Sums of reciprocals of additive functions running over short intervals

J.M. De Koninck[1] and I. Kátai[2]

Abstract

Letting $f(n) = A\log n + t(n)$, where $t(n)$ is a small additive function and A a positive constant, we obtain estimates for $\sum_{x\le n\le x+H} 1/f(Q(n))$ and $\sum_{x\le p\le x+H} 1/f(Q(p))$, where $H = H(x)$ satisfies certain growth conditions, p runs over prime numbers and Q is a polynomial with integer coefficients, whose leading coefficient is positive, and with all its roots simple.

2000 Mathematics Subject Classification: 11A25, 11N37

Édition du 14 novembre 2005

§1. Introduction

Let $t(n)$ be an additive function for which there exist two positive constants c and $\xi > 0$ such that

$$(1) \qquad |t(p^\alpha)| \le \frac{c}{p^\xi} \quad \text{for all prime powers } p^\alpha,$$

and let $A > 0$ be a fixed number; then let

$$(2) \qquad f(n) := A\log n + t(n).$$

Additive functions of the type (2) include the family of additive functions f for which Ivić [3] obtained estimates of $\sum_{n\le x,\ f(n)\ne 0} 1/f(n)$; the same is true for the family of additive functions studied by Brinitzer [1].

Let Q be a polynomial with integer coefficients, whose leading coefficient is positive, and such that all its roots are simple. Our goal here is to provide good estimates for each of the two sums

$$\sum_{x\le n\le x+H} \frac{1}{f(Q(n))} \qquad \text{and} \qquad \sum_{x\le p\le x+H} \frac{1}{f(Q(p))},$$

where $H = H(x)$ satisfies certain growth conditions and p runs over prime numbers. Let D be the discriminant of Q; for each prime p dividing D, we shall assume that there exists a positive integer $\beta_0 = \beta_0(p)$ such that $\tau(p^\beta) = \tau(p^{\beta+1}) = \ldots$ for each integer $\beta \ge \beta_0$.

[1]Research supported in part by a grant from NSERC.

[2]Research supported by the Applied Number Theory Research Group of the Hungarian Academy of Science and by a grant from OTKA.

From these estimates will follow good estimates for the more classical expressions

$$\sum_{x\le n\le x+H}\frac{1}{f(n)}\qquad\text{and}\qquad\sum_{x\le p\le x+H}\frac{1}{f(p+1)}.$$

§2. Main results

Theorem 1. *Let f be defined by (2). Let $\varepsilon<1$ be a fixed positive number and let $H=H(x)$ be an increasing function satisfying $x^{\varepsilon}\le H\le x^{1-\varepsilon}$ for all $x\ge x_0$ for a certain $x_0>0$. Moreover, let Q be as in Section 1. Then, given any positive integer r, there exist computable constants $e_1>0,e_2,\dots,e_r$ such that*

$$\sum_{x\le n\le x+H}\frac{1}{f(Q(n))}=H\sum_{j=1}^{r}\frac{e_j}{\log^j x}+O\left(\frac{H}{\log^{r+1}x}\right).$$

As usual, we define the logarithmic integral as follows

$$\mathrm{li}(x)=\int_2^x\frac{du}{\log u}.$$

Theorem 2. *Let f be defined by (2). Let $\varepsilon<1$ be a fixed positive number and let $H=H(x)$ be an increasing function satisfying $x^{\frac{7}{12}+\varepsilon}\le H\le x^{1-\varepsilon}$ for all $x\ge x_0$ for a certain $x_0>0$. Moreover, let Q be as in Section 1. Then, given any positive integer r, there exist computable constants $f_1>0,f_2,\dots,f_r$ such that*

$$\sum_{x\le p\le x+H}\frac{1}{f(Q(p))}=(li(x+H)-li(x))\sum_{j=1}^{r}\frac{f_j}{\log^j x}+O\left(\frac{H}{\log^{r+2}x}\right).$$

The following results are then consequences of the proofs of the above theorems.

Theorem 3. *Let f and A be as in (2). Let $\varepsilon<1$ be a fixed positive number and let $H=H(x)$ be an increasing function satisfying $x^{\varepsilon}\le H\le x^{1-\varepsilon}$ for all $x\ge x_0$ for a certain $x_0>0$. Then, given any positive integer r, there exist computable constants $b_1>0,b_2,\dots,b_r$ independent of A, such that*

$$\sum_{x\le n\le x+H}\frac{1}{f(n)}=H\sum_{j=1}^{r}\frac{b_j}{(A\log x)^j}+O\left(\frac{H}{\log^{r+1}x}\right).$$

Corollary. *Let g be either one of the following multiplicative functions*

$$g(n) = \sigma_k(n) := \sum_{d|n} d^k, \quad g(n) = \varphi(n) \text{ (Euler function) }, \quad g(n) = \tau^{(e)}(n),$$

where $\tau^{(e)}(n) = \tau^{(e)}(p_1^{\alpha_1} p_2^{\alpha_2} \dots p_s^{\alpha_s}) := \tau(\alpha_1)\tau(\alpha_2)\dots\tau(\alpha_s)$ stands for the number of exponential divisors of n, namely those divisors $d = p_1^{\beta_1} \dots p_s^{\beta_s}$ of n such that $\beta_i | \alpha_i$ for $i = 1, 2, \dots, r$. Let $H = H(x)$ be as in Theorem 1. Then, given any positive integer r, there exist computable constants $b_j = b_j(g)$, $1 \le j \le r$, such that

$$\sum_{x \le n \le x+H} \frac{1}{\log g(n)} = H \sum_{j=1}^{r} \frac{b_j}{\log^j x} + O\left(\frac{H}{\log^{r+1} x}\right).$$

Theorem 4. *Let f, A, ε and $H = H(x)$ be as in Theorem 2. Then, given any positive integer r, there exist computable constants $d_1 > 0, d_2, \dots, d_r$ independent of A, such that*

$$\sum_{x \le p \le x+H} \frac{1}{f(p+1)} = (li(x+H) - li(x)) \sum_{j=1}^{r} \frac{d_j}{(A \log x)^j} + O\left(\frac{H}{\log^{r+2} x}\right).$$

§3. Preliminary results

Lemma 1. *Let t be as in Section 1. Then*

$$|t(n)| \ll \frac{(\log n)^\beta}{\log\log n} \qquad (n \ge 3), \tag{3}$$

where $\beta = \max(1 - \xi, 1/4)$.

Proof. First, consider the case where $0 < \xi \le 3/4$. Then, let $t^{(1)}$ be the additive function defined on prime powers p^α by

$$t^{(1)}(p^\alpha) = \frac{c}{p^\xi}.$$

One can easily establish that

$$\max_{3 \le n \le x} t^{(1)}(n) \ll \frac{(\log x)^{1-\xi}}{\log\log x},$$

which clearly implies (3). On the other hand, if $\xi > 3/4$, then we have $|t(p^\alpha)| \le \frac{c}{p^\xi} < \frac{c}{p^{3/4}}$, so that the argument for the first case may be used again, in which case (3) follows once more.

Lemma 2. *Let $Q \in \mathbf{Z}[x]$ all the roots of which are simple. Let $\rho(m)$ be the number of solutions of $Q(n) \equiv 0 \pmod{m}$. Let D be the discriminant of Q. Then for each prime number p such that $(p, D) = 1$, we have that $\rho(p^\beta) = \rho(p)$ for each positive integer β.*

Proof. It is known that, for some positive integer x, $D = u(x)Q(x) + v(x)Q'(x)$ for some polynomials $u, v \in \mathbf{Z}[x]$. Now, given $\beta \geq 1$, let $x_1, \ldots, x_t \pmod{p^\beta}$ be the solutions of $Q(x) \equiv 0 \pmod{p^\beta}$. If $Q(y) \equiv 0 \pmod{p^{\beta+1}}$, then $y = x_i + tp^\beta \pmod{p^{\beta+1}}$ for some $t \in \{0, 1, \ldots, p-1\}$, so that $Q(x_i + tp^\beta) \equiv Q(x_i) + tp^\beta Q'(x_i) \pmod{p^{\beta+1}}$. Therefore, by $p|Q(x_i)$ and $p \not| Q'(x_i)$, we obtain that exactly one t is appropriate, which means that $\rho(p^{\beta+1}) = \rho(p^\beta)$, thus completing the proof of Lemma 2.

Let $\pi(x, k, \ell)$ denote the number of primes $p \leq x$ such that $p \equiv \ell \pmod{k}$.

Theorem A. *Let E be an arbitrary positive number and let $H = H(x)$ be as in Theorem 2. If $(k, \ell) = 1$, then uniformly for $k \leq \log^E x$,*

$$\pi(x + H, k, \ell) - \pi(x, k, \ell) = \frac{li(x + H) - li(x)}{\varphi(k)} \left(1 + O\left(\exp\{-c_1\sqrt{\log x}\}\right)\right)$$

for some positive constant c_1.

Proof. This follows directly from the Siegel-Walfisz Theorem (see Prachar [4], Chap. IX, Theorem 3.1) according to which, uniformly for $k \leq \log^E x$,

$$\psi(x + H, k, \ell) - \psi(x, k, \ell) = \frac{H}{\varphi(k)} \left(1 + O\left(\exp\{-c_1\sqrt{\log x}\}\right)\right),$$

where $\psi(x, k, \ell) := \sum_{\substack{n \leq x \\ n \equiv \ell \pmod{k}}} \Lambda(n)$, with Λ standing for the von Mangoldt function.

Remark. The exponent $\frac{7}{12}$ tied to the conditions on $H(x)$ (see statement of Theorem 2) comes from a result of Huxley [2].

§4. The proof of Theorem 1

We may clearly assume that $r + 1$ is even.

Let $x > 0$ be a large number.

Let k be the degree of the polynomial Q and let E be its leading coefficient. Let $\mathcal{J} = [x, x + H]$, $Y = Y(x) = \log^\eta x$, where η is a large number to be chosen later. Let also t_Y be the additive function defined on prime powers p^α by

$$t_Y(p^\alpha) = \begin{cases} t(p^\alpha) & \text{if } p^\alpha \le Y, \\ 0 & \text{otherwise} \end{cases}$$

and set

$$\kappa_Y(n) = t(n) - t_Y(n).$$

Finally, let $\rho(m)$ be the number of solutions of $Q(n) \equiv 0 \pmod m$ and set

$$f_1(Q(n)) = A \log Q(x) + t(Q(n)), \qquad f_2(Q(n)) = A \log Q(x) + t_Y(Q(n)).$$

Since $\dfrac{Q(x+j)}{Q(x)} = 1 + O\left(\dfrac{j}{x}\right)$ for $1 \le j \le H$, it follows that

$$\sum_{n \in \mathcal{J}} \left| \frac{1}{f(Q(n))} - \frac{1}{f_1(Q(n))} \right| \ll \frac{1}{\log^2 x} \sum_{n \in \mathcal{J}} \log \left| \frac{Q(n)}{Q(x)} \right| \ll \frac{H^2}{x \log^2 x}. \tag{4}$$

Moreover,

$$\sum_{n \in \mathcal{J}} \left| \frac{1}{f_1(Q(n))} - \frac{1}{f_2(Q(n))} \right| \ll \frac{T}{\log^2 x}, \tag{5}$$

where

$$T := \sum_{n \in \mathcal{J}} \left| \kappa_Y(Q(n)) \right|.$$

In order to estimate T, we first observe that it follows from (1) that

$$\begin{aligned} |\kappa_Y(Q(n))| &\le \sum_{\substack{p^\alpha \| Q(n) \\ p^\alpha > Y}} |t(p^\alpha)| = \sum_{\substack{p^\alpha \| Q(n) \\ Y < p^\alpha \le H}} |t(p^\alpha)| + \sum_{\substack{p^\alpha \| Q(n) \\ p^\alpha > H}} |t(p^\alpha)| \\ &= \kappa_Y^{(1)}(Q(n)) + \kappa_Y^{(2)}(Q(n)), \end{aligned}$$

say. Furthermore,

$$\begin{aligned} |\kappa_Y^{(2)}(Q(n))| &\le \sum_{\substack{p^\alpha \| Q(n) \\ p \le \sqrt{H},\ p^\alpha > H}} \frac{c}{p^\xi} + \sum_{\substack{p | Q(n) \\ \sqrt{H} < p \le H}} \frac{c}{p^\xi} + \frac{c}{H^\xi} \sum_{\substack{p | Q(n) \\ p > H}} 1 \\ &= K_1(Q(n)) + K_2(Q(n)) + K_3(Q(n)), \end{aligned} \tag{6}$$

say. Now let

$$T_j := \sum_{n \in \mathcal{J}} \kappa_Y^{(j)}(Q(n)) \qquad (j = 1, 2).$$

On the one hand,

$$\begin{aligned} T_1 &\ll H \sum_{p \ge Y} \frac{1}{p^{1+\xi}} + H \sum_{p^2 > Y} \frac{1}{p \cdot p^{1+\xi}} + H \sum_{\substack{p^\alpha \ge Y \\ \alpha \ge 3}} \frac{1}{p^{\alpha-1}} \frac{1}{p^{1+\xi}} \\ &\ll \frac{H}{Y^\xi} + \frac{H}{\sqrt{Y}}. \end{aligned} \tag{7}$$

On the other hand, it follows from (6) that

$$T_2 \le M_1 + M_2 + M_3, \tag{8}$$

where $M_\ell = \sum_{n\in\mathcal{J}} K_\ell(Q(n))$ for $\ell = 1, 2, 3$.

In order to estimate M_1, observe that the conditions $p^\alpha \| Q(n), p < \sqrt{H}, p^\alpha > H$ imply that there is a divisor p^β of p^α for which $\sqrt{H} \le p^\beta < H$ with $\beta \ge 2$. Consequently,

$$M_1 \ll H \sum_{p^\beta > \sqrt{H}} \frac{1}{p^\beta} \ll H^{3/4}, \tag{9}$$

say. Similarly, and by using Lemma 2, we infer that

$$M_2 \ll \frac{H}{H^{\xi/2}} \sum_{\sqrt{H}<p<H} \frac{\rho(p)}{p} \ll H^{1-\xi/2} \log\log H, \tag{10}$$

and that, since $\sum_{p|Q(n),\ p>H} 1$ is bounded,

$$M_3 \ll H^{1-\xi}. \tag{11}$$

Collecting (9), (10) and (11) in (8), it follows that

$$T_2 \ll H^{3/4} + H^{1-\xi/2}. \tag{12}$$

Substituting (7) and (12) into (5), we obtain that

$$\sum_{n\in\mathcal{J}} \left| \frac{1}{f_1(Q(n))} - \frac{1}{f_2(Q(n))} \right| \ll \frac{H}{\log^{\xi\eta+2} x} + \frac{H}{\log^{\frac{\eta}{2}+2}}, \tag{13}$$

provided $0 < \xi < 1$, which has indeed been assumed.

Then, letting $S(x, H) := \sum_{x\le n\le x+H} 1/f(Q(n))$ and

$$S^*(x, H) := \sum_{n\in\mathcal{J}} \frac{1}{f_2(Q(n))}, \tag{14}$$

it follows from (4) and (13) that

$$S(x, H) - S^*(x, H) = O\left(\frac{H}{\log^{r+1} x} \right),$$

provided $\eta = \eta(r, \xi, \varepsilon)$ is chosen large enough. This means that in order to complete the proof of Theorem 1, it is sufficient to prove that

$$S^*(x, H) = H \sum_{j=1}^{r} \frac{e_j}{\log^j x} + O\left(\frac{H}{\log^{r+1} x} \right). \tag{15}$$

First observe that it follows from Lemma 1 that

$$|t_Y(Q(n))| < \frac{A}{2}\log Q(x) \qquad (n \in \mathcal{J}),$$

so that

$$(16)\quad \begin{aligned} \frac{1}{f_2(Q(n))} &= \frac{1}{A\log Q(x) + t_Y(Q(n))} \\ &= \frac{1}{A\log Q(x)}\Big\{1 - \frac{t_Y(Q(n))}{A\log Q(x)} + \left(\frac{t_Y(Q(n))}{A\log Q(x)}\right)^2 + \ldots \\ &\qquad +(-1)^r\left(\frac{t_Y(Q(n))}{A\log Q(x)}\right)^r + O\left(\frac{|t_Y(Q(n))|^{r+1}}{\log^{r+1} Q(x)}\right)\Big\}. \end{aligned}$$

Now let

$$R_j(\mathcal{J}) := \sum_{n\in\mathcal{J}} t_Y^j(Q(n)),$$

so that from (14) and (16), we have

$$(17)\qquad S^*(x,H) = \sum_{j=0}^{r}(-1)^j\frac{R_j(\mathcal{J})}{(A\log Q(x))^{j+1}} + O\left(\frac{R_{r+1}(\mathcal{J})}{\log^{r+2} x}\right).$$

We shall now estimate each $R_j(\mathcal{J})$ with good accuracy. Indeed,

$$(18)\quad R_j(\mathcal{J}) = \sum_{\ell=1}^{j}\sum_{k_1+\ldots+k_\ell=j}\frac{j!}{k_1!\ldots k_\ell!}\sum_{p_1^{\alpha_1}<\ldots<p_\ell^{\alpha_\ell}\le Y} t(p_1^{\alpha_1})^{k_1}\ldots t(p_\ell^{\alpha_\ell})^{k_\ell}\Delta,$$

where $p_1,\ldots,p_\ell$ are any collection of distinct primes and $\alpha_1,\ldots,\alpha_\ell$ are positive integers such that $p_1^{\alpha_1} < \ldots < p_\ell^{\alpha_\ell} \le Y$ and

$$\Delta = \Delta(p_1^{\alpha_1},\ldots,p_\ell^{\alpha_\ell}) := \#\{n\in\mathcal{J} : p_j^{\alpha_j}\|Q(n),\ j=1,\ldots,\ell\}.$$

One easily sees that Δ may be written as

$$(19)\qquad \Delta = H\sum_{\delta|p_1\ldots p_\ell}\frac{\mu(\delta)}{\delta}\frac{\rho(p_1^{\alpha_1}\ldots p_\ell^{\alpha_\ell}\delta)}{p_1^{\alpha_1}\ldots p_\ell^{\alpha_\ell}} + O\left(\sum_{\delta|p_1\ldots p_\ell}\rho(p_1^{\alpha_1}\ldots p_\ell^{\alpha_\ell}\delta)\right)$$

Clearly the contribution of the error term in (19) to the right hand side of (18) is $O_j(1)$.

Now writing

$$\Sigma^*(Y|k_1,\ldots,k_\ell) = \sum_{p_1^{\alpha_1}<\ldots<p_\ell^{\alpha_\ell}\le Y} t(p_1^{\alpha_1})^{k_1}\ldots t(p_\ell^{\alpha_\ell})^{k_\ell}\sum_{\delta|p_1\ldots p_\ell}\frac{\mu(\delta)\rho(p_1^{\alpha_1}\ldots p_\ell^{\alpha_\ell}\delta)}{\delta p_1^{\alpha_1}\ldots p_\ell^{\alpha_\ell}},$$

it follows from (18) and (19) that

$$
\begin{aligned}
(20)\qquad R_j(\mathcal{J}) &= H\sum_{\ell=1}^{j}\sum_{k_1+\ldots+k_\ell=j}\frac{j!}{k_1!\ldots k_\ell!}\Sigma^*(Y|k_1,\ldots,k_\ell)+O_j(1)\\
&= HD_j(Y)+O_j(1),
\end{aligned}
$$

say. We shall now manage to replace $D_j(Y)$ by

$$
D_j := \sum_{\ell=1}^{j}\sum_{k_1+\ldots+k_\ell=j}\frac{j!}{k_1!\ldots k_\ell!}\Sigma^*(\infty|k_1,\ldots,k_\ell),
$$

while carefully monitoring the error term thus created by this substitution, that is by showing that

$$
(21)\qquad |\Sigma^*(\infty|k_1,\ldots,k_\ell)-\Sigma^*(Y|k_1,\ldots,k_\ell)| \ll \frac{1}{\sqrt{Y}}+\frac{1}{Y^\xi},
$$

thus enabling us, using (20), to replace (17) by

$$
(22)\qquad S^*(x,H) = H\sum_{j=0}^{r}(-1)^j\frac{D_j}{(A\log Q(x))^{j+1}}+O\left(\frac{H}{\log^{r+2}x}\right),
$$

provided η is chosen sufficiently large. Then, since

$$
A\log Q(x) = A\log(Ex^k+O(x^{k-1})) = Ak\log x + A\log E + O\left(\frac{1}{x}\right),
$$

it follows that

$$
(23)\qquad \frac{1}{(A\log Q(x))^{j+1}} = \sum_{\nu=j+1}^{r+1}\frac{u_{\nu,j}}{\log^\nu x}+O\left(\frac{1}{\log^{r+2}x}\right),
$$

with suitable constants $u_{\nu,j}$. Using (23) in (22), (15) follows.

Hence, it remains to prove (21). Indeed, using (1), it is clear that

$$
(24)\qquad \sum_{p_\ell^{\alpha_\ell}>Y}\frac{t(p_\ell^{\alpha_\ell})^{k_\ell}}{p_\ell^{\alpha_\ell}} \ll \frac{1}{\sqrt{Y}}+\frac{1}{Y^\xi}
$$

and that

$$
(25)\qquad \sum_{p_1^{\alpha_1}<\ldots<p_{\ell-1}^{\alpha_{\ell-1}}}\prod_{j=1}^{\ell-1}\frac{t(p_j^{\alpha_j})^{k_j}}{p_j^{\alpha_j}} = O(1).
$$

Since (21) clearly follows from (24) and (25), the proof of Theorem 1 is complete.

§5. The proof of Theorem 2

The proof is very similar to that of Theorem 1. As in that proof, we may clearly assume that $r+1$ is even. Using the notation introduced in Section 4 and repeating the same argument, we obtain that

$$
\begin{aligned}
(26)\ \sum_{p\in\mathcal{J}}\left|\frac{1}{f(Q(p))}-\frac{1}{f_2(Q(p))}\right| &\le \sum_{p\in\mathcal{J}}\left|\frac{1}{f(Q(p))}-\frac{1}{f_1(Q(p))}\right| \\
&\quad+\sum_{p\in\mathcal{J}}\left|\frac{1}{f_1(Q(p))}-\frac{1}{f_2(Q(p))}\right| \\
&\ll \frac{H^2}{x\log^2 x}+\frac{T}{\log^2 x}\ll\frac{H}{\log^{r+2} x},
\end{aligned}
$$

provided η is large enough. Then, proceeding as we did to obtain (16) and (17), we get that

$$
(27)\qquad \sum_{p\in\mathcal{J}}\frac{1}{f_2(Q(p))}=\sum_{j=0}^{r}(-1)^j\frac{S_j(\mathcal{J})}{(A\log Q(x))^{j+1}}+O\left(\frac{S_{r+1}(\mathcal{J})}{\log^{r+2} x}\right),
$$

where

$$
S_j(\mathcal{J})=\sum_{p\in\mathcal{J}}t_Y^j(Q(p)).
$$

Then

$$
(28)\ S_j(\mathcal{J})=\sum_{\ell=1}^{j}\sum_{k_1+\ldots+k_\ell=j}\frac{j!}{k_1!\ldots k_\ell!}\sum_{p_1^{\alpha_1}<\ldots<p_\ell^{\alpha_\ell}\le Y}t(p_1^{\alpha_1})^{k_1}\ldots t(p_\ell^{\alpha_\ell})^{k_\ell}\Delta_*,
$$

where again $p_1,\ldots,p_\ell$ are any collection of distinct primes and $\alpha_1,\ldots,\alpha_\ell$ are positive integers such that $p_1^{\alpha_1}<\ldots<p_\ell^{\alpha_\ell}\le Y$ and

$$
\Delta_*=\Delta_*(p_1^{\alpha_1},\ldots,p_\ell^{\alpha_\ell}):=\#\{p\in\mathcal{J}:p_j^{\alpha_j}\|Q(p),\ j=1,\ldots,\ell\}.
$$

Then, letting $\rho^*(m)$ be the number of residue classes $s \pmod m$ such that $Q(s)\equiv 0 \pmod m$ and $(s,m)=1$, and calling upon Theorem A, we obtain that

$$
\begin{aligned}
(29)\,\Delta_*(p_1^{\alpha_1},\ldots,p_\ell^{\alpha_\ell}) &= (\mathrm{li}(x+H)-\mathrm{li}(x))\sum_{\delta|p_1\ldots p_\ell}\frac{\mu(\delta)}{\delta}\frac{\rho^*(p_1^{\alpha_1}\ldots p_\ell^{\alpha_\ell}\delta)}{\varphi(p_1^{\alpha_1}\ldots p_\ell^{\alpha_\ell})} \\
&\quad+O\left(\frac{H}{\log x}\exp\{-c_1\sqrt{\log x}\}\frac{1}{p_1^{\alpha_1}\ldots p_\ell^{\alpha_\ell}}\right).
\end{aligned}
$$

Then, observing that the contribution of the error term in (29) to the sum in (28) is at most $O\left(\frac{H}{\log x}\exp\{-c_1\sqrt{\log x}\}\right)$, and continuing the proof as we did in Section 4, we find that

$$
(30)\qquad S_j(\mathcal{J})=(\mathrm{li}(x+H)-\mathrm{li}(x))K_j+O\left(\frac{H}{\log^{r+3} x}\right),
$$

where

$$K_j := \sum_{\ell=1}^{j} \sum_{k_1+\ldots+k_\ell=j} \frac{j!}{k_1!\ldots k_\ell!} \sum_{p_1^{\alpha_1}<\ldots<p_\ell^{\alpha_\ell}} t(p_1^{\alpha_1})^{k_1}\ldots t(p_\ell^{\alpha_\ell})^{k_\ell}$$
$$\times \sum_{\delta|p_1\ldots p_\ell} \frac{\mu(\delta)}{\delta} \frac{\rho^*(p_1^{\alpha_1}\ldots p_\ell^{\alpha_\ell}\delta)}{\varphi(p_1^{\alpha_1}\ldots p_\ell^{\alpha_\ell})}.$$

Then substituting (30) in (27), and taking into account (26), the proof of Theorem 2 is complete.

Acknowledgment. The authors would like to thank the referee for some very helpful suggestions which helped to improve the final version of this paper.

References

[1] E. Brinitzer, *Eine asymptotische Formel für Summen über die reziproken Werte additiver Funktionen*, Acta Arith. **32** (1977), 387-391.

[2] M. Huxley, *On the difference between consecutive primes*, Invent. Math. **15** (1972), 164-170.

[3] A. Ivić, *The distribution of values of some multiplicative functions*, Publ. de l'Inst. Math. (Belgrade) **22** (36) (1977), 87-94.

[4] K. Prachar, *Primzahlverteilung*, Second Edition, Springer-Verlag, Berlin, 1978.

Jean-Marie De Koninck
Département de mathématiques
Université Laval
Québec G1K 7P4
Canada
jmdk@mat.ulaval.ca

Imre Kátai
Computer Algebra Department
Eötvös Loránd University, 1117 Budapest
Pázmány Péter Sétány I/C
Hungary
katai@compalg.inf.elte.hu

JMDK, le 14 novembre 2005; fichier: reciprocal3.tex