, using the fact that $\sum_{p\le t}\frac{1}{p}=\log\log t+O(1)$ and Stirling's formula $k!=(1+o(1))k^ke^{-k}\sqrt{2\pi k}$, we have

$$\begin{aligned}
S(w,x) &= \sum_{k\ge w}\sum_{\substack{\omega(m)=k\\ m\le x}}\frac{1}{m}=\sum_{k\ge w}\frac{1}{k!}\left(\sum_{p\le x}\sum_{\alpha\ge 1}\frac{1}{p^\alpha}\right)^k\\
&= \sum_{k\ge w}\frac{1}{k!}\left(\sum_{p\le x}\frac{1}{p}+O\left(\sum_{p\ge 2}\frac{1}{p^2}\right)\right)^k \ll \sum_{k\ge w}\frac{1}{k!}\left(\log\log x+O(1)\right)^k\\
&\le \sum_{k\ge w}\left(\frac{e\log\log x+O(1)}{k}\right)^k\le\sum_{k\ge w}\left(\frac{e\log\log x+O(1)}{w}\right)^k\\
&\ll \left(\frac{e\log\log x+O(1)}{w}\right)^w\ll\frac{1}{(\log x)^{10\log(10/e)}}<\frac{1}{(\log x)^{11}} \qquad (11)
\end{aligned}$$

for $x>x_0$ because $10\log(10/e)>11$.

We now let $\mathcal{B}_3(x)$ be the set of positive integers $n\le x$ such that one of the following conditions holds:

(i) $\omega(n)\ge w$,

(ii) $p\mid n$ for some prime p for which $\omega(p\pm 1)\ge w$,

(iii) there exist primes r and p such that $p\mid n$, $p\equiv\pm 1 \pmod r$ and $\omega(r\pm 1)\ge w$.

Let $\mathcal{B}_{3,1}(x)$, $\mathcal{B}_{3,2}(x)$ and $\mathcal{B}_{3,3}(x)$ be the sets of $n\in\mathcal{B}_3(x)$ for which (i), (ii) and (iii) hold, respectively.

To bound the cardinality of $\mathcal{B}_{3,1}(x)$, note that, using (11), we have

$$\#\mathcal{B}_{3,1}(x)=\sum_{\substack{\omega(n)\ge w\\ n\le x}}1\le\sum_{\substack{\omega(n)\ge w\\ n\le x}}\frac{x}{n}=xS(w,x)<\frac{x}{(\log x)^{11}} \qquad (12)$$

for $x>x_0$. To bound the cardinality of $\mathcal{B}_{3,2}(x)$, note that each $n\in\mathcal{B}_{3,2}(x)$ admits a prime divisor p such that $\omega(p\pm 1)\ge w$. Fixing such a p, the number of such $n\le x$ does not exceed x/p. Summing up over all such p we have, again in light of (11),

$$\begin{aligned}
\#\mathcal{B}_{3,2}(x) &\le \sum_{\substack{\omega(p\pm1)\ge w\\ p\le x}}\frac{x}{p}\le x\left(\sum_{\substack{\omega(p+1)\ge w\\ p+1\le x+1}}\frac{2}{p+1}+\sum_{\substack{\omega(p-1)\ge w\\ p-1\le x}}\frac{1}{p-1}\right)\\
&\le x(2S(w,x+1)+S(w,x))<3xS(w,x)+2\\
&\ll \frac{x}{(\log x)^{11}} \qquad (13)
\end{aligned}$$

for $x > x_0$. To bound the cardinality of $\mathcal{B}_{3,3}(x)$, note that for each $n \in \mathcal{B}_{3,3}(x)$ there exist some prime r with $\omega(r \pm 1) \ge w$ and some prime $p \mid n$ such that $p \equiv \pm 1 \pmod r$. Given such r and p, the number of such $n \le x$ does not exceed x/p. Summing up over all choices of r and p given above we get, again using (11),

$$\begin{aligned}
\#\mathcal{B}_{3,3}(x) &\le \sum_{\substack{\omega(r\pm 1)\ge w\\ r\le x}} \sum_{\substack{p\equiv \pm 1 \pmod r\\ p\le x}} \frac{x}{p} = x(\log\log x) \sum_{\substack{\omega(r\pm 1)\ge w\\ r\le x}} \frac{1}{\phi(r)} \\
&= x(\log\log x) \sum_{\substack{\omega(r\pm 1)\ge w\\ r\le x}} \frac{1}{r-1} \\
&\le x(\log\log x) \left(\sum_{\substack{\omega(r-1)\ge w\\ r-1\le x}} \frac{1}{r-1} + \sum_{\substack{\omega(r+1)\ge w\\ r+1\le x}} \frac{3}{r+1} \right) \\
&\le x(\log\log x)(S(w,x) + 3S(w,x+1)) \\
&\le 4x(\log\log x)S(w,x) + O(\log\log x) \ll \frac{x(\log\log x)}{(\log x)^{11}} \qquad (14)
\end{aligned}$$

for $x > x_0$.

Hence, using estimates (12) to (14), we get

$$\#\mathcal{B}_3(x) \le \#\mathcal{B}_{3,1}(x) + \#\mathcal{B}_{3,2}(x) + \#\mathcal{B}_{3,3}(x) < \frac{x}{(\log x)^{10}} \qquad \text{for } x > x_0. \qquad (15)$$

We now let

$$\mathcal{B}_4(x) = \{n \le x \mid n \notin (\mathcal{B}_1(x) \cup \mathcal{B}_2(x)) \text{ and } p^2 \mid \phi(n) \text{ for some } p > z\}.$$

Let $n \in \mathcal{B}_4(x)$ and let $p^2 \mid \phi(n)$ for some prime p. Then it is not possible that $p^2 \mid n$ (because $n \notin \mathcal{B}_2(x)$), nor is it possible that $p^2 \mid q-1$ for some prime factor q of n (again because $n \notin \mathcal{B}_2(x)$). Thus, there must exist distinct primes q and r dividing n such that $q \equiv 1 \pmod p$ and $r \equiv 1 \pmod p$. Fixing such p, q and r, the number of acceptable values of such $n \le x$ does not exceed $x/(qr)$. Summing up over all the possible values of p, q and r we arrive at

$$\begin{aligned}
\#\mathcal{B}_4(x) &\le \sum_{z\le p\le x} \sum_{\substack{q\equiv 1 \pmod p\\ r\equiv 1 \pmod p\\ q<r,\ qr\le x}} \frac{x}{qr} \le x \sum_{z\le p\le x} \frac{1}{2} \left(\sum_{\substack{q\equiv 1 \pmod p\\ q\le x}} \frac{1}{q} \right)^2 \\
&\ll x \sum_{z\le p\le x} \frac{(\log\log x)^2}{(p-1)^2} \ll x(\log\log x)^2 \sum_{z\le p\le x} \frac{1}{p^2} \\
&\ll \frac{x(\log\log x)^2}{(\log x)^{13}},
\end{aligned}$$

where we used estimates (8) and (5). Hence,

$$\#\mathcal{B}_4(x) < \frac{x}{(\log x)^{10}} \qquad \text{for all } x > x_0. \tag{16}$$

We now let

$$\tau = \exp\left((\log\log x)^2\right)$$

and let $\mathcal{B}_5(x)$ stand for the set of $n \le x$ which are multiples of a prime p for which either $p-1$ or $p+1$ has a divisor $d > \tau$ with $P(d) < z$. Fix such a pair d and p. Then the number of $n \le x$ divisible by p is at most x/p. This shows that

$$\#\mathcal{B}_5(x) \le \sum_{\substack{P(d)<z \\ \tau<d\le x}} \sum_{\substack{p\equiv\pm 1 \pmod d \\ p\le x}} \frac{x}{p} \ll x\log\log x \sum_{\substack{P(d)<z \\ \tau<d\le x}} \frac{1}{d}. \tag{17}$$

It follows easily by partial summation from the estimates (3) for $\Psi(x, v)$, that if we write $v = \log\tau/\log z$, then

$$S = \sum_{\substack{P(d)<z \\ \tau<d\le x}} \frac{1}{d} \le \frac{\log x}{\exp((1+o(1))v\log v)}.$$

Since $v = (\log\log x)/26$, we get that

$$v\log v = (1/26 + o(1))(\log\log x)(\log\log\log x)$$

and therefore that

$$S \le \frac{\log x}{\exp((1+o(1))v\log v)} < \frac{1}{(\log x)^{11}} \tag{18}$$

for all $x > x_0$, which together with estimate (17) gives

$$\#\mathcal{B}_5(x) < \frac{x}{(\log x)^{10}} \qquad \text{for } x > x_0. \tag{19}$$

Thus, setting

$$\mathcal{B}(x) = \bigcup_{i=1}^{5} \mathcal{B}_i(x), \tag{20}$$

we get, from estimates (4), (10), (15), (16) and (19) that

$$\#\mathcal{B}(x) \le \sum_{i=1}^{5} \#\mathcal{B}_i(x) \ll \frac{x}{(\log x)^{10}} \qquad \text{for all } x > x_0. \tag{21}$$

3 The Proof of Theorem 1

We find it convenient to prove a stronger theorem.

Theorem 2. *Let a and b be any fixed positive integers. Setting*

$$\mathcal{A}_{a,b} = \{n \mid \sigma(a\phi(n)) = b\sigma(n)\},$$

then the estimate

$$\#\mathcal{A}_{a,b}(x) \ll_{a,b} \frac{x}{(\log x)^2}$$

holds for all $x \geq 3$.

Proof. Let x be large and let $\mathcal{B}(x)$ be as in (20). We assume that $n \leq x$ is a positive integer not in $\mathcal{B}(x)$. We let $\mathcal{A}_1(x)$ be the set of $n \in \mathcal{A}_{a,b}(x)\backslash\mathcal{B}(x)$ for which $(P(n)-1)/2$ is not prime but such that there exists a prime number $r > z$ and another prime number $q \mid P(n) - 1$ for which $r \mid \gcd(P(n)+1, q+1)$. To count the number of such positive integers $n \leq x$, let r and q be fixed primes such that $r \mid q+1$, and let P be a prime such that $q \mid P-1$ and $r \mid P+1$. The number of positive integers $n \leq x$ such that $P(n) = P$ does not exceed x/P. Note that the congruences $P \equiv -1 \pmod r$ and $P \equiv 1 \pmod q$ are equivalent to $P \equiv a_{q,r} \pmod{qr}$, where $a_{q,r}$ is the smallest positive integer m satisfying $m \equiv -1 \pmod r$ and $m \equiv 1 \pmod q$. We distinguish two instances:

Case 1: $qr < P$.

Let $\mathcal{A}_1'(x)$ be the set of such integers $n \in \mathcal{A}_1(x)$. Then

$$\begin{aligned}
\#\mathcal{A}_1'(x) &\leq \sum_{z<r\leq x} \sum_{\substack{q\equiv -1 \pmod r\\ q\leq x}} \sum_{\substack{P\equiv a_{q,r} \pmod{qr}\\ qr<P\leq x}} \frac{x}{P}\\
&\ll x\log\log x \sum_{z<r\leq x} \sum_{\substack{q\equiv -1 \pmod r\\ q\leq x}} \frac{1}{\phi(qr)}\\
&\ll x\log\log x \sum_{z<r\leq x} \frac{1}{r} \sum_{\substack{q\equiv -1 \pmod r\\ q\leq x}} \frac{1}{q}\\
&\ll x(\log\log x)^2 \sum_{z<r\leq x} \frac{1}{r\phi(r)}\\
&\ll x(\log\log x)^2 \sum_{z<r} \frac{1}{r^2} \ll \frac{x(\log\log x)^2}{(\log x)^{11}},
\end{aligned} \tag{22}$$

where in the above inequalities we used estimates (8) and (5).

Case 2: $qr \geq P$.

Let $\mathcal{A}_1''(x)$ be the set of such integers $n \in \mathcal{A}_1(x)$. Here we write $n = Pm$. Note that $P > P(m)$ because $y > z$ for large x and $n \notin \mathcal{B}_1(x) \cup \mathcal{B}_2(x)$. Furthermore, since $r \mid q+1$, we may write $q = r\ell - 1$. Since $q \mid P-1$, we may write $P = sq+1 = s(r\ell-1)+1 = sr\ell+1-s$. Since $r \mid P+1$, we get that $1-s \equiv -1 \pmod r$, and therefore that $s \equiv 2 \pmod r$. Hence, there exists a nonnegative integer λ such that $s = \lambda r + 2$. If $\lambda = 0$, then $s = 2$ leading to

$P = 2q+1$, which is impossible. Thus, $\lambda > 0$. Let us fix the value of λ as well as that of r. Then

$$r\ell - 1 = q \qquad \text{and} \qquad (\lambda r^2 + 2r)\ell - (\lambda r + 1) = P \tag{23}$$

are two linear forms in the variable ℓ which are simultaneously primes. Note that $P \le x/m$ and since $P \ge \ell r(\lambda r + 2)$, we get that $\ell \le x/(mr(\lambda r+2))$. In particular, $mr(\lambda r+2) \le x$. Recall that a typical consequence of Brun's sieve (see for example Theorem 2.3 in [3]), is that if

$$L_1(m) = Am + B \qquad \text{and} \qquad L_2(m) = Cm + D$$

are linear forms in m with integer coefficients such that $AD - BC \neq 0$ and if we write E for the product of all primes p dividing $ABCD(AD-BC)$, then the number of positive integers $m \le y$ such that $L_1(m)$ and $L_2(m)$ are simultaneously primes is

$$\le \frac{Ky}{(\log y)^2}\left(\frac{E}{\phi(E)}\right)^2$$

for some absolute constant K. Applying this result for our linear forms in ℓ shown at (23) for which $A = r$, $B = -1$, $C = \lambda r^2 + 2r$ and $D = -(\lambda r + 1)$, we get that the number of acceptable values for ℓ does not exceed

$$\begin{aligned} & K\frac{x}{mr(\lambda r+2)\left(\log(x/(mr(\lambda r+2)))\right)^2}\left(\frac{(\lambda r+2)(\lambda r+1)r}{\phi((\lambda r+2)(\lambda r+1)r)}\right)^2 \\ \ll\ & \frac{x(\log\log x)^2}{mr(\lambda r+2)}, \end{aligned}$$

where for the rightmost inequality we used again the minimal order of the Euler function on the interval $[1, x]$. Here, K is some absolute constant. Summing up over all possible values of λ, r and m, we get

$$\begin{aligned} \#\mathcal{A}_1''(x) &\ll \sum_{z<r\le x}\sum_{1\le\lambda\le x}\sum_{1\le m\le x}\frac{x(\log\log x)^2}{mr(\lambda r+2)} \\ &< x(\log\log x)^2\left(\sum_{z<r\le x}\frac{1}{r^2}\right)\left(\sum_{1\le\lambda\le x}\frac{1}{\lambda}\right)\left(\sum_{1\le m\le x}\frac{1}{m}\right) \\ &\ll \frac{x(\log\log x)^2(\log x)^2}{(\log x)^{13}} = \frac{x(\log\log x)^2}{(\log x)^{11}}, \end{aligned} \tag{24}$$

where we used again estimate (5).

From estimates (22) and (24), we get

$$\#\mathcal{A}_1(x) \le \#\mathcal{A}_1'(x) + \mathcal{A}_1''(x) < \frac{x}{(\log x)^{10}} \qquad \text{for all } x > x_0. \tag{25}$$

Now let $\mathcal{A}_2(x)$ be the set of those $n \in \mathcal{A}_{a,b}(x)\backslash(\mathcal{A}_1(x)\cup\mathcal{B}(x))$ and such that $(P(n)-1)/2$ is not prime. With $n \in \mathcal{A}_2(x)$, we get that $n = Pm$, where $P > P(m)$ for $x > x_0$ because

$y > z$ for large x and $n \notin \mathcal{B}_1(x) \cup \mathcal{B}_2(x)$. Then $b\sigma(n) = b\sigma(m)(P+1)$. Let d_1 be the largest divisor of $P+1$ such that $P(d_1) \le z$. Then $P+1 = d_1\ell_1$, and $b\sigma(n) = b\sigma(m)d_1\ell_1$. Furthermore, $\phi(n) = \phi(m)(P-1)$, so that $a\phi(n) = a\phi(m)(P-1)$. Let d be the largest divisor of $P-1$ which is z-smooth; that is, with $P(d) \le z$. Then $P-1 = d\ell$. Note that ℓ is squarefree (because $n \notin \mathcal{B}_2(x)$) and ℓ and $\phi(m)$ are coprime (for if not, there would exist a prime $r > z$ such that $r \mid \ell$ and $r \mid \phi(m)$, so that $r^2 \mid \phi(n)$, which is impossible because $n \notin \mathcal{B}_4(x)$). Since $z > a$ for $x > x_0$, we get that $a\phi(m)d$ and ℓ are coprime, and therefore that $\sigma(\phi(n)) = \sigma(a\phi(m)d)\sigma(\ell)$. We thus get the equation

$$\sigma(a\phi(m)d)\sigma(\ell) = b\sigma(m)d_1\ell_1.$$

Now note that ℓ_1 and $\sigma(\ell)$ are coprime. Indeed, if not, since ℓ is squarefree, there would exist a prime factor r of ℓ_1 (necessarily exceeding z) dividing $q+1$ for some prime factor q of ℓ, that is of $P-1$. But this is impossible because $n \notin \mathcal{A}_1(x)$.

Thus, $\ell_1 \mid \sigma(a\phi(m)d)$. Note now that $Pm \le x$, so that $m \le x/P \le x/y$. Furthermore, $\max\{d, d_1\} \le \tau$ because $n \notin \mathcal{B}_5(x)$. Let us now fix m, d and d_1. Then $\ell_1 \mid \sigma(a\phi(m)d)$, and therefore the number of choices for ℓ_1 does not exceed $\tau(\sigma(a\phi(m)d))$, where $\tau(k)$ is the number of divisors of the positive integer k. The above argument shows that if we write $\mathcal{M}$ for the set of such acceptable values for m, then

$$\#\mathcal{A}_2(x) \le \frac{x\tau^2}{y} \max\{\tau(\sigma(a\phi(m)d)) \mid m \in \mathcal{M},\ d \le \tau,\ d_1 \le \tau\}. \tag{26}$$

To get an upper bound on $\tau(\sigma(a\phi(m)d))$, we write $a\phi(m)d$ as

$$a\phi(m)d = AB,$$

where A is squarefull, B is squarefree, and A and B are coprime. Clearly,

$$\sigma(a\phi(m)d) = \sigma(AB) = \sigma(A)\sigma(B),$$

so that

$$\tau(\sigma(a\phi(m)d)) \le \tau(\sigma(A))\tau(\sigma(B)).$$

Since B is squarefree, it is clear that

$$\sigma(B) \Big| \prod_{q \mid a\phi(m)d} (q+1).$$

Furthermore,

$$\omega(a\phi(m)d) \le \omega(a\phi(n)) \le \omega(a) + \omega(n) + \sum_{p \mid n} \omega(p-1) \le w^2 + w + O(1), \tag{27}$$

so that we have $\omega(a\phi(m)d) \le 2w^2$ if $x > x_0$. The above inequalities follow from the fact that $n \notin \mathcal{B}_3(x)$. Also, for each one of the at most $2w^2$ prime factors q of $a\phi(m)d$, the number $q+1$ has at most w prime factors itself and its squarefull part does not exceed z, again because

$n \notin \mathcal{B}_3(x)$. In conclusion, $\sigma(B) \mid C$, where C is a number with at most $2w^3$ prime factors whose squarefull part does not exceed z^{2w^2}. Thus,

$$\tau(\sigma(B)) \le \tau(C) \le 2^{\omega(C)}\tau(\rho(C)),$$

where $\omega(C) \le 2w^3$ and $\rho(C) \le z^{2w^2}$. Clearly,

$$\tau(\rho(C)) \le \rho(C) \le z^{w^2} = \exp(w^2 \log z) = \exp(O(\log\log x)^3),$$

and since also

$$2^{\omega(C)} \le 2^{2w^3} = \exp(O(\log\log x)^3),$$

we finally get that

$$\tau(\sigma(B)) = \exp(O(\log\log x)^3). \tag{28}$$

We now deal with $\sigma(A)$. We first note that $P(A) \le z$ for $x > x_0$. Indeed, if $q > z$ divides A, then $q^2 \mid a\phi(n)$. If $x > x_0$, then $z > a$, in which case the above divisibility relation forces $q^2 \mid \phi(n)$ which is not possible because $n \notin \mathcal{B}_3(x)$. By looking at the multiplicities of the prime factors appearing in A, we easily see that

$$A \left| \rho(a)\rho(d)\prod_{p|m}\rho(p-1)\left(\prod_{\substack{q \mid a\phi(m)d\\ q\le z}} q\right)^{\omega(a\phi(m)d)}\right. .$$

As we have seen at estimate (27), $\omega(a\phi(m)d) \le 2w^2$, in which case the above relation shows that

$$A \le \rho(a)z^{\omega(a\phi(m)d)+\omega(a\phi(m)d)^2} \ll z^{4w^4+2w^2} = \exp(O(\log\log x)^5).$$

But since $\sigma(A) < A^2$ and $\tau(\sigma(A)) \ll \sigma(A) \le A^2$, we get that

$$\tau(\sigma(A)) = \exp(O(\log\log x)^5), \tag{29}$$

which together with estimate (28) gives

$$\tau(\sigma(a\phi(m)d)) \le \tau(\sigma(A))\tau(\sigma(B)) = \exp(O(\log\log x)^5).$$

Returning to estimate (26), we get that

$$\begin{aligned} \#\mathcal{A}_2(x) &\le \frac{x\tau^2}{y}\exp(O(\log\log x)^5) = x\exp\left(-\frac{\log x}{\log\log x} + O((\log\log x)^5)\right) \\ &< \frac{x}{(\log x)^{10}} \qquad \text{for all } x > x_0. \end{aligned} \tag{30}$$

Thus, writing $\mathcal{C}(x) = \mathcal{A}_{a,b}(x)\backslash(\mathcal{A}_1(x) \cup \mathcal{A}_2(x) \cup \mathcal{B}(x))$, we get that

$$\mathcal{A}_{a,b}(x) = \#\mathcal{C}(x) + O\left(\frac{x}{(\log x)^{10}}\right). \tag{31}$$

Moreover, if $n \in \mathcal{C}(x)$, then $n = Pm$, with $P > P(m)$ for $x > x_0$, and $(P-1)/2$ is a prime. Hence, $P - 1 = 2q$, where q is a Sophie Germain prime. Since also $P \le x/m$, it follows by Brun's method that the number of such values for P is

$$\ll \frac{x}{m(\log x/m)^2} \le \frac{x}{m(\log y)^2} = \frac{x(\log\log x)^2}{m(\log x)^2}.$$

In the above inequalities we used the fact that $x/m \ge P \ge y$. Summing up over all $m \le x$, we get

$$\#\mathcal{C}(x) \ll \sum_{m \le x} \frac{x(\log\log x)^2}{m(\log x)^2} \ll \frac{x(\log\log x)^2}{\log x}. \tag{32}$$

In particular,

$$\#\mathcal{A}_{a,b}(x) \ll \frac{x(\log\log x)^2}{\log x}. \tag{33}$$

This is weaker than the bound claimed by our Theorem 2. However, it implies, by partial summation, that

$$\begin{aligned}\sum_{n \in \mathcal{A}_{a,b}(x)} \frac{1}{n} &\le 1 + \int_{2_-}^{x} \frac{1}{t} d(\#\mathcal{A}_{a,b}(t)) \\ &= 1 + \frac{\#\mathcal{A}_{a,b}(t)}{t}\Big|_{t=2_-}^{t=x} + O\left(\int_{2_-}^{x} \frac{t(\log\log t)^2}{t^2 \log t} dt\right) \\ &= O((\log\log x)^3). \end{aligned} \tag{34}$$

To get some improvement, we return to $\mathcal{C}(x)$. Let $n \in \mathcal{C}(x)$ and write it as $n = Pm$, where $P > P(m)$. Write also $P = 2q + 1$. Then $\phi(n) = 2\phi(m)q$. Moreover, $q > (y-1)/2 > z$ for $x > x_0$ so that q does not divide $a\phi(m)$ (because $q > a$ and $n \notin \mathcal{B}_3(x)$). Hence, $\sigma(a\phi(n)) = \sigma(2a\phi(m))(q+1)$. On the other hand, $b\sigma(n) = b\sigma(m)(P+1) = 2b\sigma(m)(q+1)$. Thus, the equation $\sigma(a\phi(n)) = \sigma(bn)$ forces $\sigma(2a\phi(m)) = 2b\sigma(m)$, implying that $m \in \mathcal{A}_{2a,2b}(x/P) \subset \mathcal{A}_{2a,2b}(x/y)$. The argument which leads to estimate (32) now provides the better estimate

$$\#\mathcal{C}(x) \ll \sum_{m \in \mathcal{A}_{2a,2b}(x)} \frac{x(\log\log x)^2}{m(\log x)^2} \ll \frac{x(\log\log x)^5}{(\log x)^2}.$$

In particular, we get

$$\#\mathcal{A}_{a,b}(x) \ll \frac{x(\log\log x)^5}{(\log x)^2}. \tag{35}$$

This is still somewhat weaker than what Theorem 2 claims. However, it implies that the sum of the reciprocals of the numbers in $\mathcal{A}_{a,b}$ is convergent. In fact, by partial summation, it follows, as in estimates (34), that

$$\begin{aligned}\sum_{\substack{n\in\mathcal{A}_{a,b}\\ n\ge y}} \frac{1}{n} &\le \int_{y-}^{\infty} \frac{1}{t} d(\#\mathcal{A}_{a,b}(t)) \\ &= \frac{\#\mathcal{A}_{a,b}(t)}{t}\Big|_{t=y-}^{t=\infty} + O\left(\int_{y-}^{x} \frac{t(\log\log t)^5}{t^2(\log t)^2} dt\right) \\ &\ll \frac{(\log\log y)^5}{(\log y)^2} + \int_{y-}^{\infty} \frac{1}{t(\log t)^{3/2}} dt \\ &\ll \frac{1}{(\log y)^{1/2}}. \end{aligned} \tag{36}$$

We now take another look at $\mathcal{C}(x)$. Let again $n \in \mathcal{C}(x)$ and write $n = Pm$. Fixing m and using the fact that $P \le x/m$ is a prime such that $(P-1)/2$ is also a prime, we get that the number of choices for P is

$$\ll \frac{x}{m(\log(x/m))^2}.$$

Hence,

$$\#\mathcal{C}(x) \ll \sum_{m\in\mathcal{A}_{2a,2b}(x/y)} \frac{x}{m(\log(x/m))^2}.$$

We now split the above sum at $m = x^{1/2}$ and use estimate (36) to get

$$\begin{aligned}\#\mathcal{C}(x) &\le \sum_{m\in\mathcal{A}_{2a,2b}(x^{1/2})} \frac{x}{m(\log(x/m))^2} + \sum_{\substack{m\in\mathcal{A}_{2a,2b}\\ x^{1/2}\le m\le x/y}} \frac{x}{m(\log(x/m))^2} \\ &\le \frac{x}{(\log x^{1/2})^2} \sum_{m\in\mathcal{A}_{2a,2b}} \frac{1}{m} + \frac{x}{(\log y)^2} \sum_{\substack{m\in\mathcal{A}_{2a,2b}\\ m\ge x^{1/2}}} \frac{1}{m} \\ &\ll \frac{x}{(\log x)^2} + \frac{x}{(\log y)^{5/2}} = \frac{x}{(\log x)^2} + \frac{x(\log\log x)^{5/2}}{(\log x)^{5/2}} \\ &\ll \frac{x}{(\log x)^2}, \end{aligned} \tag{37}$$

which together with estimate (31) leads to the desired conclusion of Theorem 2. □

4 Acknowledgments

This work was done in April of 2006, while the second author was in residence at the Centre de recherches mathématiques of the Université de Montréal for the thematic year *Analysis and Number Theory*. This author thanks the organizers for the opportunity of participating in this program. He was also supported in part by grants SEP-CONACyT 46755, PAPIIT IN104505 and a Guggenheim Fellowship. The first author was supported in part by a grant from NSERC.

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2000 *Mathematics Subject Classification*: Primary 11A25; Secondary 11N37.
Keywords: Euler function, sum of divisors.

(Concerned with sequences A033631 and A137733.)

Received June 26 2007; revised version received February 9 2008. Published in *Journal of Integer Sequences*, February 9 2008.

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