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Integers divisible by sums of powers of their prime factors

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Abstract

For each positive integer j, let $\beta_j(n) := \sum_{p|n} p^j$. Given a fixed positive integer k, we show that there are infinitely many positive integers n having at least two distinct prime factors and such that $\beta_j(n) \mid n$ for each $j \in \{1, 2, \ldots, k\}$.

1. Introduction

Recently, the authors [2] estimated the counting function $B(x)$ of the set of integers $n \leqslant x$ which are not prime powers and which are divisible by the sum of their prime factors. They showed that, for x sufficiently large, there exist positive constants c_1 and c_2 such that

$$x \exp\{-c_1(1 + o(1))\ell(x)\} < B(x) < x \exp\{-c_2(1 + o(1))\ell(x)\},$$

where $\ell(x) := \sqrt{\log x \log\log x}$.

In this paper, we consider a smaller set. Indeed, for each positive integer j, let $\beta_j(n) := \sum_{p|n} p^j$. Given an integer $k \geqslant 2$, we are interested in the set of positive integers n having at

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least two distinct prime factors and such that $\beta_j(n) \mid n$ for each $j \in \{1, 2, \dots, k\}$. We have the following result.

Theorem 1. *For any positive integer k, there exist infinitely many positive integers n which are not prime powers and such that $\beta_j(n) \mid n$ for all $j = 1, \dots, k$.*

Let k be a large positive integer and set $s = k^3$. Throughout, $\varepsilon > 0$ is a small real number which depends on k. We use the Landau symbols O and o as well as the Vinogradov symbols $\ll$ and $\gg$ with their usual meanings. The constants and convergence implied by them might depend on k and ε.

2. Preliminary results

The following theorem can be easily deduced from Theorem 16 in Hua's book [4], p. 139.

Theorem A *(Hua). Given an integer $k \geqslant 2$, let s be an integer which is $\geqslant s_0$, where s_0 is defined according to the following table*:

k	2	3	4	5	6	7	8	9	10	$\geqslant 11$
s_0	7	19	49	113	243	417	675	1083	1773	$2k^2(3\log k + \log\log k + 4) - 21$

Let $N_1 < \cdots < N_k$ be positive integers and let also $I(N_1, \dots, N_k)$ be the set of prime solutions $p_1, \dots, p_s$ of the system of equations

$$\begin{cases} p_1 + p_2 + \cdots + p_s = N_1, \\ p_1^2 + p_2^2 + \cdots + p_s^2 = N_2, \\ \vdots \\ p_1^k + p_2^k + \cdots + p_s^k = N_k. \end{cases} \tag{1}$$

Set $P = N_k^{1/k}$. Then

$$\#I(N_1, \dots, N_k) = \frac{b_1 P^{s-\frac{1}{2}k(k+1)} \mathcal{G}(N_1, \dots, N_k)}{(\log P)^s} + \mathrm{O}\left(\frac{P^{s-\frac{1}{2}k(k+1)}}{(\log P)^{s+1}} \log\log P\right),$$

where $b_1 = b_1(N_1, \dots, N_k)$ is a non negative constant encoding the solvability in positive real numbers of system (1) *and $\mathcal{G}(N_1, \dots, N_k)$, called the singular series, encodes the condition of congruence solvability of system* (1) (*these quantities being given explicitly on pp.* 139 *and* 140 *of Hua's book* [4]).

In what follows, we will give some sufficient conditions on the parameters $N_1, \dots, N_k$ which, using Theorem A, will ensure that system (1) admits solutions in prime numbers which furthermore are in appropriate arithmetic progressions.

3. The condition of positive solvability

For $j = 1, 2, \ldots, s$, let

$$c_j := \sum_{i=1}^{s} \frac{1}{i^j}.$$

For a large positive real number X and an arbitrarily small $\varepsilon > 0$, let $I_{j,\varepsilon}(X)$ be the open interval

$$I_{j,\varepsilon}(X) := \big(c_j X^j (1-\varepsilon), c_j X^j (1+\varepsilon)\big), \quad k = 1, 2, \ldots, s.$$

Moreover, for $j = 1, 2, \ldots, s$, let $N_j \in I_{j,\varepsilon}(X)$ be positive integers and set

$$\delta_j := \frac{N_j}{X^j}, \quad k = 1, 2, \ldots, s,$$

and observe that $\delta_j - c_j = \mathrm{O}(\varepsilon)$ for $k = 1, 2, \ldots, s$.

The argument on p. 159 in Hua's book [4] shows that if we consider the equations

$$X_\ell = x_1^\ell + \cdots + x_s^\ell - \delta_\ell = 0, \quad 0 \leqslant x_\nu \leqslant 1, \ \nu = 1, \ldots, s, \ 1 \leqslant \ell \leqslant k,$$

where the x_ν's are distinct real numbers, then

$$b_1 = \underset{0 \leqslant x_\ell \leqslant 1, \ X_\ell = 0}{\int_0^1 \cdots \int_0^1} \frac{dx_{k+1} \cdots dx_s}{k! \prod_{1 \leqslant i < j \leqslant k} |x_i - x_j|},$$

where the index ℓ runs in the set $1, \ldots, k$.

Note that, by continuity and from the way we have chosen the integers $N_1, \ldots, N_k$, there exists a constant C_1 depending on k, such that, for every

$$(x_{k+1}, \ldots, x_s) \in \left(\frac{1}{k+1}(1 - C_1\varepsilon), \frac{1}{k+1}(1 + C_1\varepsilon)\right) \times \cdots \times \left(\frac{1}{s}(1 - C_1\varepsilon), \frac{1}{s}(1 + C_1\varepsilon)\right),$$

there exists $(x_1, \ldots, x_k) \in (1-\varepsilon, 1+\varepsilon) \times (\frac{1}{2}(1-\varepsilon), \frac{1}{2}(1+\varepsilon)) \times \cdots \times (\frac{1}{k}(1-\varepsilon), \frac{1}{k}(1+\varepsilon))$ such that $X_j = 0$ for $j = 1, 2, \ldots, k$.

The above argument now easily implies that there exists a positive constant C_2 depending only k such that if ε is sufficiently small, say $0 < \varepsilon < \varepsilon_0(k)$, then

$$b_1 \geqslant b'_{1,\varepsilon} := \int_{I_{k+1,C_1\varepsilon}(1)} \cdots \int_{I_{s,C_1\varepsilon}(1)} \frac{dx_{k+1} \cdots dx_s}{k! \prod |x_i - x_j|} > C_2 \varepsilon^s.$$

Note that the above lower bound does not depend on X.

Note also that our choice of parameters implies that the real solutions encoded in the multiple integral representing $b'_{1,\varepsilon}$ are of the form $(x_1, \ldots, x_s)$ with x_i and x_j being "far apart", because $x_j/x_1 = 1/j \times (1 + \mathrm{O}(\varepsilon))$ for $j = 1, 2, \ldots, s$.

4. Singular series

Given an integer $k \geqslant 2$, let

$$M = \prod_{p \leqslant k^{2k}} p^{k^3}. \tag{2}$$

Assume that ℓ is a positive integer depending on k such that the system of congruences

$$\begin{cases} y_1 + y_2 + \cdots + y_k \equiv N_1 - (s-k) \pmod{p^\ell}, \\ y_1^2 + y_2^2 + \cdots + y_k^2 \equiv N_2 - (s-k) \pmod{p^\ell}, \\ \qquad \vdots \qquad\qquad\qquad \vdots \\ y_1^k + y_2^k + \cdots + y_k^k \equiv N_k - (s-k) \pmod{p^\ell}, \end{cases} \tag{3}$$

with $1 \leqslant y_j \leqslant p^\ell - 1$, $p \nmid y_j$ for $j = 1, 2, \ldots, k$, and furthermore $p^\ell \,\|\, \prod_{1 \leqslant i<j \leqslant k}(y_i - y_j)$, admits at least one solution $(y_1, \ldots, y_k)$ for all $p \mid M$. Then the argument from Section 11.3 of Hua's book [4] shows that $\mathcal{G}(N_1, \ldots, N_k)$ is bounded below by a constant C_3 depending only on k. Furthermore, note that under our assumptions, the primes p_i for $i = k+1, \ldots, s$, may be assumed to be congruent to 1 modulo M.

Now, for each prime divisor p of M and each positive integer $j \leqslant k$, let $a_{j,p} = 1 + p^j$. Moreover, let $n_j \in \{0, \ldots, M-1\}$ be the congruence class modulo M such that

$$n_j \equiv s - k + \sum_{i=1}^{k} a_{i,p}^j \pmod{p^{k^3}} \quad \text{for all } j = 1, \ldots, k, \text{ and } p \mid M. \tag{4}$$

Note that these exist and are unique by the Chinese Remainder Theorem.

We note that if $N_j \equiv n_j \pmod{M}$, the system (3) admits the solution $y_i \equiv a_{i,p} \pmod{p^\ell}$ for $i = 1, \ldots, k$, and that the exact order at which p appears in $\prod_{1 \leqslant i<j \leqslant k}(a_{i,p} - a_{j,p})$ is precisely $k(k^2-1)/6$. This allows us to take $\ell = k(k^2-1)/6$.

To summarize the result obtained so far and using the notation introduced in this section, we have established the following theorem.

Theorem 2. *Let k be large, $s = k^3$ and $\varepsilon = \varepsilon(k) > 0$ be sufficiently small. Assume that X is large and $N_1, \ldots, N_k$ are positive integers in the intervals $I_{1,\varepsilon}(X), \ldots, I_{k,\varepsilon}(X)$, which also satisfy $N_j \equiv n_j \pmod{M}$ for $j = 1, \ldots, k$. Then there exists a positive constant C_4, depending on k but not on ε, such that if X is sufficiently large, depending on the choice of both k and ε, the system of equations* (1) *admits a prime solution $p_1, \ldots, p_s$ such that $p_i \equiv 1 \pmod{M}$ for $i = 1, \ldots, s$, and furthermore $|p_j/p_1 - 1/j| < C_4\varepsilon$ for $j = 1, \ldots, s$.*

We shall now be using the powerful Theorem 2 to prove Theorem 1.

Proof of Theorem 1. Let M be as in (2) and again let $s = k^3$. Let b_j be a positive integer with $2M - n_j + 1 + 2(-1)^{j-1}$ prime factors all congruent to 1 modulo M for all $j = 1, \ldots, k$, where the n_j's are the ones appearing in (4). Furthermore, let $\varepsilon > 0$ be small and assume that

$$b_j^j \in I_{j,\varepsilon}(b_1/c_1), \quad j = 1, 2, \ldots, k. \tag{5}$$

It is easy to see that such numbers exist provided b_1 is chosen sufficiently large with respect to k and ε.

Choose a number m of the form $m = pq$, where both p and q are congruent to -1 modulo M and moreover $p - q = \mathrm{O}(p/\log p)$. Assume furthermore that $p \in [\sqrt{b_1}, 2\sqrt{b_1}\,]$. It is clear that such primes p and q exist if b_1 is large enough. Now let

$$N_j := b_j^j \cdot m^j - \beta_j\bigl(b_j^j \cdot m^j\bigr), \quad j = 1, 2, \ldots, k \tag{6}$$

and observe that

$$N_j \equiv 1 - 2(-1)^j - \omega(b_j) \equiv n_j \pmod{M}, \quad j = 1, 2, \ldots, k.$$

Here, $\omega(n)$ stands for the number of distinct prime factors of n.

Furthermore, observe that in light of (6), $m = N_1/b_1(1 + \mathrm{O}(\varepsilon))$, and also that, in light of (5), $(b_j c_1)^j/(c_j b_1^j) = 1 + \mathrm{O}(\varepsilon)$. Hence it follows from (6) that, by choosing $X = N_1/c_1$,

$$N_j = b_j^j m^j \left(1 + \mathrm{O}\left(\frac{1}{X^{j/2}}\right)\right) = \frac{b_j^j c_1^j}{b_1^j} X^j \left(1 + \mathrm{O}\left(\varepsilon + \frac{1}{X^{1/2}}\right)\right) = c_j X^j\bigl(1 + \mathrm{O}\bigl(\varepsilon + X^{-1/2}\bigr)\bigr)$$

holds for $j = 1, \ldots, k$, as X becomes large. From Theorem 2, it follows that if ε is sufficiently small and X is large enough, then there exist prime numbers $p_1, \ldots, p_s$ such that $N_j = \sum_{i=1}^{s} p_i^j$ and such that also $p_i \equiv 1 \pmod{M}$ for all $i = 1, \ldots, s$, and $p_i \gg X$ for all $i = 1, \ldots, s$.

Since the prime factors of m are congruent to -1 modulo M, it follows that $p_i \nmid m$ for $i = 1, \ldots, s$ and also that if X is sufficiently large, then $p_i \nmid b_j$ for all $i = 1, \ldots, s$ and $j = 1, \ldots, k$.

Finally, observing that the number

$$n = \prod_{j=1}^{k} b_j^j \cdot m^k \cdot \prod_{i=1}^{s} p_i$$

satisfies $\beta_j(n) \mid n$ for all $j = 1, \ldots, k$, Theorem 1 is proved. $\square$

5. Further remarks

In Hua's book [4], page 157, it is shown that one may choose $s \gg k^2 \log k$ once the arithmetic conditions and the conditions for positive solvability are satisfied. We only took k large and $s = k^3$, so that the above inequality is clearly met. Note that there is recent work by Arkhipov and Chubarikov [1] in which they improve somewhat upon Hua's work.

Now let

$$\kappa(x) = \max\bigl\{k \geqslant 1\colon\ \beta_j(n) \mid n \text{ for all } j = 1, \ldots, k \text{ for some } n \leqslant x \text{ with } \omega(n) \geqslant 2\bigr\}.$$

Our Theorem 1 shows that $\kappa(x) \to \infty$ as $x \to \infty$. One may inquire about the growth rate of $\kappa(x)$. A lower bound on $\kappa(x)$ could be deduced from the proofs of our Theorems 1 and 2 provided that the constants in Theorem A are made explicit. A trivial upper bound is $\kappa(x) < (\log x)/(\log 3)$

and follows by observing that if $n \leqslant x$ is such that $\beta_k(n) \mid n$ for $k = \kappa(x)$ then, since $\omega(n) \geqslant 2$, we have

$$3^k < \beta_k(n) \leqslant n \leqslant x,$$

so that $k < (\log x)/(\log 3)$. In this section, we give a nontrivial upper bound for $\kappa(x)$.

Theorem 3. *The estimate $\kappa(x) \leqslant (\log x)/(4 \log\log\log x)$ holds for all sufficiently large x.*

Proof. Let x be large, put $k = \kappa(x)$ and let $n \leqslant x$ with $\omega(n) \geqslant 2$ be such that $\beta_j(n) \mid n$ for $j = 1, \ldots, k$. Let $n = p_1^{\alpha_1} \cdots p_t^{\alpha_t}$, where $p_1 < \cdots < p_t$ are primes and α_i are positive integers for $i = 1, \ldots, t$. Let $y = p_t$. Since

$$y^k = p_t^k \leqslant \beta_k(n) \leqslant n \leqslant x,$$

we get that $k \leqslant \log x / \log y$. Moreover, note that each one of the relations

$$p_1^j + p_2^j + \cdots + p_t^j - \beta_j(n) = 0$$

provides a solution to the equation

$$x_1 + x_2 + \cdots + x_{t+1} = 0 \tag{7}$$

in unknowns $x_1, \ldots, x_{t+1}$ which are integers all the prime factors of which are contained in the set $\{p_1, \ldots, p_t\}$ and $\gcd(x_1, \ldots, x_{t+1}) = 1$. Furthermore, $t + 1 \geqslant 3$, and $x_i > 0$ for all $i = 1, \ldots, t$, implying that all such solutions are *non-degenerate* in the sense that $\sum_{i \in I} x_i \neq 0$ if I is any proper subset of $\{1, 2, \ldots, t\}$. A result of Evertse [3] shows that the total number of such solutions of (7) does not exceed

$$\left(2^{35}(t+1)^2\right)^{(t+1)^4},$$

from which it follows that

$$k \leqslant \left(2^{35}(t+1)^2\right)^{(t+1)^4}. \tag{8}$$

Assume first that $y \leqslant (\log\log x)^{1/4}$. Then since $t \leqslant \pi(y)$, estimate (8) on the size of k shows that

$$k \leqslant \exp\bigl(\mathrm{O}(t^4 \log t)\bigr) = \exp\bigl(\mathrm{O}\bigl(\pi(y)^4 \log \pi(y)\bigr)\bigr) = \exp\bigl(\mathrm{o}(y^4)\bigr) = (\log x)^{\mathrm{o}(1)} \quad (x \to \infty),$$

allowing us to conclude that, in this case, $k < (\log x)/(4 \log\log\log x)$ for all sufficiently large x. On the other hand, if $y > (\log\log x)^{1/4}$, then $k < (\log x)/(\log y) < (\log x)/(4 \log\log\log x)$, thereby covering the other case.

6. Numerical data

For each positive integer k, let $\widetilde{n_k}$ be the smallest positive integer n with $\omega(n) \geqslant 2$ and such that $\beta_j(n) \mid n$ for $j = 1, 2, \ldots, k$.

In the following table, we give the values of $\widetilde{n_1}$, $\widetilde{n_2}$ and $\widetilde{n_3}$, and also what we believe to be the values of $\widetilde{n_4}$ and $\widetilde{n_5}$.

k	$n = \widetilde{n_k}$	$\beta_i(n)$
1	$30 = 2 \cdot 3 \cdot 5$	$\beta_1(n) = 10 = 2 \cdot 5$
2	$99528 = 2^3 \cdot 3 \cdot 11 \cdot 13 \cdot 29$	$\beta_1(n) = 58 = 2 \cdot 29$ $\beta_2(n) = 1144 = 2^3 \cdot 11 \cdot 13$
3	$12192180 = 2^2 \cdot 3 \cdot 5 \cdot 7^2 \cdot 11 \cdot 13 \cdot 29$	$\beta_1(n) = 70 = 2 \cdot 5 \cdot 7$ $\beta_2(n) = 1218 = 2 \cdot 3 \cdot 7 \cdot 29$ $\beta_3(n) = 28420 = 2^2 \cdot 5 \cdot 7^2 \cdot 29$
4	$\widetilde{n_4} \leqslant n = 2078479331940068525081053440$ $= 2^8 \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 31 \cdot 37$ $\cdot 41^2 \cdot 47 \cdot 53 \cdot 61 \cdot 71 \cdot 83 \cdot 89$ (a 28-digit number)	$\beta_1(n) = 2 \cdot 3^3 \cdot 11$ $\beta_2(n) = 2^2 \cdot 7 \cdot 17 \cdot 71$ $\beta_3(n) = 2^3 \cdot 3 \cdot 31 \cdot 37 \cdot 83$ $\beta_4(n) = 2^8 \cdot 17 \cdot 23 \cdot 41^2$
5	$\widetilde{n_5} \leqslant n = 2^4 \cdot 3 \cdot 5 \cdot 7^4 \cdot 11 \cdot 13 \cdot 23 \cdot 29 \cdot 31 \cdot 41$ $\cdot 43 \cdot 47 \cdot 53 \cdot 67 \cdot 73 \cdot 79 \cdot 83 \cdot 89 \cdot 97$ $\cdot 101 \cdot 103 \cdot 107 \cdot 109 \cdot 113 \cdot 127 \cdot 131$ $\cdot 137 \cdot 151 \cdot 163 \cdot 167 \cdot 173 \cdot 179$ $\cdot 181 \cdot 191 \cdot 199 \cdot 211 \cdot 223$ (a 70-digit number)	$\beta_1(n) = 2 \cdot 13 \cdot 137$ $\beta_2(n) = 2^3 \cdot 3 \cdot 7 \cdot 41 \cdot 73$ $\beta_3(n) = 2^4 \cdot 7^4 \cdot 13 \cdot 163$ $\beta_4(n) = 2^2 \cdot 3 \cdot 7 \cdot 47 \cdot 109 \cdot 173 \cdot 191$ $\beta_5(n) = 2^3 \cdot 11 \cdot 89 \cdot 97 \cdot 127 \cdot 151 \cdot 179$

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