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Counting the number of twin Niven numbers

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Abstract Given an integer $q \geq 2$, we say that a positive integer is a *q-Niven number* if it is divisible by the sum of its digits in base q. Given an arbitrary integer $r \in [2, 2q]$, we say that $(n, n+1, \dots, n+r-1)$ is a *q-Niven r-tuple* if each number $n+i$, for $i = 0, 1, \dots, r-1$, is a q-Niven number. We show that there exists a positive constant $c = c(q, r)$ such that the number of q-Niven r-tuples whose leading component is $< x$ is asymptotic to $cx/(\log x)^r$ as $x \to \infty$.

Keywords Niven numbers

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1 Introduction

A positive integer n is said to be a *Niven number* (or a Harshad number) if it is divisible by the sum of its decimal digits. For instance, 24 is a Niven number, while 25 is not. More generally, given an integer $q \geq 2$, we say that a positive integer is a

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q-Niven number if it is divisible by the sum of its digits in base q. Let $\mathcal{N}_q(x)$ stand for the number of q-Niven numbers $< x$.

In [4], we established that, as $x \to \infty$,

$$\mathcal{N}_q(x) = (\eta_q + o(1))\frac{x}{\log x} \quad \text{with } \eta_q = \frac{2\log q}{(q-1)^2}\sum_{j=1}^{q-1}(j, q-1).$$

Independently, Mauduit, Pomerance and Sárközy [6] obtained the same result, but with an error term.

We shall say that the positive integers $n, n+1$ are *twin Niven numbers* in base q if they are both q-Niven numbers. More generally, given an integer $r \geq 2$, we shall say that $(n, n+1, \ldots, n+r-1)$ is a *q-Niven r-tuple* if each number $n+i$, for $i = 0, 1, \ldots, r-1$, is a q-Niven number.

In 1993, Cooper and Kennedy [2] showed that it is not possible to have a sequence of more than 20 consecutive Niven numbers. In 1994, Grundman [5] showed that, given any integer $q \geq 2$, no q-Niven r-tuple, with $r > 2q$, exists. Moreover, she conjectured that, for each integer r, $2 \leq r \leq 2q$, there exist infinitely many q-Niven r-tuples. In 1996, Cai [1] proved that, if $q = 2$ or 3, this conjecture is true and in 1997, Wilson [7] showed that the conjecture is true for any integer $q \geq 2$. In this paper, we prove a quantitative version of this.

Given fixed integers $q \geq 2$ and $2 \leq r \leq 2q$, let $\mathcal{N}_q^{(r)}(x)$ stand for the number of q-Niven r-tuples whose leading component is $< x$. Recently, the first two authors [3] proved that

$$\frac{x}{\log^2 x} \ll T(x) \ll \frac{x \log\log x}{\log^2 x},$$

where $T(x) := \mathcal{N}_{10}^{(2)}(x)$ stands for the number of twin Niven numbers $n, n+1$ such that $n < x$. Here, we show not only that one may remove the "$\log\log x$" factor, but also that one can obtain an asymptotic expression for $T(x)$. More generally, we prove that, given $q \geq 2$ and $2 \leq r \leq 2q$, there exists a positive constant $c = c(q, r)$ such that $\mathcal{N}_q^{(r)}(x)$ is asymptotic to $cx/(\log x)^r$ as $x \to \infty$.

2 Main result

Theorem *Given fixed integers $q \geq 2$ and $2 \leq r \leq 2q$, then, as $x \to \infty$, there exists a constant $c = c(q, r)$ such that*

$$\mathcal{N}_q^{(r)}(x) = (c + o(1))\frac{x}{\log^r x}.$$

3 Notations and preliminary results

Let $\mathbf{N}$ and $\mathbf{N}_0$ stand for the set of positive integers and non negative integers, respectively. Given r non negative integers $x_1, x_2, \ldots, x_r$, we denote by $[x_1, x_2, \ldots, x_r]$ the least common multiple of those $x_i's$ which are not zero, and for $j \in \mathbf{N}_0$, we set

$$\Lambda_j := \Lambda_j(x_1, x_2, \ldots, x_r) = [x_1 + j, x_2 + j, \ldots, x_r + j].$$

We let as usual $\tau(n)$ and $\omega(n)$ stand for the number of positive divisors of n and the number of distinct prime divisors of n, respectively.

Throughout this paper, $q \geq 2$ is a fixed integer. The *q-ary expansion* of a non negative integer n is defined as the unique sequence $\epsilon_0(n), \epsilon_1(n), \epsilon_2(n), \ldots$ for which

$$n = \sum_{j=0}^{\infty} \epsilon_j(n) q^j, \quad \epsilon_j(n) \in \{0, 1, 2, \ldots, q-1\}.$$

Let $\alpha(n) = \alpha_q(n)$ be the sum of the digits of n in base q, that is

$$\alpha(n) = \epsilon_0(n) + \epsilon_1(n) + \epsilon_2(n) + \cdots.$$

Given a real number $x > 1$, we set $N_x = N_{q,x} := [\frac{\log x}{\log q}]$, where as usual $[y]$ stands for the largest integer not exceeding y, and set also $t_x = t_{q,x} := (q-1)N_x$, the maximum possible value of $\alpha(n)$ for $n < x$. As in [4], given $k \in \mathbf{N}$, $t \in \mathbf{N}_0$ and an integer ℓ, we set

$$A(x|k, \ell, t) := \#\{0 \leq n < x : n \equiv \ell \pmod{k} \text{ and } \alpha(n) = t\}.$$

Now consider the positive integers t satisfying

$$\log x \ll t \ll \log x$$

and, for such t's, define

$$L_x(t) := \frac{1}{\sqrt{N_x}} \varphi\left(\frac{t - \frac{q-1}{2} N_x}{\sigma \sqrt{N_x}}\right),$$

where $\sigma^2 := \frac{1}{q} \sum_{j=1}^{q-1} j^2 - (\frac{q-1}{2})^2 = \frac{q^2-1}{12}$ and $\varphi(y) := \frac{1}{\sqrt{2\pi}} e^{-y^2/2}$ is the density function of the Gaussian law.

Lemma 1 *There exists a constant κ depending only on q such that if $\ell \equiv t \pmod{(k, q-1)}$*

$$\left| A(x|k, \ell, t) - \frac{(q-1, k)}{k} A(x|1, 0, t) \right| < \frac{\kappa}{k} A(x|1, 0, t) (\log x)^{-1/8},$$

and $A(x|k, \ell, t) = 0$ otherwise.

Proof This is Theorem C of Mauduit, Pomerance and Sárközy [6]. □

Lemma 2 *Consider the interval*

$$I = I(x, q) := \left[\frac{q-1}{2} N_x - (N_x)^{5/8}, \frac{q-1}{2} N_x + (N_x)^{5/8}\right].$$

Then, for any fixed integer $r \geq 2$,

$$\sum_{t \in I} \sum_{\substack{0 \leq n < x \\ \alpha(n)=t}} 1 = x + O\left(\frac{x}{\log^{2r} x}\right).$$

Proof This follows immediately from Lemma 4 of Mauduit, Pomerance and Sárközy [6]. □

Lemma 3 *Let x be a positive real number and $t \in I$. Then*

$$A(x|1,0,t) = xL_x(t)\left(1 + O\left((\log x)^{-1/8}\right)\right).$$

Proof This is Lemma 5 of Mauduit, Pomerance and Sárközy [6]. □

Lemma 4 *Given r distinct positive integers $x_1, x_2, \ldots, x_r$ and $H \in \mathbf{N}$ such that $\max_{1 \leq i < j \leq r} |x_i - x_j| \leq H$, then*

$$[x_1, x_2, \ldots, x_r] \geq \frac{x_1 \cdot x_2 \cdots x_r}{H^{\frac{r(r-1)}{2}}}.$$

Proof We prove this result by induction on r. The result is certainly true for $r = 2$, since

$$[x_1, x_2] = \frac{x_1 x_2}{(x_1, x_2)} \geq \frac{x_1 x_2}{|x_1 - x_2|} \geq \frac{x_1 x_2}{H}.$$

Hence, let us show that if it holds for a certain integer $r \geq 2$, then it will hold for $r + 1$. Indeed,

$$\begin{aligned}
[x_1, x_2, \ldots, x_r, x_{r+1}] &= [[x_1, x_2, \ldots, x_r], x_{r+1}] \\
&= \frac{[x_1, x_2, \ldots, x_r] x_{r+1}}{([x_1, x_2, \ldots, x_r], x_{r+1})} \\
&= \frac{[x_1, x_2, \ldots, x_r] x_{r+1}}{[(x_1, x_{r+1}), (x_2, x_{r+1}), \ldots, (x_r, x_{r+1})]} \\
&\geq \frac{[x_1, x_2, \ldots, x_r] x_{r+1}}{H^r}.
\end{aligned}$$

By our induction hypothesis, the later is larger than

$$\frac{x_1 \cdot x_2 \cdots x_r}{H^{\frac{r(r-1)}{2}}} \cdot \frac{x_{r+1}}{H^r} = \frac{x_1 \cdot x_2 \cdots x_{r+1}}{H^{\frac{(r+1)r}{2}}},$$

and the proof of Lemma 4 is thus complete. □

Lemma 5 *Given r distinct positive integers $x_1, x_2, \ldots, x_r$, let $Y = \{y_1, y_2, \ldots, y_m\}$ be the set of values of $|x_\nu - x_\rho|$, where $1 \leq \rho < \nu \leq r$, and set $\gamma := [y_1, y_2, \ldots, y_m]$.*

Then, if $t \equiv j \pmod{\gamma}$,

$$\frac{\Lambda_j}{(x_1+j)(x_2+j)\cdots(x_r+j)} = \frac{\Lambda_t}{(x_1+t)(x_2+t)\cdots(x_r+t)}.$$

Proof Once again, we can carry the proof by induction on r. The result clearly holds for $r=2$. To carry the induction, we observe that

$$\begin{aligned}\frac{\Lambda_j}{(x_1+j)(x_2+j)\cdots(x_r+j)} &= \frac{[x_1+j, x_2+j, \ldots, x_r+j]}{(x_1+j)(x_2+j)\cdots(x_r+j)}\\ &= \frac{[[x_1+j, x_2+j, \ldots, x_{r-1}+j], x_r+j]}{(x_1+j)(x_2+j)\cdots(x_r+j)}\\ &= \frac{[x_1+j, x_2+j, \ldots, x_{r-1}+j]}{(x_1+j)(x_2+j)\cdots(x_{r-1}+j)([x_1+j, x_2+j, \ldots, x_{r-1}+j], x_r+j)}. \quad (1)\end{aligned}$$

By our induction hypothesis we have

$$\frac{[x_1+j, x_2+j, \ldots, x_{r-1}+j]}{(x_1+j)(x_2+j)\cdots(x_{r-1}+j)} = \frac{[x_1+t, x_2+t, \ldots, x_{r-1}+t]}{(x_1+t)(x_2+t)\cdots(x_{r-1}+t)}. \quad (2)$$

On the other hand, one can see that

$$\begin{aligned}&([x_1+j, x_2+j, \ldots, x_{r-1}+j], x_r+j)\\ &\quad = [(x_1+j, x_r+j), (x_2+j, x_r+j), \ldots, (x_{r-1}+j, x_r+j)].\end{aligned}$$

Since for each integer v, $1 \le v \le r-1$, we have that $(x_v+j, x_r+j) \mid |x_v - x_r|$ and by hypothesis $t \equiv j \pmod{|x_v - x_r|}$, it follows that $(x_v+j, x_r+j) = (x_v+t, x_r+t)$, so that

$$([x_1+j, x_2+j, \ldots, x_{r-1}+j], x_r+j) = ([x_1+t, x_2+t, \ldots, x_{r-1}+t], x_r+t). \quad (3)$$

Combining (1), (2), and (3), we obtain

$$\begin{aligned}&\frac{\Lambda_j}{(x_1+j)(x_2+j)\cdots(x_r+j)}\\ &\quad = \frac{[x_1+j, x_2+j, \ldots, x_r+j]}{(x_1+j)(x_2+j)\cdots(x_r+j)}\\ &\quad = \frac{[x_1+t, x_2+t, \ldots, x_r+t]}{(x_1+t)(x_2+t)\cdots(x_r+t)} = \frac{\Lambda_t}{(x_1+t)(x_2+t)\cdots(x_r+t)},\end{aligned}$$

thus completing the proof of Lemma 5. □

The following lemma guarantees the existence of infinitely many q-Niven r-tuples for any integer $2 \le r \le 2q$.

Lemma 6 *Given an integer* $q \ge 2$, *there exist infinitely many positive integers* n *such that* $\alpha(n+j) | n+j$ *for* $j = 0, 1, \ldots, 2q-1$.

Proof See Wilson [7]. □

Given two integers r and a, with $2 \le r \le q-1$ and $0 \le a \le q-r$, let $N_{r,a,0}(x)$ stand for the number of integers $n < x$ such $n, n+1, \dots, n+r-1$ are all q-Niven numbers and such that $n \equiv a \pmod{q}$. Observe that $N_{r,a,0}(x)$ counts those r-tuples of consecutive q-Niven numbers $n+i$, with $n < x$, such that none ends with a zero, with the possible exception of the first one which ends with the digit a (which can possibly be 0).

Let $h \in \mathbf{N}$. Then, given two integers r and a, with $q \le r \le 2q$ and $0 \le a \le 2q-r$, or with $2 \le r \le q-1$ and $q-r+1 \le a \le q-1$, let $N_{r,a,h}(x)$ stand for the number of positive integers $n < x$ such $n, n+1, \dots, n+r-1$ are all Niven numbers and such that $q^h \| n+q-a$. Observe that $N_{r,a,h}(x)$ counts those r-tuples of consecutive q-Niven numbers $n+i$, with $n < x$, such that the first one ends with the digit a while one (and exactly one) of the other ones ends with h consecutive zeros.

Remark There exists no r-tuple of consecutive Niven numbers such that $q \le r \le 2q$ and such that the first one begins with a, $a \ge 2q-r+1$. If it were the case, two of the Niven numbers would end with the digit 0. As in the proof of Wilson [7], we can show that it leads to a contradiction.

With these notations, we have

$$\mathcal{N}_q^{(r)}(x) = \sum_{a=0}^{q-r} N_{r,a,0}(x) + \sum_{h=1}^{N_x} \sum_{a=\max(0,q-r+1)}^{q-1} N_{r,a,h}(x),$$

where the first sum is taken as 0 whenever $q-r < 0$.

Lemma 7 *There exist positive constants $c_0(q,r,a)$ and $c_h(q,r,a)$ for $h = 1, 2, \dots$ such that*

$$N_{r,a,0}(x) = (1+o(1))c_0(q,r,a)\frac{x}{\log^r x} \quad (\textit{as } x \to \infty),$$

$$N_{r,a,h}(x) = (1+o(1))c_h(q,r,a)\frac{x}{\log^r x} \quad (\textit{uniformly for } h < (\log\log x)^2 \textit{ as } x \to \infty).$$

Proof We begin with the study of $N_{r,a,0}(x)$. Let n be counted by $N_{r,a,0}(x)$, so that $\alpha(n) = t$, $\alpha(n+1) = t+1$, $\alpha(n+2) = t+2$, $\dots$, $\alpha(n+r-1) = t+r-1$. Let $\gamma := [2, 3, \dots, r-1]$ and $\Lambda_t := \Lambda_t(0, 1, 2, \dots, r-1) = [t, t+1, t+2, \dots, t+r-1]$.

First observe that, assuming that $t \equiv j \pmod{\gamma}$, it follows from Lemma 5 that

$$\Lambda_t = \frac{t(t+1)(t+2)\dots(t+r-1)}{j(j+1)(j+2)\dots(j+r-1)}\Lambda_j$$
$$= \frac{(j-1)!}{(j+r-1)!}\Lambda_j \cdot t^r \prod_{i=1}^{r-1}\left(1+\frac{i}{t}\right).$$

Thus,

$$\Lambda_t = \frac{(j-1)!}{(j+r-1)!}\Lambda_j \cdot t^r \left(1 + O\left(\frac{1}{t}\right)\right) \quad (t \in I). \tag{4}$$

In light of Lemma 2, we only need to consider those $t \in I$, so that in particular we may assume that $t \gg \log x$.

Now, by the definition of $N_{r,a,0}(x)$, we have

$$N_{r,a,0}(x) = \sum_{j=1}^{[\gamma,q,q-1]} \sum_{\substack{1 \le t \le t_x \\ t \equiv j \pmod{[\gamma,q,q-1]}}} \sum_{\substack{1 \le n < x \\ \alpha(n)=t \\ n \equiv b \pmod{[\Lambda_t,q]}}} \delta(q,t,r,a), \tag{5}$$

where $\delta(q,t,r,a) = 1$ if there exists a (unique) positive solution $b < [\Lambda_t, q]$ to the system of congruences

$$\begin{aligned} b &\equiv 0 \pmod{t}, \\ b &\equiv -1 \pmod{t+1}, \\ &\vdots \\ b &\equiv -r+1 \pmod{t+r-1}, \\ b &\equiv a \pmod{q}, \end{aligned}$$

and $\delta(q,t,r,a) = 0$ if this system has no solution. Since the above system of congruences is equivalent to the two congruences

$$\begin{aligned} b &\equiv t \pmod{[t, t+1, \ldots, t+r-1]}, \\ b &\equiv a \pmod{q}, \end{aligned}$$

it follows that $\delta(q,t,r,a) = 1$ if and only if $a \equiv t \pmod{(\Lambda_t, q)}$. Assuming that $j \equiv t \pmod{q}$, we get $(\Lambda_t, q) = (\Lambda_j, q)$ so that $\delta(q,t,r,a) = \delta(q,j,r,a)$. Hence, from Lemmas 1 and 2, it follows that

$$\begin{aligned} N_{r,a,0}(x) &= (1+o(1)) \sum_{j=1}^{[\gamma,q,q-1]} \delta(q,j,r,a) \\ &\quad \sum_{\substack{t \equiv j \pmod{[\gamma,q,q-1]} \\ t \in I}} \frac{A(x|1,0,t)([\Lambda_t,q], q-1)}{[\Lambda_t,q]} + O\left(\frac{x}{\log^{2r} x}\right) \\ &= (1+o(1)) \sum_{\substack{j=1 \\ j \equiv a \pmod{(\Lambda_j,q)}}}^{[\gamma,q,q-1]} \\ &\quad \sum_{\substack{t \equiv j \pmod{[\gamma,q,q-1]} \\ t \in I}} \frac{A(x|1,0,t)([\Lambda_t,q], q-1)}{[\Lambda_t,q]} + O\left(\frac{x}{\log^{2r} x}\right). \end{aligned} \tag{6}$$

Clearly, $([\Lambda_t, q], q-1) = (\Lambda_t, q-1)$. Moreover, because $t \equiv j \pmod{q(q-1)}$, we have that $(\Lambda_t, q-1) = (\Lambda_j, q-1)$ and $[\Lambda_t, q] = \frac{\Lambda_t \cdot q}{(\Lambda_j, q)}$. Consequently, using (4) and Lemma 3, it follows from (6) that

$$N_{r,a,0}(x) = (1+o(1))x \sum_{\substack{j=1 \\ j\equiv a \pmod{(\Lambda_j,q)}}}^{[\gamma,q,q-1]} \frac{1}{[\gamma,q,q-1]}\left(\frac{2}{q-1}\frac{\log q}{\log x}\right)^r \times \frac{(j+r-1)!}{(j-1)!\Lambda_j}\frac{(\Lambda_j, q(q-1))}{q},$$

which we shall write as

$$N_{r,a,0}(x) = (1+o(1))c_0(q,r,a)\frac{x}{\log^r x},$$

say, thus establishing the first estimate of Lemma 7.

We now estimate $N_{r,a,h}(x)$ for each $h \geq 1$. Let n be counted by $N_{r,a,h}(x)$ and set $t = \alpha(n+q-a)$. It follows that $\alpha(n) = t+h(q-1)-(q-a)$, $\alpha(n+1) = t+h(q-1)-(q-a)+1$, $\alpha(n+2) = t+h(q-1)-(q-a)+2, \ldots, \alpha(n+(q-a)-1) = t+h(q-1)-1$, $\alpha(n+q-a) = t, \ldots, \alpha(n+r-1) = t+r-1-(q-a)$. Then, set

$$\gamma := [2,3,\ldots,\max(q, h(q-1)-1)]$$

and

$$\begin{aligned}\Lambda_t :&= \Lambda_t(h(q-1)-(q-a),\ldots,h(q-1)-1,0,1,\ldots,r-1-(q-a)) \\ &= [t+h(q-1)-(q-a),\ldots,t+h(q-1)-1,t,t+1,\ldots, \\ &\quad t+r-1-(q-a)].\end{aligned}$$

As in the case $h=0$, if $t \equiv j \pmod{\gamma}$, we have, because of Lemma 5,

$$\begin{aligned}\Lambda_t &= \frac{(t+h(q-1)-1)!}{(t+h(q-1)-(q-a)-1)!}\frac{(t+r-1-(q-a))!}{(t-1)!} \\ &\quad\times \frac{(j+h(q-1)-(q-a)-1)!}{(j+h(q-1)-1)!}\frac{(j-1)!}{(j+r-1-(q-a))!}\Lambda_j \\ &= (t+h(q-1))^{q-a}\prod_{i=1}^{q-a}\left(1-\frac{i}{t+h(q-1)}\right)t^{r-(q-a)}\prod_{i=0}^{r-1-(q-a)}\left(1+\frac{i}{t}\right) \\ &\quad\times \frac{(j+h(q-1)-(q-a)-1)!}{(j+h(q-1)-1)!}\frac{(j-1)!}{(j+r-1-(q-a))!}\Lambda_j.\end{aligned}$$

Thus,

$$\Lambda_t = t^r\left(1+O\left(\frac{h}{t}\right)\right)\cdot\frac{(j+h(q-1)-(q-a)-1)!}{(j+h(q-1)-1)!}\frac{(j-1)!}{(j+r-1-(q-a))!}\Lambda_j$$
$$(t\in I, h\leq(\log\log x)^2). \qquad (7)$$

By the definition of $N_{r,a,h}(x)$, we have

$$N_{r,a,h}(x) = \sum_{k=1}^{q-1} \sum_{j=1}^{[\gamma,q^{h+1},q-1]} \sum_{t\equiv j \pmod{[\gamma,q^{h+1},q-1]}} \sum_{\substack{n<x \\ \alpha(n+q-a)=t \\ n\equiv b \pmod{[\Lambda_t,q^{h+1}]}}} \delta(q,t,r,a,h,k), \tag{8}$$

where $\delta(q,t,r,a,h,k) = 1$ if there exists a (unique) positive solution $b < [\Lambda_t, q^{h+1}]$ to the system of congruences

$$\begin{aligned}
b &\equiv 0 \pmod{t + h(q-1) - (q-a)},\\
b &\equiv -1 \pmod{t + h(q-1) - (q-a) + 1},\\
&\ \vdots\\
b &\equiv -q + a + 1 \pmod{t + h(q-1) - 1},\\
b &\equiv -q + a \pmod{t},\\
&\ \vdots\\
b &\equiv -r + 1 \pmod{t + r - 1 - (q-a)},\\
b &\equiv -q + a \pmod{q^h},\\
b &\equiv -q + a + kq^h \pmod{q^{h+1}}.
\end{aligned}$$

Now this system of congruences is equivalent to the three congruences

$$\begin{aligned}
b &\equiv t + h(q-1) - (q-a) \quad (\mathrm{mod}\ [t + h(q-1) - (q-a),\\
&\qquad t + h(q-1) - (q-a) + 1, \dots, t + h(q-1) - 1]),\\
b &\equiv t - q + a \quad (\mathrm{mod}\ [t, t+1, \dots, t + r - 1 - (q-a)]),\\
b &\equiv -q + a + kq^h \pmod{q^{h+1}},
\end{aligned}$$

a solution of which exists if and only if the following three congruences are true:

$$\begin{cases}
h(q-1) \equiv 0 \quad (\mathrm{mod}\ ([t + h(q-1) - (q-a), \dots, t + h(q-1) - 1],\\
\qquad [t, t+1, \dots, t + r - 1 - (q-a)])),\\
t \equiv kq^h \quad (\mathrm{mod}\ ([t, t+1, \dots, t + r - 1 - (q-a)], q^{h+1})),\\
t + h(q-1) \equiv kq^h \quad (\mathrm{mod}\ ([t + h(q-1) - (q-a), \dots,\\
\qquad t + h(q-1) - 1], q^{h+1})).
\end{cases} \tag{9}$$

One can easily show that, given any positive integers $a_1, a_2, \dots, a_r, b_1, b_2, \dots, b_s$,

$$\begin{aligned}
&([a_1, a_2, \dots, a_r], [b_1, b_2, \dots, b_s])\\
&\quad = [(a_1, b_1), (a_1, b_2), \dots, (a_1, b_s), (a_2, b_1), (a_2, b_2), \dots, (a_2, b_s), \dots,\\
&\qquad (a_r, b_1), (a_r, b_2), \dots, (a_r, b_s)],
\end{aligned}$$

so that assuming that $t \equiv j \pmod{[\gamma, q^{h+1}, q-1]}$, we obtain that

$$\begin{aligned}
&\big([t+h(q-1)-(q-a),\dots,t+h(q-1)-1],[t,t+1,\dots,t+r-1-(q-a)]\big)\\
&\quad=[(t+h(q-1)-(q-a),t),(t+h(q-1)-(q-a),t+1),\dots,\\
&\qquad(t+h(q-1)-(q-a),t+r-1-(q-a)),\dots,\\
&\qquad(t+h(q-1)-1,t),(t+h(q-1)-1,t+1),\dots,\\
&\qquad(t+h(q-1)-1,t+r-1-(q-a)]\\
&\quad=[(j+h(q-1)-(q-a),j),(j+h(q-1)-(q-a),j+1),\dots,\\
&\qquad(j+h(q-1)-(q-a),j+r-1-(q-a)),\dots,\\
&\qquad(j+h(q-1)-1,j),(j+h(q-1)-1,j+1),\dots,\\
&\qquad(j+h(q-1)-1,j+r-1-(q-a)]\\
&\quad=\big([j+h(q-1)-(q-a),\dots,j+h(q-1)-1],\\
&\qquad[j,j+1,\dots,j+r-1-(q-a)]\big),
\end{aligned}$$

$$\big([t,t+1,\dots,t+q-a-1],q^{h+1}\big)=\big([j,j+1,\dots,j+q-a-1],q^{h+1}\big)$$

and

$$\begin{aligned}
&\big([t+h(q-1)-(q-a),\dots,t+h(q-1)-1],q^{h+1}\big)\\
&\quad=\big([j+h(q-1)-(q-a),\dots,j+h(q-1)-1],q^{h+1}\big),
\end{aligned}$$

so that under this hypothesis, $\delta(q,t,r,a,h,k)=\delta(q,j,r,a,h,k)$.

Now by Lemma 6, for $q \ge 2$ and $r \in [2, 2q]$, there exist $j, a, h,$ and k such that $\delta(q,j,r,a,h,k)=1$. Hence, using Lemmas 1 and 2, (8) becomes

$$\begin{aligned}
N_{r,a,h}(x) &= \left(1+O\left(\frac{1}{\log^{1/8} x}\right)\right)\\
&\quad\times\sum_{k=1}^{q-1}\sum_{\substack{j=1\\ j \text{ satisfies (9)}}}^{[\gamma,q^{h+1},q-1]}\sum_{\substack{t\in I\\ t\equiv j \pmod{[\gamma,q^{h+1},q-1]}}}\frac{A(x|1,0,t)}{[\Lambda_t,q^{h+1}]}([\Lambda_t,q^{h+1}],q-1)\\
&\quad+O\left(\frac{x}{\log^{2r} x}\right).
\end{aligned}\tag{10}$$

Observing that

$$\frac{([\Lambda_t,q^{h+1}],q-1)}{[\Lambda_t,q^{h+1}]}=\frac{(\Lambda_t,q-1)}{[\Lambda_t,q^{h+1}]}=\frac{(\Lambda_t,q-1)\cdot(\Lambda_t,q^{h+1})}{\Lambda_t\cdot q^{h+1}}=\frac{(\Lambda_j,q^{h+1}(q-1))}{\Lambda_t\cdot q^{h+1}}$$

and using (7) to estimate Λ_t and Lemma 3 to estimate $A(x|1,0,t)$, relation (10) becomes

$$N_{r,a,h}(x) = \left(1 + O\left(\frac{1}{\log^{1/8} x}\right)\right) x \sum_{k=1}^{q-1} \sum_{\substack{j=1 \\ j \text{ satisfies (9)}}}^{[\gamma, q^{h+1}, q-1]} \sum_{\substack{t \in I \\ t \equiv j \pmod{[\gamma, q^{h+1}, q-1]}}} R_1 R_2 + O\left(\frac{x}{\log^{2r} x}\right), \tag{11}$$

where

$$R_1 = \frac{L_x(t)(t + h(q-1) - (q-a))!t!}{(t + h(q-1) - 1)!(t + r - 1 - (q-a))!} = \frac{L_x(t)}{t^r(1 + O(\frac{h}{t}))},$$

$$R_2 = R_2(j) = \frac{(j + h(q-1) - 1)!(j + r - 1 - (q-a))!}{(j + h(q-1) - (q-a) - 1)!(j-1)!} \cdot \frac{(\Lambda_j, q^{h+1}(q-1))}{q^{h+1}\Lambda_j}.$$

We may therefore write (11) as

$$\begin{aligned} N_{r,a,h}(x) &= \left(1 + O\left(\frac{h}{\log^{1/8} x}\right)\right) x \left(\frac{2}{q-1} \frac{\log q}{\log x}\right)^r \sum_{\substack{j=1 \\ j \text{ satisfies (9)}}}^{[\gamma, q^{h+1}, q-1]} \frac{R_2}{[\gamma, q^{h+1}, q-1]} \\ &\quad + O\left(\frac{x}{\log^{2r} x}\right) \\ &= \left(1 + O\left(\frac{(\log\log x)^2}{\log^{1/8} x}\right)\right) x \left(\frac{2}{q-1} \frac{\log q}{\log x}\right)^r \sum_{\substack{j=1 \\ j \text{ satisfies (9)}}}^{[\gamma, q^{h+1}, q-1]} \frac{R_2}{[\gamma, q^{h+1}, q-1]} \\ &\quad + O\left(\frac{x}{\log^{2r} x}\right), \end{aligned}$$

an estimate which holds uniformly for $1 \le h \le (\log\log x)^2$ and which we shall write as

$$N_{r,a,h}(x) = \left(1 + O\left(\frac{(\log\log x)^2}{\log^{1/8} x}\right)\right) c_h(q, r, a) \frac{x}{\log^r x}, \tag{12}$$

say, which clearly proves the second estimate of Lemma 7. □

Lemma 8 *Given $r \ge 2$,*

$$\lim_{T \to \infty} \lim_{x \to \infty} \sum_{T < h < (\log\log x)^2} \sum_{a=\max(0, q-r+1)}^{q-1} \frac{N_{r,a,h}(x)}{x} \log^r x = 0. \tag{13}$$

Proof Assume $h < (\log\log x)^2$ and that n is counted by $N_{r,a,h}(x)$. Then there exists $m \in \mathbf{N_0}$ such that $n = mq^h - q + a$, $q \nmid m$, and we also have $\alpha(n+j) | n+j$ for $j = 0, 1, \ldots, r-1$. Setting $t = \alpha(m)$, we get

$$\begin{aligned} \alpha(n+j) &= \alpha(m-1) + (h-1)(q-1) + a + j \\ &\quad (j = 0, 1, \ldots, q-a-1), \\ \alpha(n + (q-a) + \ell) &= \alpha(m) + \ell \quad (\ell = 0, 1, \ldots, r-q+a-1) \end{aligned}$$

and

$$\begin{cases} mq^h - q + a + j \equiv 0 \pmod{t - 1 + (h-1)(q-1) + a + j} \\ \qquad\qquad (j = 0, 1, \ldots, q-a-1), \\ mq^h + \ell \equiv 0 \pmod{t+\ell} \ (\ell = 0, 1, \ldots, r-q+a-1). \end{cases} \tag{14}$$

If we fix t, the solutions m of (14) form a residue class m_0 mod $K_t^{(h)}$, where

$$K_t^{(h)} := \left[\frac{t+\ell}{(t+\ell, q^h)} : \ell \in \mathcal{B} \right],$$

with

$$\mathcal{B} = \{0, 1, \ldots, r-q+a-1, (h-1)(q-1)+a-1, \ldots, (h-1)(q-1)+q-2\}.$$

Hence

$$K_t^{(h)} \geq \frac{\Psi_t^{(h)}}{\xi_t^{(h)}},$$

where

$$\Psi_t^{(h)} := [t+\ell : \ell \in \mathcal{B}]$$

and

$$\xi_t^{(h)} := \prod_{\ell \in \mathcal{B}} \left((t+\ell), q^h \right).$$

But it follows from Lemma 4 that

$$\Psi_t^{(h)} \geq \frac{t^r}{(hq)^{\frac{r(r-1)}{2}}}. \tag{15}$$

Let $t \in I = [N_{(x/q^h)} - N_{(x/q^h)}^{5/8}, N_{(x/q^h)} + N_{(x/q^h)}^{5/8}]$. Using Lemma 1, we may write that

$$\#\left\{ m \leq \frac{x}{q^h} : \alpha(m) = t,\ m \equiv m_0 \pmod{K_t^{(h)}} \right\} = A\left(\frac{x}{q^h} | K_t^{(h)}, m_0, t \right)$$

and

$$\left| A\left(\frac{x}{q^h} | K_t^{(h)}, m_0, t \right) - \frac{(q-1, K_t^{(h)})}{K_t^{(h)}} A\left(\frac{x}{q^h} | 1, 0, t \right) \right|$$

$$\leq \frac{\kappa}{K_t^{(h)}} A\left(\frac{x}{q^h}|1,0,t\right)(\log x)^{-1/8},$$

so that in particular

$$A\left(\frac{x}{q^h}|K_t^{(h)},m_0,t\right) < \frac{2(q-1,K_t^{(h)})}{K_t^{(h)}} A\left(\frac{x}{q^h}|1,0,t\right).$$

Therefore, summing over all t, we obtain that

$$N_{r,a,h}(x) < \sum_{t\in I} \frac{2(q-1,K_t^{(h)})}{K_t^{(h)}} A\left(\frac{x}{q^h}|1,0,t\right) + \#\{m < x/q^h : \alpha(m) \notin I\}. \tag{16}$$

Moreover, using (15) we get

$$\begin{aligned}
&\sum_{t\in I} \frac{(q-1,K_t^{(h)})}{K_t^{(h)}} A\left(\frac{x}{q^h}|1,0,t\right)\\
&\quad \leq \sum_{t\in I} \frac{(q-1)(hq)^{\frac{r(r-1)}{2}}\xi_t^{(h)}}{t^r} A\left(\frac{x}{q^h}|1,0,t\right)\\
&\quad \leq \frac{(q-1)(hq)^{\frac{r(r-1)}{2}}}{\left(N_{(x/q^h)} - N_{(x/q^h)}^{5/8}\right)^r} \sum_{t\in I} \xi_t^{(h)} A\left(\frac{x}{q^h}|1,0,t\right).
\end{aligned} \tag{17}$$

Using the fact that $\xi_t^{(h)}$ depends only on the congruence class of t mod q^h, we can write

$$\sum_{t\in I} \xi_t^{(h)} A\left(\frac{x}{q^h}|1,0,t\right) = \sum_{1\leq D\leq q^h} \xi_D^{(h)} \sum_{\substack{t\in I\\ t\equiv D \pmod{q^h}}} A\left(\frac{x}{q^h}|1,0,t\right). \tag{18}$$

We also have

$$\sum_{\substack{t\in I\\ t\equiv D \pmod{q^h}}} A\left(\frac{x}{q^h}|1,0,t\right) \leq \frac{x}{q^{2h}} + A\left(\frac{x}{q^h}|1,0,N_{x/q^h}\right) \tag{19}$$

and for a constant κ_2 depending only on q

$$A\left(\frac{x}{q^h}|1,0,N_{x/q^h}\right) < \frac{\kappa_2 x}{q^h\sqrt{\log(x/q^h)}}. \tag{20}$$

From (18), (19) and (20), it follows that

$$\sum_{t\in I} \xi_t^{(h)} A\left(\frac{x}{q^h}|1,0,t\right) \leq \sum_{1\leq D\leq q^h} \frac{x\xi_D^{(h)}}{q^{2h}} + \sum_{1\leq D\leq q^h} \xi_D^{(h)} \frac{\kappa_2 x}{q^h\sqrt{\log(x/q^h)}}. \tag{21}$$

Writing the standard factorization of q as

$$q = \prod_{i=1}^{s} p_i^{\beta_i}$$

and defining

$$\xi_{t,p_i}^{(h)} = \left(\xi_t^{(h)}, p_i^{\beta_i h}\right) \quad (i = 1, \dots, s),$$

we then have

$$\sum_{1 \le D \le q^h} \xi_D^{(h)} = \sum_{1 \le D \le q^h} \prod_{i=1}^{s} \xi_{D,p_i}^{(h)} = \prod_{i=1}^{s} \left(\sum_{1 \le D \le p_i^{\beta_i h}} \xi_{D,p_i}^{(h)} \right). \tag{22}$$

Moreover, since $\#\mathcal{B} = r$ and the difference of two elements in $\mathcal{B}$ is at most qh, it is clear that

$$\xi_{D,p_i}^{(h)} \le \max_{\ell \in \mathcal{B}} (D + \ell, p_i^{\beta_i h})(qh)^{r-1} \quad (i = 1, \dots, s),$$

from which it follows that, for $i = 1, \dots, s$,

$$\begin{aligned}
\sum_{1 \le D \le p_i^{\beta_i h}} \xi_{D,p_i}^{(h)} &\le \sum_{0 \le F \le \beta_i h} \sum_{\substack{D \le p_i^{\beta_i h} \\ \max_{\ell \in \mathcal{B}} (D+\ell, p_i^{\beta_i h}) = p^F}} p_i^F (qh)^{r-1} \\
&\le \sum_{0 \le F \le \beta_i h} \sum_{\ell \in \mathcal{B}} \sum_{\substack{D \le p_i^{\beta_i h} \\ (D+\ell, p_i^{\beta_i h}) = p^F}} p_i^F (qh)^{r-1} \\
&\le \sum_{0 \le F \le \beta_i h} \sum_{\ell \in \mathcal{B}} p_i^{\beta_i h - F} p_i^F (qh)^{r-1} \\
&\le r \beta_i h p_i^{\beta_i h} (qh)^{r-1}.
\end{aligned} \tag{23}$$

From (22), (23) and the fact that $\beta_i \le \log q / \log 2$ for $1 \le i \le s$, we obtain that

$$\sum_{1 \le D \le q^h} \xi_D^{(h)} \le \prod_{i=1}^{s} r \beta_i h p_i^{\beta_i h} (qh)^{r-1} \le q^h \left(\frac{\log q}{\log 2} rh \right)^{\omega(q)} (qh)^{(r-1)\omega(q)}.$$

Using this, it follows from (21) that

$$\sum_{t \in I} \xi_t^{(h)} A\left(\frac{x}{q^h} | 1, 0, t \right) \le \left(\frac{\log q}{\log 2} hr \right)^{\omega(q)} (qh)^{(r-1)\omega(q)} \left(\frac{x}{q^h} + \frac{\kappa_2 x}{\sqrt{\log(x/q^h)}} \right). \tag{24}$$

On the other hand, under the hypothesis $h < (\log\log x)^2$ for x sufficiently large, we have that

$$\frac{\kappa_2 x}{\sqrt{\log(x/q^h)}}\left(\frac{\log q}{\log 2}hr\right)^{\omega(q)}(qh)^{(r-1)\omega(q)} < \frac{x}{\log^{1/4} x}. \tag{25}$$

Finally, observing that for x sufficiently large we have for any $h < (\log\log x)^2$

$$\left(N_{(x/h)} - N_{(x/h)}^{5/8}\right)^r \geq \frac{q-1}{4\log q}\log^r x$$

and combining (16), (17), (24) and (25), we obtain

$$\begin{aligned}
&\sum_{T \leq h \leq (\log\log x)^2} N_{r,a,h}\\
&\quad < \frac{\kappa_3 x}{\log^r x}\sum_{T \leq h \leq (\log\log x)^2}\left((q-1)(hq)^{\frac{r(r-1)}{2}}\frac{\left(\frac{\log q}{\log 2}hr\right)^{\omega(q)}(qh)^{(r-1)\omega(q)}}{q^h}\right.\\
&\qquad \left. + \frac{1}{\log^{1/4} x}\right),
\end{aligned}$$

where κ_3 is a constant depending only on q. Since the sum

$$\sum_{T \leq h \leq (\log\log x)^2}(q-1)(hq)^{\frac{r(r-1)}{2}}\frac{\left(\frac{\log q}{\log 2}hr\right)^{\omega(q)}(qh)^{(r-1)\omega(q)}}{q^h}$$

tends to 0 as $T \to \infty$, the proof of Lemma 8 is complete. □

Lemma 9 *The sum of the constants appearing in the second estimate of Lemma 7 converges, that is*

$$\sum_{h=1}^{\infty}\sum_{a=0}^{q-1} c_h(q,r,a) < \infty. \tag{26}$$

Proof Let $\varepsilon > 0$. By Lemma 8 there exists a number W such that

$$\lim_{x\to\infty}\sum_{W<h<(\log\log x)^2}\ \sum_{a=\max(0,q-r+1)}^{q-1}\frac{N_{r,a,h}(x)}{x}\log^r x \leq \varepsilon. \tag{27}$$

Now assume that the left hand side of (26) diverges, implying that there exists $W_1 > W$ such that

$$\sum_{W<h<W_1}\sum_{a=0}^{q-1} c_h(q,r,a) > 2\epsilon.$$

Then it would follow from Lemma 7 that

$$\begin{aligned}\lim_{x\to\infty} \sum_{W<h<W_1} \sum_{a=\max(0,q-r+1)}^{q-1} \frac{N_{r,a,h}(x)}{x}\log^r x &= \sum_{W<h<W_1} \sum_{a=\max(0,q-r+1)}^{q-1} \lim_{x\to\infty} \frac{N_{r,a,h}(x)}{x}\log^r x \\ &= \sum_{W<h<W_1} \sum_{a=\max(0,q-r+1)}^{q-1} c_h(q,r,a) > 2\varepsilon,\end{aligned}$$

contradicting (27) and thus completing the proof of Lemma 9. □

4 The proof of the theorem

To prove the theorem, we only need to show that for any $\varepsilon > 0$ there exists a real number x_0 such that for any $x > x_0$,

$$\left|\mathcal{N}_q^{(r)}(x) - \frac{cx}{\log^r x}\right| \le \varepsilon \frac{x}{\log^r x}.$$

By Lemma 8, there exists a number T_0 such that

$$\lim_{x\to\infty} \sum_{T_0<h<(\log\log x)^2} \sum_{a=\max(0,q-r+1)}^{q-1} \frac{N_{r,a,h}(x)}{x}\log^r x \le \frac{\varepsilon}{8}.$$

By Lemma 9, there exists a number T_1 such that

$$\sum_{h>T_1} \sum_{a=\max(0,q-r+1)}^{q-1} c_h(q,r,a) < \frac{\varepsilon}{4}.$$

Let $T_2 := \max(T_0, T_1)$ and choose x_0 so that for any $x > x_0$ we have

$$\sum_{T_2<h<(\log\log x)^2} \sum_{a=\max(0,q-r+1)}^{q-1} \frac{N_{r,a,h}(x)}{x}\log^r x \le \frac{\varepsilon}{4}.$$

By Lemma 7, there exists a number x_1 such that for any $h < T_2$ and any $x > x_1$ we have

$$\left|\sum_{a=\max(0,q-r+1)}^{q-1} N_{r,a,h}(x) - \sum_{a=\max(0,q-r+1)}^{q-1} c_h(q,r,a)\frac{x}{\log^r x}\right| \le \frac{\varepsilon}{4T_2}\frac{x}{\log^r x}.$$

Moreover, choose x_2 in such a way that for any $x > x_2$, $\frac{x}{q^{(\log\log x)^2}} \le \frac{\varepsilon}{4}\frac{x}{\log^r x}$ and set $x_3 := \max(x_0, x_1, x_2)$. We thus obtain, for $x > x_3$,

$$\left|\mathcal{N}_q^{(r)}(x) - \left(\sum_{a=0}^{q-r} c_0(q,r,a) + \sum_{h=1}^{T_2}\sum_{a=\max(0,q-r+1)}^{q-1} c_h(q,r,a)\right)\frac{x}{\log^r x}\right| \le \Sigma_1 + \Sigma_2 + \Sigma_3, \tag{28}$$

where

$$\Sigma_1 = \sum_{h=T_2}^{[(\log\log x)^2]}\sum_{a=\max(0,q-r+1)}^{q-1} N_{r,a,h}(x) \le \frac{\varepsilon}{4}\frac{x}{\log^r x},$$

$$\Sigma_2 = \sum_{h=[(\log\log x)^2]+1}^{\infty}\sum_{a=\max(0,q-r+1)}^{q-1} N_{r,a,h}(x) \le \frac{x}{q^{(\log\log x)^2}} \le \frac{\varepsilon}{4}\frac{x}{\log^r x},$$

$$\Sigma_3 = \sum_{h=0}^{T_2}\sum_{a=0}^{q-1}\left(N_{r,a,h}(x) - c_h(q,r,a)\frac{x}{\log^r x}\right) \le T_2\frac{\varepsilon}{4T_2}\frac{x}{\log^r x} = \frac{\varepsilon}{4}\frac{x}{\log^r x}.$$

Thus we finally obtain

$$\left|\mathcal{N}_q^{(r)}(x) - \frac{x}{\log^r x}\sum_{h=0}^{\infty}\sum_{a=0}^{q-1} c_h(q,r,a)\right| \le \frac{3\varepsilon}{4}\frac{x}{\log^r x} + \frac{x}{\log^r x}\sum_{h=T_2}^{\infty}\sum_{a=0}^{q-1} c_h(q,r,a) \le \frac{\varepsilon x}{\log^r x}.$$

Setting $c(q,r) := \sum_{h=0}^{\infty}\sum_{a=0}^{q-1} c_h(q,r,a)$, which is possible by Lemma 9, the proof of the theorem is complete.

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